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Some Remarks on the Classification of Poisson Lie Groups

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ABSTRACT. We describe some results in the problem of classifying the bialgebra structures on a given finite dimensional Lie algebra. We consider two aspects of this problem. One is to see which Lie algebras arise (up to isomorphism) as the big algebra in a Manin triple, and the other is to try and determine all the exact Poisson structures for a given semisimple Lie algebra. We follow here the presentation of the talk that one of us gave at the Yokohama Symposium; in particular, we recall many well known properties so that it is essentially self-contained.

1. Introduction

A Poisson structure on a manifold M is a Lie algebra structure on $C^\infty(M)$, denoted by $\{\ ,\ \}$, satisfying $\{uv, w\} = u\{v, w\} + v\{u, w\}$. It is defined by a contravariant skew symmetric 2-tensor P on M by $\{u, v\} = \langle du \wedge dv, P\rangle$. This satisfies $[P, P] = 0$ where $[\ ,\ ]$ is the Schouten bracket – the natural extension to contravariant tensor fields of the usual bracket of vector fields; for instance:

$$[X\wedge Y, Z\wedge T] = X\wedge[Y,Z]\wedge T - X\wedge[Y,T]\wedge Z - Y\wedge[X,Z]\wedge T + Y\wedge[X,T]\wedge Z \quad (1)$$

for X, Y, Z, T vector fields on M.

DEFINITION 1.1 ([**4**]). A *Poisson Lie group* (G, P) is a Lie group G with a Poisson structure P such that the multiplication $(m : G\times G \to G,\ (x, y) \mapsto xy)$ is a Poisson map (where $G \times G$ is endowed with the product Poisson structure). This is equivalent to the fact that P is multiplicative:

$$P_{xy} = L_{x*}P_y + R_{y*}P_x \qquad \forall x, y \in G$$

where L_x (resp. R_x) denotes the left (resp. right) translation by x in G and L_{x*} (resp. R_{x*}) denotes the differential of this map applied to contravariant tensors.

Observe that, if (G, P) is a Poisson Lie group, then

1°) $P_e = 0$ where e is the identity of G;

2°) the inverse map $\nu : G \to G$, $x \mapsto x^{-1}$ is a Poisson map.

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Example.

1) Any Poisson Lie structure on the abelian group $\mathbb{R}^n$ has the form
$$P_x = 1/2 \sum_{i,j,k=1}^{n} P_k^{ij} x^k \partial_i \wedge \partial_j$$
where the P_k^{ij} are the structure constants of an n-dimensional Lie algebra.
2) The only Poisson Lie structure on the torus T^n is the trivial zero structure.

Definition 1.2 ([**4**]). A *Lie bialgebra* $(\mathfrak{g}, p)$ is a Lie algebra $\mathfrak{g}$ with a 1-cocycle $p : \mathfrak{g} \to \Lambda^2\mathfrak{g}$ (relative to the adjoint action) such that $p^* : \mathfrak{g}^* \times \mathfrak{g}^* \to \mathfrak{g}^*(\xi, \eta) \to [\xi, \eta]$ with
$$\langle [\xi, \eta], X\rangle = \langle \xi \wedge \eta, p(X)\rangle$$
is a Lie bracket on $\mathfrak{g}^*$. One also denotes the bialgebra by $(\mathfrak{g}, \mathfrak{g}^*)$.

Proposition 1.1 (Drinfeld [**4**]). *If (G, P) is a Poisson Lie group and $\mathfrak{g} =$ Lie(G) then $p : \mathfrak{g} \to \Lambda^2\mathfrak{g}$ given by $X \to (\mathcal{L}_{\widetilde{X}} P)(e)$ is a Lie bialgebra (said to be associated to (G, P)). Here $\widetilde{X}$ is the left invariant vector field on G corresponding to $X \in \mathfrak{g} \simeq T_eG$.*

If $(\mathfrak{g}, p)$ is a Lie bialgebra and G is the connected simply-connected Lie group with Lie algebra $\mathfrak{g}$, then there exists a unique structure of Poisson Lie group on G, (G, P) such that $(\mathfrak{g}, p)$ is the associated Lie bialgebra.

Definition 1.3 ([**4**]). A Lie bialgebra $(\mathfrak{g}, p)$ is said to be *exact* if the 1-cocycle p is a coboundary, $p = \partial Q$, for $Q \in \Lambda^2\mathfrak{g}$.

