Natural star products on symplectic manifolds and quantum moment maps

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It is a pleasure to dedicate this paper to Alan Weinstein on the occasion of his sixtieth birthday.

Abstract

We define a natural class of star products: those which are given by a series of bidifferential operators which at order k in the deformation parameter have at most k derivatives in each argument. We show that any such star product on a symplectic manifold defines a unique symplectic connection. We parametrise such star products, study their invariance and give necessary and sufficient conditions for them to yield a quantum moment map.

We show that Kravchenko's sufficient condition [18] for a moment map for a Fedosov star product is also necessary.

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1 Introduction

The relation between a star product on a symplectic manifold and a symplectic connection on that manifold appears in many contexts. In particular, when one studies properties of invariance of star products, results are much easier when there is an invariant connection. We show here that there is a natural class of star products which define a unique symplectic connection. We study the invariance of such products and the conditions for them to have a moment map.

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2 Natural star products

Let (M, P) be a Poisson manifold. Let $C^\infty(M)$ be the space of $\mathbb{K}$-valued smooth functions on M, where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$. The space of Hochschild $(p+1)$-cochains on $C^\infty(M)$ with values in $C^\infty(M)$ which are given by differential operators in each argument (**differential** $(p+1)$**-cochains**) will be denoted by $\mathscr{D}^p$. Those which are differential operators of order at most k in each argument will be denoted by $\mathscr{D}^p_k$. $\mathfrak{d} : \mathscr{D}^p \to \mathscr{D}^{p+1}$ denotes the Hochschild coboundary operator, and $[A, B]$ denotes the Gerstenhaber bracket of cochains which we also write $\operatorname{ad} A\ B$. If $m(u, v) = uv$ denotes the multiplication of functions and $br(u, v) = \{u, v\} = P(du, dv)$ the Poisson bracket then $m \in \mathscr{D}^1_0$ and $br \in \mathscr{D}^1_1$. Finally we have the relation $\mathfrak{d}\, A = -\operatorname{ad} m\ A$ and $[\mathscr{D}^p_r, \mathscr{D}^q_s] \subset \mathscr{D}^{p+q}_{r+s-1}$.

Definition 2.1 A **natural star product** on (M, P) is a bilinear map

$$C^\infty(M) \times C^\infty(M) \to C^\infty(M)[\![\nu]\!], \qquad (u, v) \mapsto u * v := \sum_{r \geq 0} \nu^r C_r(u, v),$$

which defines a formally associative product on $C^\infty(M)[\![\nu]\!]$ when the map is extended $\mathbb{K}[\![\nu]\!]$-linearly (i.e. $(u * v) * w = u * (v * w)$) and such that

- $C_0 = m$;
- the skewsymmetric part of C_1 is the Poisson bracket: $C_1(u, v) - C_1(v, u) = 2\{u, v\}$;
- $1 * u = u * 1 = u$;

- each C_r is in $\mathscr{D}^1_r$ (i.e. is a **bidifferential operator** on M **of order at most** r **in each argument**).

Remark 2.2 It would be enough for some purposes to assume that C_1 be of order 1, and C_2 of order at most 2 in each argument; c.f. [22] where natural is defined this way, except that C_1 is the Poisson bracket. The main results still hold with the appropriate modifications.

Proposition 2.3 *Two natural star products $*$ and $*'$ on (M, P) are equivalent if and only if there is a series*

$$E = \sum_{r=1}^{\infty} \nu^r E_r$$

where the E_r are differential operators of order at most $r+1$, such that

$$f *' g = \operatorname{Exp} E\, ((\operatorname{Exp} -E)\, f * (\operatorname{Exp} -E)\, g)), \tag{1}$$

where Exp *denotes the exponential series.*

Proof We have, for any $E \in \mathscr{D}^0$, $C \in \mathscr{D}^1$

$$(\operatorname{ad} E\, C)(u, v) = E(C(u, v)) - C(Eu, v) - C(u, Ev).$$

and so

$$(\operatorname{Exp}(\operatorname{ad} E)\, C)\, (u, v) = \operatorname{Exp} E\, (C\, ((\operatorname{Exp} -E)\, u, (\operatorname{Exp} -E)\, v)).$$

Consider now a natural star product $* = \sum_{r \geq 0} \nu^r C_r$ and a series $E = \sum_{r=1}^{\infty} \nu^r E_r$ where the E_r are differential operators of order at most $r+1$. From the observation above and $[\mathscr{D}^0_r, \mathscr{D}^1_s] \subset \mathscr{D}^1_{r+s-1}$ we have

$$f *' g = \operatorname{Exp} E\, (\operatorname{Exp} -Ef * \operatorname{Exp} -Eg) = (\operatorname{Exp}(\operatorname{ad} E) *)(u, v)$$

is a natural star product on M.

