Homogeneous symplectic manifolds with Ricci-type curvature

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Abstract

We consider invariant symplectic connections ∇ on homogeneous symplectic manifolds (M, ω) with curvature of Ricci type. Such connections are solutions of a variational problem studied by Bourgeois and Cahen, and provide an integrable almost complex structure on the bundle of almost complex structures compatible with the symplectic structure. If M is compact with finite fundamental group then (M, ω) is symplectomorphic to $\mathbb{P}_n(\mathbb{C})$ with a multiple of its Kähler form and ∇ is affinely equivalent to the Levi-Civita connection.

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The space of curvature tensors of symplectic connections on a symplectic manifold (M, ω) of dimension $2n \geq 4$ splits under the action of the symplectic group $Sp(2n, \mathbb{R})$ as a direct sum of two subspaces on which $Sp(2n, \mathbb{R})$ acts irreducibly [1, 7, 5]. For a given curvature tensor R we shall denote by E and W its projections onto these two subspaces. The E-component is determined by the Ricci tensor of the connection. When the W-component vanishes identically we say that the curvature is of Ricci type.

The motivation for looking at such connections is two-fold. They provide critical points of a functional which has been introduced in [1] to select preferred symplectic connections, and $W = 0$ is the integrability condition for an almost complex structure which a symplectic connection determines on the total space of the bundle $J(M, \omega)$ of almost complex structures compatible with the symplectic structure [6, 8, 9].

The simplest framework in which one can study the $W = 0$ condition is the compact homogeneous one. Our main result is

Theorem 1 *Let (M, ω) be a compact homogeneous symplectic manifold with finite fundamental group. If (M, ω) admits a homogeneous symplectic connection ∇ with Ricci-type curvature then (M, ω) is symplectomorphic to $(\mathbb{P}_n(\mathbb{C}), \omega_0)$, where ω_0 is a multiple of the Kähler form of the Fubini–Study metric, and ∇ is affinely equivalent to the Levi-Civita connection.*

When we do not impose any restriction on the fundamental group, we were only able to prove

Theorem 2 *Let (M, ω) be a compact homogeneous symplectic manifold of dimension 4. If (M, ω) admits a homogeneous symplectic connection ∇ with Ricci-type curvature then ∇ is locally symmetric.*

In §1 we prove some general identities which hold for any symplectic connection with Ricci-type curvature. In §2 we deduce some easy consequences of these identities in the homogeneous (respectively compact homogeneous) framework. In §3 we prove Theorem 1 in the simply connected case and show how to extend this to a finite fundamental group. Finally §4 is devoted to the proof of Theorem 2.

1 Let (M, ω) be a symplectic manifold and ∇ be a symplectic connection (a torsion-free connection on TM with $\nabla\omega = 0$). The curvature endomorphism R of ∇ is defined by

$$R(X, Y)Z = \left(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}\right) Z$$

for vector fields X, Y, Z on M. The symplectic curvature tensor

$$R(X, Y; Z, T) = \omega(R(X, Y)Z, T)$$

is antisymmetric in its first two arguments, symmetric in its last two and satisfies the first Bianchi identity

$$\oint_{X,Y,Z} R(X,Y;Z,T) = 0$$

where $\oint$ denotes the sum over the cyclic permutations of the listed set of elements. The Ricci tensor r is the symmetric 2-tensor

$$r(X,Y) = \mathrm{Trace}[Z \mapsto R(X,Z)Y].$$

R also obeys the second Bianchi identity

$$\oint_{X,Y,Z} (\nabla_X R)(Y,Z) = 0.$$

The Ricci part E of the curvature tensor is given by

$$\begin{aligned} E(X,Y;Z,T) &= \frac{-1}{2(n+1)}\Big[2\omega(X,Y)r(Z,T) + \omega(X,Z)r(Y,T) + \omega(X,T)r(Y,Z) \\ &\qquad - \omega(Y,Z)r(X,T) - \omega(Y,T)r(X,Z)\Big]. \end{aligned} \tag{1.1}$$

The curvature is of Ricci type when $R = E$.