This means that $\partial Q_X = [X, Q]$ and then the condition for $(\mathfrak{g}, \partial Q)$ to be a Lie bialgebra is that the bracket $[Q, Q]$ be invariant under the adjoint action in $\Lambda^3\mathfrak{g}$ where the bracket in $\Lambda^2\mathfrak{g}$ is obtained by a formula similar to (1):
$$[X \wedge Y, Z \wedge T] = X \wedge [Y, Z] \wedge T - X \wedge [Y, T] \wedge Z - Y \wedge [X, Z] \wedge T + Y \wedge [X, T] \wedge Z \quad (1')$$
for $X, Y, Z, T \in \mathfrak{g}$.

In the case (G, P) is an exact Poisson Lie group (i.e a Lie group whose associated Lie bialgebra is exact) then
$$P_x = L_{x_*}Q - R_{x_*}Q.$$

Proposition 1.2 (De Smedt [**3**]). *Any Lie algebra $\mathfrak{g}$ admits a structure of Lie bialgebra $(\mathfrak{g}, p)$ with $p \neq 0$.*

Definition 1.4 ([**4**]). A *Manin triple* consists of three Lie algebras $(\mathfrak{L}, \mathfrak{g}_1, \mathfrak{g}_2)$ and a symmetric invariant non-degenerate bilinear form $\langle\langle\ ,\ \rangle\rangle$ on $\mathfrak{L}$ such that

1) $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are subalgebras of $\mathfrak{L}$;
2) $\mathfrak{L} = \mathfrak{g}_1 + \mathfrak{g}_2$ as vector spaces;
3) $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are isotropic for $\langle\langle\ ,\ \rangle\rangle$.

We shall call the Lie algebra $\mathfrak{L}$ the associated *Manin algebra*.

Proposition 1.3 (Drinfeld [**4**]). *There is a bijective correspondence between Lie bialgebras and Manin triples:*

- *if* $(\mathfrak{g}, p)$ *is a Lie bialgebra then* $\mathfrak{L} = \mathfrak{g}+\mathfrak{g}^*$ *with* $\langle\langle\ ,\ \rangle\rangle$ *and bracket defined by*

$$\langle\langle (X,\alpha),(Y,\beta)\rangle\rangle = \langle \alpha, Y\rangle + \langle \beta, X\rangle$$
$$[(X,\alpha),(Y,\beta)]$$
$$= ([X,Y] - \mathrm{ad}^*\beta \cdot X + \mathrm{ad}^*\alpha \cdot Y, [\alpha,\beta] + \mathrm{ad}^* X\cdot\beta - \mathrm{ad}^* Y \cdot \alpha)$$

  $(\alpha, \beta \in \mathfrak{g}^*, X, Y \in \mathfrak{g})$;
- *if* $(\mathfrak{L}, \mathfrak{g}_1, \mathfrak{g}_2)$ *is a Manin triple and* $\mathfrak{g}_2$ *is identified with* $\mathfrak{g}_1^*$ *via* $\langle\langle\ ,\ \rangle\rangle$ *then* $\langle\langle\ ,\ \rangle\rangle$ *and* $[\ ,\ ]$ *on* $\mathfrak{L}$ *are given as above; the fact that* $\mathfrak{L}$ *is a Lie algebra implies that* $(\mathfrak{g}_1, \mathfrak{g}_1^*)$ *is a bialgebra.*

An open problem is to classify all bialgebra structures on a given Lie algebra $\mathfrak{g}$ (up to isomorphisms of $\mathfrak{g}$)[1].

2. A notion of isomorphism between Manin triples

Remark 2.1. If $\mathfrak{g}$ is any Lie algebra with $p = 0$ then the corresponding Manin triple is

$$\mathfrak{L} \simeq \mathrm{Lie}(T^*G) \cong \mathfrak{g}\times\mathfrak{g}^*, \qquad \mathfrak{g}_1 = \mathfrak{g}, \qquad \mathfrak{g}_2 = \mathfrak{g}^*,$$

with $[(X,\alpha),(Y,\beta)] = ([X,Y], \mathrm{ad}^* X\cdot\beta - \mathrm{ad}^* Y\cdot\alpha)$ and $\langle\langle (X,\alpha),(Y,\beta)\rangle\rangle = \langle\alpha, Y\rangle + \langle\beta, X\rangle$.

If instead the Poisson-Lie structure is exact $p = \partial Q$ with $[Q,Q] = 0$ one also has $\mathfrak{L} \cong \mathrm{Lie}(T^*G)$ with the same symmetric invariant non degenerate bilinear form.