Reciprocally, if $*$ and $*'$ are two natural star products which are equivalent, we shall show that the equivalence is necessarily given by $f *' g = \operatorname{Exp} E\, (\operatorname{Exp} -E\, f * \operatorname{Exp} -E\, g) = (\operatorname{Exp}(\operatorname{ad} E) *)(u, v)$ where $E = \sum_{r=1}^{\infty} \nu^r E_r$ with $E_r \in \mathscr{D}^0_{r+1}$. The fact that $*$ and $*'$ are equivalent can be written $f *' g = \operatorname{Exp} E\, (\operatorname{Exp} -Ef * \operatorname{Exp} -Eg)$ where $E = \sum_{r=1}^{\infty} \nu^r E_r$ is a formal series of linear maps from $C^\infty(M)$ to $C^\infty(M)$. We shall prove by induction that all the E_r are in D^0_{r+1}. Assume the E_r are in $\mathscr{D}^0_{r+1}$ for $r \leq k$. Define $E^{(k)} = \sum_{r=1}^{k} \nu^r E_r$ and $*'' = (\operatorname{Exp}(\operatorname{ad} E^{(k)}) *) =: \sum_{r \geq 0} \nu^r C''_r$. It coincides with $*'$ up to and including order k, and it is a natural star product on M. The difference $C''_{k+1} - C'_{k+1}$ is a bidifferential operator which is a Hochschild 2-cocycle whose skewsymmetric part vanishes because it is given by ∂E_{k+1}; hence it is the coboundary of a differential operator; since this bidifferential operator is of order at most $k+1$ in each argument, it is the coboundary $= \partial E'_{k+1}$ of an element $E'_{k+1} \in \mathscr{D}^0_{k+2}$. Thus $E_{k+1} - E'_{k+1}$ is a 1-cocycle, i.e. a vector field. This proves that E_{k+1} is in $\mathscr{D}^0_{k+2}$. □

Remark 2.4 The space of formal differential operators considered in Proposition 2.3, $E = \sum_{r=1}^{\infty} \nu^r E_r$ where the E_r are differential operators of order at most $r+1$, is a pronilpotent Lie algebra.

Remark 2.5 All the explicit constructions of star products are natural:

- star products on cotangent bundles [5, 6, 7];
- star products given by Fedosov's construction [8, 9];
- star products on the dual of a Lie algebra or coadjoint orbits [1, 11];
- star products with separation of variables on a Kaehler manifold [16];
- star products given by Kontsevich's construction [17].

3 Connections

The link between the notion of star product on a symplectic manifold and symplectic connections already appears in the seminal paper of Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer [2], and was further developed by Lichnerowicz [19] who showed that any so called Vey star product (i.e. a star product defined by bidifferential operators whose principal symbols at each order coincide with those of the Moyal star product) determines a unique symplectic connection. Fedosov gave a construction of Vey star products starting from a symplectic connection and a series of closed two forms on the manifold. It was shown that any star product is equivalent to a Fedosov star product. Nevertheless, many star products which appear in natural contexts (cotangent bundles, Kaehler manifolds...) are not Vey star products (but are natural in the sense defined above). The aim of this section is to generalise the result of Lichnerowicz and to show that on any symplectic manifold, a natural star product determines a unique symplectic connection.

Given any torsion free linear connection ∇ on (M,P), the term of order 1 of a natural star product can be written

$$C_1 = \{\ ,\ \} - \mathfrak{d}\, E_1 = \{\ ,\ \} + (\operatorname{ad} E_1)\, m \qquad \text{where } E_1 \in \mathscr{D}^0_2$$

and the term of order 2 can be written in a chart

$$\begin{aligned} C_2(u,v) \;=\;& \tfrac{1}{2}((\operatorname{ad} E_1)^2\, m)(u,v) + ((\operatorname{ad} E_1)\,\{\ ,\ \})(u,v) \\ &+ \tfrac{1}{2} P^{ij} P^{i'j'}\, \nabla^2_{ii'} u\, \nabla^2_{jj'} v \\ &\quad + \tfrac{1}{6}(P^{rk}\nabla_r P^{jl} + P^{rl}\nabla_r P^{jk})(\nabla^2_{kl} u\, \nabla_j v + \nabla_j u\, \nabla^2_{kl} v) \\ &\quad - \mathfrak{d}\, E_2(u,v) + c_2(u,v) \end{aligned}$$

where $E_2 \in \mathscr{D}^0_3$ and where $c_2 \in \mathscr{D}^1_1$ is skewsymmetric.

Remark that E_1 is not uniquely defined; two choices differ by an element $X \in \mathscr{D}^0_1$. Observe that the first lines in the definition of C_2 for two such different choices only differ by an element in $\mathscr{D}^1_1$. Indeed

$$\begin{aligned} \tfrac{1}{2}\left(\operatorname{ad}(E_1+X)^2\, m\right)(u,v) + ((\operatorname{ad} E_1 + X)\,\{\ ,\ \})\,(u,v) \;=\;& \\ \tfrac{1}{2}((\operatorname{ad} E_1)^2\, m)(u,v) + ((\operatorname{ad} E_1)\,\{\ ,\ \})(u,v)& \\ + \tfrac{1}{2}((\operatorname{ad} X)\circ(\operatorname{ad} E_1)\, m)(u,v) + ((\operatorname{ad} X)\,\{\ ,\ \})(u,v)& \end{aligned}$$

and $(\operatorname{ad} E_1)\, m$, $(\operatorname{ad} X)\,\{\ ,\ \}$ are in $\mathscr{D}^1_1$, so also is $((\operatorname{ad} X)\circ(\operatorname{ad} E_1)\, m)$.