Lemma 1 *Let (M,ω) be a symplectic manifold of dimension $2n \geq 4$. If the curvature of a symplectic connection ∇ on M is of Ricci type then there is a 1-form u such that*

$$(\nabla_X r)(Y,Z) = \frac{1}{2n+1}\left(\omega(X,Y)u(Z) + \omega(X,Z)u(Y)\right). \tag{1.2}$$

Conversely, if there is such a 1-form u then the W part of the curvature satisfies

$$\oint_{X,Y,Z} (\nabla_X W)(Y,Z;T,U) = 0. \tag{1.3}$$

Proof When the curvature is of Ricci type, the second Bianchi identity for R becomes an identity for E. Since ω is parallel, covariantly differentiating equation (1.1) and summing cyclically, we get

$$\begin{aligned} 0 &= \oint_{X,Y,Z} 2\omega(Y,Z)(\nabla_X r)(T,U) + \omega(Y,T)(\nabla_X r)(Z,U) + \omega(Y,U)(\nabla_X r)(Z,T) \\ &\qquad - \omega(Z,T)(\nabla_X r)(Y,U) - \omega(Z,U)(\nabla_X r)(Y,T). \end{aligned} \tag{1.4}$$

Choose local frames $\{V_a\}_{a=1}^{2n}$, $\{W_a\}_{a=1}^{2n}$ on M such that $\omega(V_a,W_b) = \delta_{ab}$. Substitute $Y = V_a$ and $Z = W_a$ in equation (1.4) and sum over a to obtain

$$\begin{aligned} 0 &= 2n(\nabla_X r)(T,U) - (\nabla_T r)(X,U) - (\nabla_U r)(X,T) \\ &\qquad + \omega(X,T)\sum_a (\nabla_{W_a} r)(V_a,U) + \omega(X,U)\sum_a (\nabla_{W_a} r)(V_a,T). \end{aligned} \tag{1.5}$$

If we cyclically permute X, T, U in equation (1.5) and sum we get

$$(2n-2)\oint_{X,T,U}(\nabla_X r)(T,U)=0 \tag{1.6}$$

and since $n \geq 2$ we have

$$\oint_{X,T,U}(\nabla_X r)(T,U)=0 \tag{1.7}$$

Using equation (1.7) in equation (1.5) gives

$$(2n+1)(\nabla_X r)(T,U)+\omega(X,T)\sum_a(\nabla_{W_a} r)(V_a,U)+\omega(X,U)\sum_a(\nabla_{W_a} r)(V_a,T)=0$$

which is of the desired form if

$$u(X)=-\sum_a(\nabla_{W_a} r)(V_a,X).$$

Conversely, if one substitutes (1.2) into the covariant derivative of (1.1) and cyclically sums then one obtains

$$\oint_{X,Y,Z}(\nabla_X E)(Y,Z,T,U)=0.$$

Combining this with the second Bianchi identity, gives the second part of the Lemma. ■

Corollary *A symplectic manifold with a symplectic connection whose curvature is of Ricci type is locally symmetric if and only if the* 1*-form* u*, defined in the Lemma, vanishes.*

Remark 1 It will be useful to have an equivalent form of formula (1.2). Denote by A the linear endomorphism such that

$$r(X,Y)=\omega(X,AY). \tag{1.8}$$

The symmetry of r is equivalent to saying that A is in the Lie algebra of the symplectic group of ω. Denote by $\overline{u}$ the vector field such that

$$u=i(\overline{u})\omega \tag{1.9}$$

then (1.2) is equivalent to

$$\nabla_X A=\frac{-1}{2n+1}(X\otimes u+\overline{u}\otimes i(X)\omega). \tag{1.10}$$

Lemma 2 *Let* (M,ω) *be a symplectic manifold with a symplectic connection* ∇ *with Ricci-type curvature. Then, keeping the above notation, the following identities hold:*

(i) There is a function b such that

$$\nabla u = -\frac{1+2n}{2(1+n)} \overset{(2)}{r} + b\omega \tag{1.11}$$

where $\overset{(2)}{r}$ *is the* 2*-form*

$$\overset{(2)}{r}(X,Y) = \omega(X, A^2Y). \tag{1.12}$$

(ii) The differential of the function b is given by

$$db = \frac{1}{1+n} i(\overline{u}) r. \tag{1.13}$$

(iii) The covariant differential of db is given by

$$\nabla db = \frac{1}{1+n}\left[-\frac{1}{1+2n} u \otimes u - \frac{1+2n}{2(1+n)} \overset{(3)}{r} + br\right]. \tag{1.14}$$

where

$$\overset{(3)}{r}(X,Y) = \omega(X, A^3Y). \tag{1.15}$$

Proof We can compute the action of the curvature on endomorphisms in two different ways. On the one hand it is

$$\begin{aligned} R(X,Y)\cdot A &= [R(X,Y), A] \\ &= R(X,Y)A - AR(X,Y) \\ &= -\frac{1}{2(n+1)}[X \otimes \omega(A^2Y, \,.\,) - Y \otimes \omega(A^2X, \,.\,) \\ &\qquad + A^2Y \otimes \omega(X, \,.\,) - A^2X \otimes \omega(Y, \,.\,)]. \end{aligned}$$