We want to see when two bialgebra structures on a given Lie algebra $\mathfrak{g}$ yield isomorphic algebras $\mathfrak{L}$ in the corresponding Manin triple. To see this we consider a larger class of Lie algebras containing $\mathfrak{g}$ as a subalgebra: the set of Manin pairs $(\mathfrak{L}, \mathfrak{g})$ [2].

Let $\mathfrak{g}$ be a Lie algebra of dimension n. Consider any vector space $\mathfrak{L}$ of dimension $2n$ with a nondegenerate symmetric bilinear form $\langle\langle\ ,\ \rangle\rangle$ and a skewsymmetric bilinear map $[\ ,\ ] : \mathfrak{L}\times\mathfrak{L}\to\mathfrak{L}$ such that

i) $\mathfrak{L}$ contains $\mathfrak{g}$;
ii) the bracket restricted to $\mathfrak{g}\times\mathfrak{g}$ is the Lie bracket of $\mathfrak{g}$;
iii) $\mathfrak{g}$ is isotropic;
iv) $\langle\langle [X,Y],Z\rangle\rangle + \langle\langle Y,[X,Z]\rangle\rangle = 0$, $\forall X,Y,Z\in\mathfrak{L}$.

Then, choosing an isotropic subspace supplementary to $\mathfrak{g}$ in $\mathfrak{L}$ and identifying it with $\mathfrak{g}^*$ via $\langle\langle\ ,\ \rangle\rangle$, $\mathfrak{L} = \mathfrak{g}+\mathfrak{g}^*$ as vector spaces and one has:

1) $\langle\langle (X,\alpha),(Y,\beta)\rangle\rangle = \langle\alpha, Y\rangle + \langle\beta, X\rangle$;
2) $[(X,\alpha),(Y,\beta)] = ([X,Y] + C_1(\alpha,Y) - C_1(\beta,X) + \bar{S}(\alpha,\beta), \mathrm{ad}^* X\cdot\beta - \mathrm{ad}^* Y\cdot\alpha + T(\alpha,\beta))$.

The invariance condition becomes:

3) $S(\alpha,\beta,\gamma) \stackrel{\mathrm{def}}{=} \langle\gamma, \bar{S}(\alpha,\beta)\rangle$ is totally skewsymmetric;
4) $\langle T(\alpha,\beta), Z\rangle = \langle\alpha, C_1(\beta,Z)\rangle$.

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[1] This has been studied for small dimensional groups by Dazord, Ohn, Zakrzewski...

[2] We thank the referee for introducing us to ref [**5**].

The bracket defined on $\mathfrak{L}$ is then a Lie bracket (i.e. satisfies Jacobi's identity) if and only if:

5) $\partial p = 0$ where $p = {}^tT : \mathfrak{g} \to \Lambda^2\mathfrak{g}$;

6) $[X,S](\alpha,\beta,\gamma) + \langle \underset{\alpha\beta\gamma}{\sigma} T(T(\alpha,\beta),\gamma)), X\rangle = 0$ where σ denotes the sum over cyclic permutations;

7) $\underset{\alpha\beta\gamma}{\sigma}(S(T(\alpha,\beta),\gamma,\delta) + S(T(\alpha,\delta),\beta,\gamma)) = 0.$

Notation. Let $\mathfrak{L}_{S,p}$, where $p = {}^tT$, denote $\mathfrak{L} = \mathfrak{g} + \mathfrak{g}^*$ with $\langle\langle\ ,\ \rangle\rangle$ and $[\ ,\ ]$ defined by 1 and 2 with the conditions 3 and 4.

DEFINITION 2.1 ([**5**]). A *Manin pair* is a pair of Lie algebras $(\mathfrak{L},\mathfrak{g})$ and a non degenerate symmetric bilinear form $\langle\langle\ ,\ \rangle\rangle$ on $\mathfrak{L}$ such that the conditions i), ii), iii) and iv) are satisfied.

A *quasi Lie bialgebra* is a triple $(\mathfrak{g},p,S)$ where $\mathfrak{g}$ is a Lie algebra, $p : \mathfrak{g} \to \Lambda^2\mathfrak{g}$ is a cocycle and $S \in \Lambda^3\mathfrak{g}$ with the equations 6) and 7) satisfied.