Changing the torsion free linear connection gives a modification of the terms of the second line of C_2; writing $\nabla' = \nabla + S$, this modification involves terms of order 2 in one argument and 1 in the other given by

$$\begin{aligned} \left(-\tfrac{1}{2}P^{rk}P^{sl}S^j_{rs} + \tfrac{1}{3}(P^{rk}S^j_{rs}P^{sl} + P^{rk}S^l_{rs}P^{js})\right)(\nabla^2_{kl}u\,\nabla_j v + \nabla_j u\,\nabla^2_{kl} v) \;=\;& \\ -\underset{jkl}{\circlearrowleft\!\!\!\!\bigoplus}\, \tfrac{1}{6}P^{rk}P^{sl}S^j_{rs}(\nabla^2_{kl}u\,\nabla_j v + \nabla_j u\,\nabla^2_{kl} v)& \end{aligned}$$

as well as terms of order 1 in each argument, where $\circlearrowleft\!\!\!\!\bigoplus$ denotes a cyclic sum over the indicated variables.

Notice that the terms above coincide with the terms of the same order in the coboundary of the operator $E' = \frac{1}{6}\underset{jkl}{\circlearrowleft\!\!\!\!\bigoplus} P^{rk}P^{sl}S^j_{rs}\nabla^3_{jkl}$.

If the Poisson tensor is invertible (i.e. in the symplectic situation), the symbol of any differential operator of order 3 can be written in this form E', hence we have:

Proposition 3.1 *A star product* $* = \sum_{r\geq 0} \nu^r C_r$ *on a symplectic manifold* (M, ω)*, so that* C_1 *is a bidifferential operator of order 1 in each argument and* C_2 *of order at most 2 in each argument, determines a unique symplectic connection* ∇ *such that*

$$C_1 = \{\ ,\ \} - \mathfrak{d}\, E_1 \qquad C_2 = \tfrac{1}{2}(\mathrm{ad}\, E_1)^2\, m + ((\mathrm{ad}\, E_1)\,\{\ ,\ \}) + \tfrac{1}{2}P^2(\nabla^2\cdot, \nabla^2\cdot) + A_2 \tag{2}$$

where $A_2 \in \mathscr{D}^1_1$ *and* $P^2(\nabla^2 u, \nabla^2 v)$ *denotes the bidifferential operator which is given by* $P^{ij} P^{i'j'}\, \nabla^2_{ii'} u\, \nabla^2_{jj'} v$ *in a chart.*

Remark 3.2 This shows, in particular, that any natural star product $* = \sum_{r\geq 0} \nu^r C_r$ on a symplectic manifold (M, ω) determines a unique symplectic connection ∇.

4 Symmetry and Invariance

Symmetries in quantum theories are automorphisms of the algebra of observables. Thus we define a symmetry σ of a star product $* = \sum_r \nu^r C_r$ as an automorphism of the $\mathbb{K}[\![\nu]\!]$-algebra $C^\infty(M)[\![\nu]\!]$ with multiplication given by $*$:

$$\sigma(u * v) = \sigma(u) * \sigma(v), \qquad \sigma(1) = 1,$$

where σ, being determined by what it does on $C^\infty(M)$, will be a formal series

$$\sigma(u) = \sum_{r\geq 0} \nu^r \sigma_r(u)$$

of linear maps $\sigma_r\colon C^\infty(M) \to C^\infty(M)$. In terms of the components of σ, the conditions to be an automorphism are

- $\displaystyle\sum_{r+s=k} \sigma_r(C_s(u,v)) = \sum_{r+s+t=k} C_r(\sigma_s(u), \sigma_t(v)), \quad k \geq 0, u, v \in C^\infty(M);$ (3)
- σ_0 is invertible;
- $\sigma_0(1) = 1, \qquad \sigma_r(1) = 0, \quad r \geq 1.$ (4)

Lemma 4.1 *If* $*$ *is a star product on a Poisson manifold* (M, P) *and* σ *is an automorphism of* $*$ *then it can be written* $\sigma(u) = T(u \circ \tau)$ *where* τ *is a Poisson diffeomorphism of* (M, P) *and* $T = \mathrm{Id} + \sum_{r\geq 1} \nu^r T_r$ *is a formal series of linear maps. If* $*$ *is differential, then the* T_r *are differential operators; if* $*$ *is natural, then* $T = \mathrm{Exp}\, E$ *with* $E = \sum_{r\geq 1} \nu^r E_r$ *and* E_r *is a differential operator of order at most* $r + 1$.