On the other hand the curvature is of Ricci type so that (1.10) gives

$$R(X,Y)\cdot A = \frac{1}{2n+1}[X \otimes \nabla_Y u - Y \otimes \nabla_X u + \nabla_Y \overline{u} \otimes \omega(X, \,.\,) - \nabla_X \overline{u} \otimes \omega(Y, \,.\,)].$$

If we define an endomorphism B of TM by

$$BY = \frac{2n+1}{2(n+1)} A^2Y + \nabla_Y \overline{u}$$

then equality of the two right hand sides yields

$$X \otimes \omega(BY, \,.\,) - Y \otimes \omega(BX, \,.\,) + BY \otimes \omega(X, \,.\,) - BX \otimes \omega(Y, \,.\,) = 0$$

whose only solution is

$$B = b\,\mathrm{Id}\,.$$

This gives

$$\nabla_Y u = -\frac{2n+1}{2(n+1)} \omega(A^2Y, \,.\,) + b\omega(Y, \,.\,)$$

which is equation (1.11).

Antisymmetrising (1.11) we get

$$du = -\frac{2n+1}{n+1} \overset{(2)}{r} + 2b\omega.$$

Taking the exterior derivative gives

$$0 = -\frac{2n+1}{n+1} d \overset{(2)}{r} + 2db \wedge \omega.$$

But

$$\begin{aligned} d \overset{(2)}{r}(X,Y,Z) &= \oint_{X,Y,Z} \omega(\nabla_X A^2 Y, Z) \\ &= -\frac{1}{2n+1} \oint_{X,Y,Z} \omega(u(AY)X + \omega(X,AY)\overline{u}, Z) \\ &\qquad + \omega(u(Y)AX + \omega(X,Y)A\overline{u}, Z) \\ &= \frac{2}{2n+1} \oint_{X,Y,Z} \omega(X,Y) r(\overline{u}, Z). \end{aligned}$$

Substituting,

$$\left[-\frac{1}{n+1} r(\overline{u}, \,.\,) + db \right] \wedge \omega = 0.$$

and in dimension 4 or higher this implies

$$db = \frac{1}{n+1} r(\overline{u}, \,.\,)$$

which is (1.13). Covariantly differentiating

$$\begin{aligned} (\nabla_X db)(Y) &= \frac{1}{n+1} \left[(\nabla_X r)(\overline{u}, Y) + r(\nabla_X \overline{u}, Y) \right] \\ &= \frac{1}{n+1} \left[\frac{1}{1+2n} \omega(X, \overline{u}) u(Y) + r\left(-\frac{2n+1}{2(n+1)} A^2 X + bX, Y \right) \right] \end{aligned}$$

which is (1.14). ∎

2 Assume (M, ω) is a G-homogeneous symplectic manifold and ∇ is a G-invariant symplectic connection with Ricci-type curvature. If ∇ is not locally symmetric the G-invariant 1-form u is everywhere different from zero and the function b is also G-invariant and hence constant. Putting these two facts into (1.13) we see that r as a bilinear form is necessarily degenerate

$$r(\overline{u}, \,.\,) = 0. \tag{2.1}$$

Also (1.14) implies

$$\frac{1}{2n+1} u \otimes u + \frac{1+2n}{2(1+n)} \overset{(3)}{r} - br = 0 \tag{2.2}$$

or equivalently

$$\frac{1}{2n+1}\overline{u}\otimes u-\frac{1+2n}{2(1+n)}A^3+bA=0. \tag{2.3}$$

Applying A to (2.3) and using (2.1)

$$-\frac{1+2n}{2(1+n)}A^4+bA^2=0. \tag{2.4}$$

It follows that the only possible non-zero eigenvalues of A are $\pm\sqrt{\frac{2(1+n)}{1+2n}b}$ and so are real or imaginary.

Lemma 3 *If (M,ω) is a compact homogeneous symplectic manifold admitting a homogeneous symplectic connection ∇ with Ricci-type curvature which is not locally symmetric then $b=0$.*

Proof Recall that for any vector field X, Cartan's identity gives

$$\operatorname{div} X\omega^n \stackrel{\text{def}}{=\!=} \mathcal{L}_X\omega^n = n\,d\left((i(X)\omega)\wedge\omega^{n-1}\right)$$

and

$$\mathcal{L}_X\omega^n=(\mathcal{L}_X-\nabla_X)\,\omega^n=n\,(\omega(\nabla_{\cdot}X,\,.\,)+\omega(\,.\,,\nabla_{\cdot}X))\wedge\omega^{n-1}$$

so that

$$\operatorname{div} X=\operatorname{Trace}[Z\mapsto\nabla_Z X].$$

In particular, by (1.11)

$$\operatorname{div}\overline{u}=-\frac{2n+1}{2(n+1)}\operatorname{Trace}A^2+2nb.$$

G-invariance implies that $\operatorname{div}\overline{u}$ is constant. But M compact with no boundary implies $\int_M\operatorname{div}\overline{u}\omega^n=0$ since the argument is exact; hence the constant is zero. Thus

$$b=\frac{2n+1}{4n(n+1)}\operatorname{Trace}A^2.$$

On the other hand, (2.4) implies that A^2 is a multiple of a projection and with A symplectic this has even rank $2p$ say; using 2.1 we get $2p<2n$. Thus

$$\operatorname{Trace}A^2=\frac{4pb(1+n)}{1+2n}$$

so

$$b=\frac{2n+1}{4n(n+1)}\cdot\frac{4pb(1+n)}{1+2n}=\frac{p}{n}b$$

and hence $b=0$. ∎

It follows that $A^4=0$ so A is nilpotent; moreover (2.3) tells us that A^3 has rank 1.

Lemma 4 *Let (M,ω) be a 4-dimensional homogeneous symplectic manifold admitting a homogeneous symplectic connection ∇ with Ricci-type curvature which is not locally symmetric. Let A be the endomorphism associated to the Ricci tensor. Then*

(i) either A is nilpotent, $b\neq 0$, $A^2=0$, and A has rank 1 at any point;

(ii) or A is nilpotent, $b=0$, and A^3 has rank 1 at any point;

(iii) or A has a non zero eigenvalue so $b\neq 0$. Then A admits a pair of non zero eigenvalues of opposite sign (real or imaginary) with multiplicity 1 and 0 is an eigenvalue of multiplicity 2 at any point. Furthermore, A has necessarily a nilpotent part.

PROOF The dimension – at any point $x\in M$ – of the generalised 0 eigenspace of A is even and non-zero, so is 2 or 4. If it is 4 then A is nilpotent and $A^4=0$ in dimension 4. Thus, by (2.3), $bA^2=0$. If $b\neq 0$ then $A^2=0$ so, by (2.4), A has rank 1 at any point. Otherwise $b=0$ and A^3 has rank 1 at any point.

When the generalised 0 eigenspace V_0 is 2-dimensional at any point, then $\pm\sqrt{\frac{2(1+n)}{1+2n}b}$ are eigenvalues with multiplicity 1. Choose a globally defined vector field $v\in V_0$ so that $\omega(v,\overline{u})=1$. Set $Av=p\overline{u}$. Then

$$\nabla_X(Av)=\nabla_X(p\overline{u})=(Xp)\overline{u}+p\left(-\frac{1+2n}{2(1+n)}A^2X+bX\right)$$

but it is also equal to

$$\nabla_X(Av)=(\nabla_XA)v+A(\nabla_Xv)=-\frac{1}{1+2n}\left(Xu(v)+\overline{u}\omega(X,v)\right)+A(\nabla_Xv).$$

Observe that $\omega(A^2X,\overline{u})=\omega(A(\nabla_Xv),\overline{u})=0$, so that

$$p\omega(bX,\overline{u})=\omega(-\frac{1}{1+2n}u(v)X,\overline{u})=\frac{1}{1+2n}\omega(X,\overline{u}).$$

Hence $pb=\frac{1}{1+2n}$ which implies that $p\neq 0$. Thus A has a nilpotent part. ∎