Proof Taking $k = 0$ in (3), we have $\sigma_0(uv) = \sigma_0(u)\sigma_0(v)$ so σ_0 is an automorphism of $C^\infty(M)$ and hence is composition with a diffeomorphism τ of M. Taking $k = 1$ in

(3), and antisymmetrising in u and v we have $\sigma_0(\{u,v\}) = \{\sigma_0(u), \sigma_0(v)\}$, and so τ is a Poisson map.

Set $T(u) = \sigma(u \circ \tau^{-1})$ so $T = \sigma \circ \sigma_0^{-1}$ which has the stated form. Define a new star product by $u *' v = \sigma_0(\sigma_0^{-1}(u) * \sigma_0^{-1}(v))$ then $*'$ is differential (resp. natural) if $*$ is. On the other hand $\sigma(u * v) = \sigma(u) * \sigma(v)$ implies $T(\sigma_0(u * v)) = T(\sigma_0(u)) * T(\sigma_0(v))$ and hence that $T(u *' v) = T(u) * T(v)$. Thus T is an equivalence between $*'$ and $*$. The result now follows from Theorem 2.22 of [12] and Proposition 2.3. $\square$

If σ_t is a one-parameter group of symmetries of the star product $*$, then its generator D will be a derivation of $*$. Denote the Lie algebra of derivations of $*$ by $\mathrm{Der}(M, *)$. Moreover by differentiating the statement of the Lemma, $D = \sum_{r \geq 0} \nu^r D_r$ with $D_0 = X$, a Poisson vector field ($\mathscr{L}_X P = 0$), and if $*$ is natural then each D_r for $r \geq 1$ is a differential operator of order at most $r + 1$. In the case we have a Lie group G acting by symmetries, then there will be an induced action of G on M and the infinitesimal automorphisms will give a homomorphism of Lie algebras $D\colon \mathfrak{g} \to \mathrm{Der}(M, *)$ from its Lie algebra $\mathfrak{g}$ into the derivations of the star product of the above form. For each $\xi \in \mathfrak{g}$, $D_\xi = \widetilde{\xi} + \sum_{r\geq 1} \nu^r D_\xi^r$ where $\widetilde{\xi}$ is the vector field generating the induced action of $\exp t\xi$ on M.

Definition 4.2 A star product $* = m + \sum_{r \geq 1} \nu^r C_r$ on a Poisson manifold (M, P) is said to be **invariant** under a diffeomorphism τ of M if $u \mapsto u \circ \tau$ is a symmetry of $*$.

Observe that if $*$ is τ-invariant then τ preserves each cochain C_r and hence the Poisson bracket. Invariance is a much stronger condition than being the leading term of a symmetry.

5 Quantum moment maps

A derivation $D \in \mathrm{Der}(M, *)$ is said to be **essentially inner** or **Hamiltonian** if $D = \frac{1}{\nu} \mathrm{ad}_* u$ for some $u \in C^\infty(M)[\![\nu]\!]$. We denote by $\mathrm{Inn}(M, *)$ the essentially inner derivations of $*$. It is a linear subspace of $\mathrm{Der}(M, *)$ and is the quantum analogue of the Hamiltonian vector fields.

By analogy with the classical case, we call an action of a Lie group **almost $*$-Hamiltonian** if each D_ξ is essentially inner, and call a linear choice of functions u_ξ satisfying

$$D_\xi = \tfrac{1}{\nu} \mathrm{ad}_* u_\xi, \qquad \xi \in \mathfrak{g}$$

a (quantum) Hamiltonian. We say the action is **$*$-Hamiltonian** if u_ξ can be chosen to make $\xi \mapsto u_\xi \colon \mathfrak{g} \to C^\infty(M)[\![\nu]\!]$ a homomorphism of Lie algebras. When $D_\xi = \widetilde{\xi}$, this map is called **a quantum moment map** [22].

Considering the map $a \colon C^\infty(M)[\![\nu]\!] \to \operatorname{Der}(M,*)$ given by

$$a(u)(v) = \tfrac{1}{\nu} \operatorname{ad}_* u(v) = \tfrac{1}{\nu}(u * v - v * u).$$

and defining a bracket on $C^\infty(M)[\![\nu]\!]$ by

$$[u,v]_* = \tfrac{1}{\nu}(u * v - v * u)$$

then, by associativity of the star product, a is a homomorphism of Lie algebras whose image is $\operatorname{Inn}(M,*)$. Since $D \circ a(u) - a(u) \circ D = a(Du)$, $\operatorname{Inn}(M,*)$ is an ideal in $\operatorname{Der}(M,*)$ and so there is an induced Lie bracket on the quotient $\operatorname{Der}(M,*)/\operatorname{Inn}(M,*)$.

Lemma 5.1 *([4]) If $*$ is a star product on a symplectic manifold (M,ω) then the space of derivations modulo inner derivations,* $\operatorname{Der}(M,*)/\operatorname{Inn}(M,*)$, *can be identified with $H^1(M,\mathbb{R})[\![\nu]\!]$ and the induced bracket is zero.*

Proof The first part is well known; let us recall that locally any derivation $D \in \operatorname{Der}(M,*)$ is inner, and that the ambiguity in the choice of a corresponding function u is locally constant so that the exact 1-forms du agree on overlaps and yield a globally defined (formal) closed 1-form α_D. The map $\operatorname{Der}(M,*) \to Z^1(M)[\![\nu]\!]$ defined by $D \mapsto \alpha_D$ if $D|_U = \frac{1}{\nu} \operatorname{ad}_* u$ and $\alpha_D|_U = du$ is a linear isomorphism with the space of formal series of closed 1-forms and maps essentially inner derivations to exact 1-forms inducing a bijection $\operatorname{Der}(M,*)/\operatorname{Inn}(M,*) \to Z^1(M)[\![\nu]\!]/d(C^\infty(M)[\![\nu]\!]) = H^1(M,\mathbb{R})[\![\nu]\!]$.

Let D_1 and D_2 be two derivations of $*$. $(D_1 \circ D_2 - D_2 \circ D_1)|_U = a([u_1,u_2]_*)$. But $[u_1,u_2]_*$ does not change if we add a local constant to either function, so is the restriction to U of a globally defined function which depends only on D_1 and D_2. We denote this function by $b(D_1,D_2)$ and have the identity

$$D_1 \circ D_2 - D_2 \circ D_1 = \tfrac{1}{\nu} \operatorname{ad}_* b(D_1,D_2).$$

This shows that $[\operatorname{Der}(M,*),\operatorname{Der}(M,*)] \subset \operatorname{Inn}(M,*)$ and hence that the induced bracket on $H^1(M,\mathbb{R})[\![\nu]\!]$ is zero. $\square$

The kernel of a consists of the locally constant formal functions $H^0(M,\mathbb{R})[\![\nu]\!]$ and hence:

Remark 5.2 If $*$ is a differential star product on a symplectic manifold (M,ω) then there is an exact sequence of Lie algebras

$$0 \to H^0(M,\mathbb{R})[\![\nu]\!] \hookrightarrow C^\infty(M)[\![\nu]\!] \xrightarrow{a} \operatorname{Der}(M,*) \xrightarrow{c} H^1(M,\mathbb{R})[\![\nu]\!] \to 0$$

where $c(D) = [\alpha_D]$.

Corollary 5.3 *(see also [22]) Let G be a Lie group of symmetries of a star product $*$ on (M,ω) and $d\sigma\colon \mathfrak{g} \to \mathrm{Der}(M,*)$ the induced infinitesimal action. If $H^1(M,\mathbb{R}) = 0$ or $[\mathfrak{g},\mathfrak{g}] \subset \mathfrak{g}$ then the action is almost $*$-Hamiltonian.*

Indeed, by definition, the action is almost $*$-Hamiltonian if $d\sigma(\mathfrak{g}) \subset \mathrm{Inn}(M,*)$. This is the case under either of the two conditions.

6 Moment Maps for a Fedosov Star Product

In this section we examine the necessary and sufficient conditions for a Fedosov star product to have a moment map. In order to do this it is necessary to examine the Fedosov construction in detail, which we therefore repeat here.

Having chosen a series of closed 2-forms $\Omega \in \nu\Lambda^2(M)[\![\nu]\!]$ and a symplectic connection ∇ on a symplectic manifold (M,ω), we consider the Fedosov star product associated to these data. This star product is obtained by identifying $C^\infty(M)[\![\nu]\!]$ with the space of flat sections of the Weyl bundle $\mathscr{W}$ (which is bundle of associative algebras) endowed with a flat connection (the Fedosov connection).

Sections of the Weyl bundle have the form of formal series

$$a(x,y,\nu) = \sum_{2k+l\geq 0} \nu^k a_{k,i_1,\dots,i_l}(x) y^{i_1} \cdots y^{i_l}$$

where the coefficients $a_{k,i_1,\dots,i_l}$ are symmetric covariant tensor fields on M; $2k+l$ is the degree in $\mathscr{W}$ of the corresponding homogeneous component.

The product of two sections taken pointwise makes the space of sections into an algebra, and in terms of the above representation of sections the multiplication has the form

$$(a \circ b)(x,y,\nu) = \left(\mathrm{Exp}\left(\tfrac{\nu}{2}\Lambda^{ij}\tfrac{\partial}{\partial y^i}\tfrac{\partial}{\partial z^j}\right) a(x,y,\nu) b(x,z,\nu)\right)\bigg|_{y=z},$$

with $\Lambda^{ij}\omega_{jk} = \delta^i_k$ (thus $\{f,g\} = \Lambda^{ij}\partial_i f \partial_j g$).