3 We first prove Theorem 1 in the simply-connected case. It is standard that a compact simply-connected homogeneous symplectic manifold (M,ω) is symplectomorphic to a coadjoint orbit of a simply-connected compact semisimple Lie group G. Such a Lie group G is a product of simple groups and the orbit is a product of orbits. We may throw away any factors where the orbit is zero dimensional as the remaining group will still act transitively. A G-invariant symplectic connection ∇ on such an orbit is compatible with the product structure. If the curvature of ∇ is of Ricci type, then it was shown in [3] that the curvature is zero when (M,ω,∇) is a product of more than one factor. But a non-trivial compact coadjoint orbit of a simple Lie group does not admit a flat connection since it has a non-zero Euler characteristic. It follows that we can assume G is simple and (M,ω) is a coadjoint

orbit $(\mathcal{O}, \omega^{\mathcal{O}})$ with its Kirillov–Kostant–Souriau symplectic structure and with an invariant symplectic connection ∇ with curvature of Ricci type.

Further, the Euler characteristic of such an orbit is non-zero. If the vector field $\overline{u}$ were non-zero, then invariance would imply that it is everywhere non-zero and this cannot happen. Thus $\overline{u} = 0$ and hence ∇ is locally symmetric ($\nabla R = 0$).

Pick a point $\xi_0 \in \mathcal{O}$ and construct a symmetric symplectic triple $(\mathfrak{l}, \sigma, \Omega)$ as follows: Let $\mathfrak{a} = \{R_{\xi_0}(X, Y) \in \mathrm{End}(T_{\xi_0}\mathcal{O}) \mid X, Y \in T_{\xi_0}\mathcal{O}\}$ and $\mathfrak{l} = T_{\xi_0}\mathcal{O} \oplus \mathfrak{a}$. The bracket is defined by

$$
\begin{aligned}
[X, Y] &= R_{\xi_0}(X, Y), \qquad X, Y \in T_{\xi_0}\mathcal{O}; && (3.1)\\
[B, X] &= BX, \qquad B \in \mathfrak{a}, X \in T_{\xi_0}\mathcal{O}; && (3.2)\\
[B, C] &= BC - CB, \qquad B, C \in \mathfrak{a}, && (3.3)
\end{aligned}
$$

σ by

$$\sigma = -\,\mathrm{Id}_{T_{\xi_0}\mathcal{O}} \oplus \mathrm{Id}_{\mathfrak{a}},$$

and Ω by

$$\Omega(X + Y, X' + Y') = \omega_{\xi_0}(X, X'), \qquad X, X' \in T_{\xi_0}, Y, Y' \in \mathfrak{a}.$$

Lemma 5 *$(\mathfrak{l}, \sigma, \Omega)$ is, indeed, a symmetric symplectic triple.*

Proof There are two things to check to see that $\mathfrak{l}$ is a Lie algebra. Firstly that the brackets defined above belong to $\mathfrak{l}$. The only ones in doubt are the brackets of two elements of $\mathfrak{a}$. But $\mathfrak{a}$ is in fact the linear infinitesimal holonomy. This follows since the latter is spanned by the values of the curvature endomorphism and its covariant derivatives. The latter vanish by the local symmetry condition.

The second thing to check is the Jacobi identity. Obviously this holds if all three elements are in $\mathfrak{a}$ since this is a Lie algebra. If all three are in $T_{\xi_0}\mathcal{O}$ then $[X, [Y, Z]] = -R_{\xi_0}(Y, Z)X$ and the Jacobi identity is satisfied for these elements by the first Bianchi identity. When one element is in $T_{\xi_0}\mathcal{O}$ and two in $\mathfrak{a}$ we have

$$[X, [B, C]] + [B, [C, X]] + [C, [X, B]] = -[B, C]X + BCX - CBX = 0.$$

Finally, if two elements are in $T_{\xi_0}\mathcal{O}$ and one in $\mathfrak{a}$ we have

$$
\begin{aligned}
[X, [Y, B]] + [Y, [B, X]] + [B, [X, Y]] &= -R_{\xi_0}(X, BY) - R_{\xi_0}(BX, Y)\\
&\qquad + BR_{\xi_0}(X, Y) - R_{\xi_0}(X, Y)B\\
&= (B \cdot R_{\xi_0})(X, Y)
\end{aligned}
$$

where $B \cdot R_{\xi_0}$ denotes the natural action of the holonomy Lie algebra $\mathfrak{a}$ on curvature tensors. But $\nabla R = 0$ if and only if $B \cdot R_{\xi_0} = 0, \forall B \in \mathfrak{a}$.