If we introduce the $\mathscr{W}$-valued 1-form $\overline{\Gamma}$ given by

$$\overline{\Gamma} = \tfrac{1}{2}\omega_{ki}\Gamma^k_{rj} y^i y^j dx^r$$

then the connection in $\mathscr{W}$ is given by

$$\partial a = da - \tfrac{1}{\nu}[\overline{\Gamma}, a].$$

More generally one looks at forms with values in the Weyl bundle, and locally sections of $\mathscr{W} \otimes \Lambda^q$ have the form

$$\sum_{2k+p\geq 0} \nu^k a_{k,i_1,\ldots,i_l,j_1,\ldots,j_q}(x) y^{i_1} \ldots y^{i_p}\, dx^{j_1} \wedge \ldots \wedge dx^{j_q}$$

where the coefficients are again covariant tensors, symmetric in $i_1, \ldots, i_p$ and anti-symmetric in $j_1, \ldots, j_q$. Such sections can be multiplied using the product in $\mathscr{W}$ and simultaneously exterior multiplication $a \otimes \omega \circ b \otimes \omega' = (a \circ b) \otimes (\omega \wedge \omega')$. The space of $\mathscr{W}$-valued forms $\Gamma(\mathscr{W} \otimes \Lambda^*)$ is then a graded Lie algebra with respect to the bracket

$$[s, s'] = s \circ s' - (-1)^{q_1 q_2} s' \circ s.$$

As usual, the connection ∂ in $\mathscr{W}$ extends to a covariant exterior derivative on all of $\Gamma(\mathscr{W} \otimes \Lambda^*)$, also denoted by ∂ by using the Leibnitz rule:

$$\partial(a \otimes \omega) = \partial(a) \wedge \omega + a \otimes d\omega.$$

The curvature of ∂ is then given by $\partial \circ \partial$ which is a 2-form with values in $\mathrm{End}(\mathscr{W})$. In this case it admits a simple expression in terms of the curvature R of the symplectic connection ∇:

$$\partial \circ \partial a = \tfrac{1}{\nu}[\overline{R}, a]$$

where

$$\overline{R} = \tfrac{1}{4}\omega_{rl} R^l_{ijk} y^r y^k\, dx^i \wedge dx^j. \tag{5}$$

Define

$$\delta(a) = dx^k \wedge \frac{\partial a}{\partial y^k}.$$

Note that δ can be written in terms of the algebra structure by

$$\delta(a) = \tfrac{1}{\nu}\left[-\omega_{ij} y^i dx^j, a\right].$$

With these preliminaries we construct a connection D on $\mathscr{W}$ of the form

$$Da = \partial a - \delta(a) - \tfrac{1}{\nu}[r, a] \tag{6}$$

which is flat: $D \circ D = 0$. Since

$$D \circ Da = \tfrac{1}{\nu}\left[\overline{R} + \delta r - \partial r + \tfrac{\nu}{2}[r, r], a\right]$$

one takes the solution r so that

$$\overline{R} + \delta r - \partial r + \tfrac{1}{2\nu}[r, r] = \Omega$$

and hence so that

$$Dr = \overline{R} - \Omega - \tfrac{1}{2\nu}[r, r]. \tag{7}$$

This solution is given by the iterative process

$$r = \delta^{-1}\left(-\overline{R} + \partial r - \tfrac{1}{2\nu}[r, r] + \Omega\right)$$

with r a 1-form with values in elements of $\mathscr{W}$ of degree at least 2. In the formula above $\delta^{-1}a_{pq} = \frac{1}{p+q}y^k i(\frac{\partial}{\partial x^k})a_{pq}$ if $p+q > 0$ and $\delta^* a_{pq} = 0$ if $p+q = 0$ where a_{pq} denotes the terms in a corresponding to a q-form with p y's. Remark then that $(\delta^{-1}\circ\delta + \delta\circ\delta^{-1})a = a - a_{00}$.

One has, for any smooth vector field X on M:

$$\delta \circ i(X) + i(X) \circ \delta = \tfrac{1}{\nu}\,\mathrm{ad}_*(\omega_{ij}X^i y^j)$$

$$\mathrm{ad}_* r \circ i(X) + i(X) \circ \mathrm{ad}_* r = \mathrm{ad}_*(i(X)r)$$

and

$$\partial \circ i(X) + i(X) \circ \partial = \mathscr{L}_X - (\nabla_i X)^j y^i \partial_{y^j}$$

which can be rewritten as

$$\partial \circ i(X) + i(X) \circ \partial = \mathscr{L}_X + \tfrac{1}{\nu}\,\mathrm{ad}_*\left(-\tfrac{1}{2}(\nabla_i(i(X)\omega))_j y^i y^j\right) + \tfrac{1}{2}(di(X)\omega)_{ip} y^i \Lambda^{jp} \partial_{y^j}.$$

This gives the generalised Cartan formula (which coincides with the one given by Neumaier [21])

$$\begin{aligned} \mathscr{L}_X \;&=\; D \circ i(X) + i(X) \circ D + \tfrac{1}{\nu}\,\mathrm{ad}_*(\omega_{ij}X^i y^j) + \tfrac{1}{\nu}\,\mathrm{ad}_*(i(X)r) && (8) \\ &\quad + \tfrac{1}{\nu}\,\mathrm{ad}_*\left(\tfrac{1}{2}(\nabla_i(i(X)\omega))_j y^i y^j\right) - \tfrac{1}{2}(di(X)\omega)_{ip} y^i \Lambda^{jp} \partial_{y^j}. && (9) \end{aligned}$$

The last term obviously drops out when X is a symplectic vector field.