The other two properties follow immediately from the definitions. ∎

If L is the simply connected Lie group associated to $\mathfrak{l}$ and K the Lie subgroup associated to the subalgebra $\mathfrak{a}$ then K is the connected component of the fixed point set of the automorphism of L induced by σ and $M_1 = L/K$ is a simply connected symmetric space. Ω induces a symplectic form ω_1 on M_1 which is parallel for the canonical connection ∇_1.

Consider the point $\xi_0 \in M = \mathcal{O}$ and the point $\xi_1 = eK \in M_1$. There is a linear isomorphism ϕ from the tangent space $T_{\xi_0}M$ to the tangent space $T_{\xi_1}M_1$ so that

$$\phi\,(R_0(X,Y)Z) = R_1(\phi X, \phi Y)\phi Z.$$

This implies [4, p. 259, thm. 7.2] that there exists an affine symplectic diffeomorphism ψ of a neighbourhood U_0 of ξ_0 in $\mathcal{O}$ onto a neighbourhood U_1 of ξ_1 in M_1 such that $\psi_{*\xi_0} = \phi$.

Both $(\mathcal{O}, \omega^{\mathcal{O}}, \nabla)$ and $(M_1, \omega_1, \nabla_1)$ are real analytic, as is ψ, and $\mathcal{O}$ is simply connected whilst ∇_1 is complete. Hence [4, p. 252, thm. 6.1] there exists a unique affine map $\widetilde{\psi} : \mathcal{O} \to M_1$ such that $\widetilde{\psi}|_{U_0} = \psi$. This map $\widetilde{\psi}$ is symplectic since it is an analytic extension of the map ψ which is symplectic. Symplectic maps are immersions, and hence local diffeomorphisms when the dimensions are equal as they are in this case. Hence $\widetilde{\psi}(\mathcal{O})$ is open. On the other hand, $\mathcal{O}$ is compact, so $\widetilde{\psi}(\mathcal{O})$ is compact and thus closed. Hence $\widetilde{\psi}$ is surjective. It follows that M_1 is compact.

From the preceding arguments we see that $(M_1, \omega_1, \nabla_1)$ is a compact simply connected symmetric symplectic space whose curvature is of Ricci type. The only such space is $\mathbb{P}_n(\mathbb{C})$ with a multiple of its standard Kähler form ω_0 and the Levi-Civita connection ∇_0 of the Fubini–Study metric. Since $\mathcal{O}$ and M_1 are both simply connected they are diffeomorphic and hence we have proved Theorem 1 in the simply connected case.

Next we consider the case where M has a finite fundamental group, (M, ω) is G-homogeneous symplectic with a G-invariant symplectic connection ∇ with curvature of Ricci type. Then the simply connected covering space $\widetilde{M}$ is compact and carries such data $\widetilde{\omega}, \widetilde{\nabla}$ for the simply connected covering group $\widetilde{G}$.

It follows that $(\widetilde{M}, \widetilde{\omega}, \widetilde{\nabla})$ is diffeomorphic to $(\mathbb{P}_n(\mathbb{C}), \omega_0, \nabla_0)$ and hence that M is diffeomorphic to $\mathbb{P}_n(\mathbb{C})/\Gamma$ where Γ is a discrete subgroup of $PU(n+1)$ acting properly discontinuously on $\mathbb{P}_n(\mathbb{C})$. But non-trivial elements of $PU(n+1)$ always have fixed points, so Γ must be trivial. This proves Theorem 1.

4 We now proceed to give the proof of Theorem 2 indicating along the way why we restrict ourselves to dimension 4 and why we only obtain a local result.

Recall that when (M, ω) is homogeneous and admits a non-locally-symmetric invariant symplectic connection with Ricci-type curvature we have the non-zero vector field $\overline{u}$ and the Ricci endomorphism satisfies

$$\begin{gathered} A\overline{u} = 0, \\ \frac{1}{1+2n}\overline{u} \otimes u - \frac{1+2n}{2(1+n)}A^3 + bA = 0, \end{gathered} \tag{4.1}$$

$$\frac{1+2n}{2(1+n)}A^4 - bA^2 = 0.$$

Furthermore, if M is compact Lemma 3 tells us that $b = 0$ so that $A^4 = 0$, and A^3 has rank 1:

$$A^3 = \frac{2(1+n)}{(1+2n)^2}\overline{u} \otimes u.$$

The 1-form u is everywhere non-zero so there is a globally defined vector field e_1 with $u(e_1)$ everywhere $\neq 0$. The vector fields $e_1, e_2 = Ae_1, e_3 = A^2e_1, e_4 = A^3e_1$ form at each point $x \in M$ a basis of a 4-dimensional subspace V_x of the tangent space T_xM. Furthermore, by equation (4.1)

$$e_4 = \frac{2(1+n)}{(1+2n)^2}u(e_1)\overline{u}.$$

If we choose the vector field e_1 so that $\omega(e_1, e_4) = \epsilon$ with $\epsilon^2 = 1$, we get $-\frac{2(1+n)}{(1+2n)^2}(u(e_1))^2 = \epsilon$ so that $\epsilon = -1$ and $(u(e_1))^2 = \frac{(1+2n)^2}{2(1+n)}$ so that $\overline{u} = u(e_1)e_4$. Remark that we can always assume that $\omega(e_1, e_2) = 0$ (by adding to e_1 a multiple of e_3). So the symplectic form restricted to V_x writes in the chosen basis

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The tangent space at each point $x \in M$ writes

$$T_xM = V_x \oplus V_x'$$

where V_x' is the ω_x-orthogonal to V_x; it is stable under A and, since A^3 has rank 1, $A^3|_{V_x'} = 0$ but this is not enough to describe the behaviour of A on V_x'.

From now on, we restrict ourselves to the 4-dimensional case. We define 1-forms $\alpha, \beta, \gamma, \delta$ such that

$$\nabla_X e_1 = \alpha(X)e_1 + \beta(X)e_2 + \gamma(X)e_3 + \delta(X)e_4.$$

Using formula (1.10) for ∇A (i.e. $\nabla_X A = \frac{-1}{2n+1}(X \otimes u + u(e_1)e_4 \otimes i(X)\omega)$) we obtain

$$\begin{aligned}
\nabla_X e_2 &= \frac{-u(e_1)}{2n+1}X^1 e_1 + \left(\alpha(X) - \frac{u(e_1)}{2n+1}X^2\right)e_2 \\
&\quad + \left(\beta(X) - \frac{u(e_1)}{2n+1}X^3\right)e_3 + \left(\gamma(X) - \frac{2u(e_1)}{2n+1}X^4\right)e_4, \\
\nabla_X e_3 &= \frac{-u(e_1)}{2n+1}X^1 e_2 + \left(\alpha(X) - \frac{u(e_1)}{2n+1}X^2\right)e_3 + \beta(X)e_4, \\
\nabla_X e_4 &= \frac{-u(e_1)}{2n+1}X^1 e_3 + \left(\alpha(X) - \frac{2u(e_1)}{2n+1}X^2\right)e_4.
\end{aligned}$$

On the other hand, formula (1.11) gives

$$\nabla_X e_4 = \frac{-u(e_1)}{2n+1} A^2 X$$

so that

$$\alpha(X) = \frac{u(e_1)}{2n+1} X^2.$$

The fact that ∇ is symplectic gives the additional condition that

$$\gamma(X) = \frac{u(e_1)}{2n+1} X^4.$$

The connection is thus determined by the two 1-forms β and δ. The vanishing of the torsion gives the expression of the brackets of the vector fields e_j.

We can now compute the action of the curvature endomorphism on the vector fields e_j in two different ways: using the formulas above or using the fact that the curvature is of Ricci type.

This yields two identities

$$d\beta = \frac{3}{2(n+1)}\omega + \frac{u(e_1)}{2n+1} e_*^2 \wedge \beta + \frac{1}{2(n+1)} e_*^1 \wedge e_*^4,$$

$$d\delta = 2\gamma \wedge \beta - \frac{2}{2(n+1)} e_*^3 \wedge e_*^4 + 2\alpha \wedge \delta$$

where the e_*^j are 1-forms so that $e_*^j(e_k) = \delta_k^j$ at each point. Using the formulas for the bracket of vector fields we have

$$de_*^3 = \frac{2u(e_1)}{2n+1} e_*^1 \wedge e_*^4 - \frac{u(e_1)}{2n+1} e_*^2 \wedge e_*^3 + e_*^2 \wedge \beta,$$

and substituting $e_*^2 \wedge \beta$ in $d\beta$ yields

$$d(\beta - \frac{u(e_1)}{2n+1} e_*^3) = \frac{2}{n+1}\omega$$

which is impossible on a compact manifold. This contradiction tells us that u must vanish and hence that ∇ is locally symmetric.

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