If X is a symplectic vector field preserving the connection and preserving the series of 2-forms Ω, then $\mathscr{L}_X r = 0$ so

$$-Di(X)r = i(X)Dr + \tfrac{1}{\nu}\left[\omega_{ij}X^i y^j + \tfrac{1}{2}(\nabla_i(i(X)\omega))_j y^i y^j + i(X)r\ ,\ r\right]$$

Using equation (7), this gives

$$-Di(X)r = i(X)\overline{R} - i(X)\Omega + \tfrac{1}{\nu}\left[\omega_{ij}X^i y^j + \tfrac{1}{2}(\nabla_i(i(X)\omega))_j y^i y^j, r\right].$$

On the other hand, using the fact that $Da = \partial a - \delta(a) - \frac{1}{\nu}[r, a]$ one has

$$D(\omega_{ij}X^i y^j) = -i(X)\omega + \partial(\omega_{ij}X^i y^j) + \tfrac{1}{\nu}[\omega_{ij}X^i y^j, r]$$

and

$$\begin{aligned} D\left(\tfrac{1}{2}(\nabla_i(i(X)\omega))_j y^i y^j\right) &= -\nabla_i(i(X)\omega))_j dx^i y^j + \partial\left(\tfrac{1}{2}(\nabla_i(i(X)\omega))_j y^i y^j\right) \\ &\quad + \tfrac{1}{\nu}\left[\tfrac{1}{2}(\nabla_i(i(X)\omega))_j y^i y^j, r\right]. \end{aligned}$$

Since X is an affine vector field, one has $(i(X)R)(Y)Z = (\nabla^2 X)(Y,Z)$ so that

$$\partial\left(\tfrac{1}{2}(\nabla_i(i(X)\omega))_j y^i y^j\right) = -\tfrac{1}{2}((\nabla^2 X)^p_{ki}\omega)_{jp} y^i y^j dx^k = i(X)\overline{R}.$$

Hence

$$D\left(-i(X)r - \omega_{ij}X^i y^j - \tfrac{1}{2}(\nabla_i(i(X)\omega))_j y^i y^j\right) = i(X)\omega - i(X)\Omega.$$

So, for any vector field X so that $\mathscr{L}_X\omega = 0, \mathscr{L}_X\Omega = 0$ and $\mathscr{L}_X\nabla = 0$, one has

$$\mathscr{L}_X = D \circ i(X) + i(X) \circ D + \tfrac{1}{\nu}\,\mathrm{ad}_*(T(X))$$

with $T(X) = i(X)r + \omega_{ij}X^i y^j + \frac{1}{2}(\nabla_i(i(X)\omega))_j y^i y^j$ and

$$DT(X) = -i(X)\omega + i(X)\Omega.$$

In particular, if there exists a series of smooth functions λ_X so that

$$i(X)\omega - i(X)\Omega = d\lambda_X \tag{10}$$

one can write

$$\mathscr{L}_X = D \circ i(X) + i(X) \circ D + \tfrac{1}{\nu}\,\mathrm{ad}_*(\lambda_X + T(X))$$

with

$$D(\lambda_X + T(X)) = 0.$$

Thus $\lambda_X + T(X)$ is the flat section associated to the series of smooth function on M obtained by taking the part of $\lambda_X + T(X)$ with no y terms hence λ_X (notice that $i(X)r$ has no terms without a y from the construction of r). If Q denotes the quantisation map associating a flat section to a series in ν of smooth functions, the above yields

$$\mathscr{L}_X = D \circ i(X) + i(X) \circ D + \tfrac{1}{\nu}\,\mathrm{ad}_*(Q(\lambda_X)).$$

Since in those assumptions the map Q commutes with $\mathscr{L}_X$ one has

$$Q(Xf) = \mathscr{L}_X Q(f) = \tfrac{1}{\nu}[Q(\lambda_X), Q(f)]$$

so that for any smooth function f, one has

$$Xf = \tfrac{1}{\nu}(\mathrm{ad}_*\,\lambda_X)(f).$$

This proves Proposition 4.3 of [18].

We now aim to show that the condition (10) is not only sufficient, but also necessary. Observe that any Fedosov star product has the Poisson bracket for the term of order 1 in ν and has a second term which is of order at most 2 in each argument so, as was mentioned before, it uniquely defines a symplectic connection (which is the connection used in the construction) so that invariance of ∇ is a necessary condition for the invariance of $*_{\nabla,\Omega}$. In [4] it is shown that Ω can also be recovered, so in fact we have the following well known Lemma:

Lemma 6.1 *A vector field X is a derivation of $*_{\nabla,\Omega}$ if and only if $\mathscr{L}_X\omega = 0$, $\mathscr{L}_X\Omega = 0$, and $\mathscr{L}_X\nabla = 0$.*

We have seen above that such a vector field X is an inner derivation if $i(X)(\omega - \Omega)$ is exact. We shall show now that this is also a necessary condition.

Assume X is a vector field on M such that there exists a series of smooth functions λ_X with

$$X(u) = \tfrac{1}{\nu}(\mathrm{ad}_* \lambda_X)(u) \tag{11}$$

for every smooth function u on M. Then X is a derivation of $*$ so $\mathscr{L}_X\omega = 0$, $\mathscr{L}_X\Omega = 0$, $\mathscr{L}_X\nabla = 0$ and

$$Q(Xf) = \mathscr{L}_X Q(f) = \tfrac{1}{\nu}[T(X), Q(f)]$$

with $T(X) = i(X)r + \omega_{ij}X^i y^j + \frac{1}{2}(\nabla_i(i(X)\omega))_j y^i y^j$ and

$$DT(X) = -i(X)\omega + i(X)\Omega.$$

Taking a contractible open set U in M, there exists a series of smooth locally defined functions λ_X^U on U so that

$$(i(X)\omega - i(X)\Omega)|_U = d\lambda_X^U$$

and, everything being local, we have on U

$$D(\lambda_X^U + T(X))|_U = 0,$$

thus $\lambda_X^U + T(X)$ is the flat section on U associated to the series of smooth functions on U obtained by taking the part of $\lambda_X^U + T(X)$ with no y terms (which is λ_X^U) and

$$Q(X(u))|_U = \mathscr{L}_X Q(u)|_U = \tfrac{1}{\nu}[Q(\lambda_X^U), Q(u)]\big|_U$$

so that

$$X(u)|_U = \tfrac{1}{\nu}(\mathrm{ad}_{*_{\nabla,\Omega}} \lambda_X^U)(u)\big|_U$$

for any smooth function u. Comparing this with equation (11) shows that

$$\lambda_X^U - \lambda_X$$

is a constant on U and hence that

$$i(X)\omega - i(X)\Omega = d\lambda_X.$$

Thus we have proved the converse of Kravchenko's result. In summary:

Theorem 6.2 *A vector field X is an inner derivation of $* = *_{\nabla,\Omega}$ if and only if $\mathscr{L}_X \nabla = 0$ and there exists a series of functions λ_X such that*

$$i(X)\omega - i(X)\Omega = d\lambda_X.$$

In this case

$$X(u) = \tfrac{1}{\nu}(\mathrm{ad}_* \lambda_X)(u).$$

7 Moment Maps for an invariant Star Product with an invariant connection

Let (M, ω) be endowed with a differential star product $*$,

$$u * v = uv + \sum_{r \geq 1} \nu^r C_r(u, v).$$

Consider an algebra $\mathfrak{g}$ of vector fields on M consisting of derivations of $*$ and assume that there is a symplectic connection ∇ which is invariant under $\mathfrak{g}$ (i.e. $\mathscr{L}_X \nabla = 0$, $\forall X \in \mathfrak{g}$). This is of course automatically true if the star product is natural and invariant.

It was proven in [3] that $*$ is equivalent, through an equivariant equivalence

$$T = \mathrm{Id} + \sum_{r \geq 1} \nu^r T_r$$

(i.e. $\mathscr{L}_X T = 0$), to a Fedosov star product built from ∇ and a series of invariant closed 2-forms Ω which give a representative of the characteristic class of $*$.

Observe that

$$X(u) = \tfrac{1}{\nu}(\mathrm{ad}_* \mu_X)(u)$$

for any $X \in \mathfrak{g}$ if and only if

$$X(u) = T \circ X \circ T^{-1}(u) = T(\tfrac{1}{\nu}(\mathrm{ad}_* T\mu_X)(T^{-1}u)) = \tfrac{1}{\nu}(\mathrm{ad}_{*_{\nabla,\Omega}} T\mu_X)(u).$$

Hence the Lie algebra $\mathfrak{g}$ consists of inner derivations for $*$ if and only if this is true for the Fedosov star product $*_{\nabla,\Omega}$ and this true if and only if there exists a series of functions λ_X such that

$$i(X)\omega - i(X)\Omega = d\lambda_X.$$

In this case

$$X(u) = \tfrac{1}{\nu}(\mathrm{ad}_* \, \mu_X)(u) \quad \text{with} \quad \mu_X = T\lambda_X.$$

In particular, this yields

Theorem 7.1 *Let G be a compact Lie group of symplectomorphisms of (M, ω) and $\mathfrak{g}$ the corresponding Lie algebra of symplectic vector fields on M. Consider a star product $*$ on M which is invariant under G. The Lie algebra $\mathfrak{g}$ consists of inner derivations for $*$ if and only if there exists a series of functions λ_X and a representative $\frac{1}{\nu}(\omega - \Omega)$ of the characteristic class of $*$ such that*

$$i(X)\omega - i(X)\Omega = d\lambda_X.$$

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