From the expressions above, we get:

LEMMA 2.1 (DRINFELD, [**5**]). *If $(\mathfrak{L},\mathfrak{g})$ is a Manin pair, then a choice of an isotropic subspace in $\mathfrak{L}$ supplementary to $\mathfrak{g}$ identifies $\mathfrak{L}$ with a Lie algebra $\mathfrak{L}_{S,p}$ so that $(\mathfrak{g},p,S)$ is a quasi Lie bialgebra. Reciprocally, any quasi Lie bialgebra $(\mathfrak{g},p,S)$ yields a Manin pair $(\mathfrak{L}_{S,p},\mathfrak{g})$.*

A map $\varphi : \mathfrak{L}_{S,p} \to \mathfrak{L}_{S',p'}$ which is linear, maps $\mathfrak{g}$ to $\mathfrak{g}$ and preserves $\langle\langle\ ,\ \rangle\rangle$ is necessarily of the form

$$\varphi(X,\alpha) = (A(X + \widehat{Q}(\alpha)),\ {}^tA^{-1}(\alpha))$$

where $A : \mathfrak{g} \to \mathfrak{g}$ is linear and bijective and where $\widehat{Q} : \mathfrak{g}^* \to \mathfrak{g}$ is induced by an element $Q \in \Lambda^2\mathfrak{g}$ through

$$\langle\beta,\widehat{Q}(\alpha)\rangle = Q(\alpha,\beta).$$

Then $\varphi[(X,\alpha),(Y,\beta)]_{S,p} = [\varphi(X,\alpha),\varphi(Y,\beta)]_{S',p'}$ if and only if

i) A is a Lie automorphism of $\mathfrak{g}$;

ii) $A^{-1}\cdot p' - p = -\partial Q$;

iii) $(A^{-1}\cdot S' - S)(\alpha,\beta,\gamma) = \underset{\alpha\beta\gamma}{\sigma}(Q(T(\alpha,\beta),\gamma) + \langle\alpha,[\widehat{Q}(\beta),\widehat{Q}(\gamma)]\rangle)$
$= 1/2[Q,Q](\alpha,\beta,\gamma) + \underset{\alpha\beta\gamma}{\sigma} Q(T(\alpha,\beta),\gamma)$

where $(A\cdot p')_X(\alpha,\beta) = p'_{A^{-1}(X)}({}^tA\alpha,{}^tA\beta)$ and $(A\cdot S)(\alpha,\beta,\gamma) = S({}^tA\alpha,{}^tA\beta,{}^tA\gamma)$. We then say that $\mathfrak{L}_{S,p}$ and $\mathfrak{L}_{S',p'}$ are *isomorphic* under φ.

Remark 2.2. In particular, if $\mathfrak{L}_{S,p}$ is a Lie algebra (i.e. $(\mathfrak{g},p,S)$ is a quasi Lie bialgebra), if A is an automorphism of $\mathfrak{g}$ and if $Q \in \Lambda^2(\mathfrak{g})$, then $\mathfrak{L}_{S',p'}$ where

- $p' = A(p - \partial Q)$
- $S'(\alpha,\beta,\gamma) = (S + 1/2[Q,Q])({}^tA\alpha,{}^tA\beta,{}^tA\gamma) + \underset{\alpha\beta\gamma}{\sigma} Q({}^tp({}^tA\alpha,{}^tA\beta),{}^tA\gamma)$

is a Lie algebra (i.e. $(\mathfrak{g},p',S')$ is a quasi Lie bialgebra).

The $(\mathfrak{g},p',S')$ obtained as above with $A = \mathrm{Id}$ is called by Drinfeld [**5**] a *twisting* of $(\mathfrak{g},p,S)$ by Q.

Remark 2.3. A Manin pair $(\mathfrak{L}, \mathfrak{g})$ yields a Manin triple $(\mathfrak{L}, \mathfrak{g}, \mathfrak{g}^*)$ if and only if there is an isotropic subspace supplementary to $\mathfrak{g}$ in $\mathfrak{L}$ which is a subalgebra of $\mathfrak{L}$. Hence, a bialgebra structure on $\mathfrak{g}$ yields as its corresponding Manin algebra an algebra $\mathfrak{L}_{S,p'}$ which is isomorphic to a Lie algebra $\mathfrak{L}_{0,p}$ and vice versa.

Observe that $\mathfrak{L}_{0,p}$ is a Lie algebra if and only if

1) $\partial p = 0$;
2) $\underset{\alpha\beta\gamma}{\sigma} T(T(\alpha,\beta),\gamma) = 0$

and that 2) means that $(\mathfrak{g}^*, T)$ is a Lie algebra and we get back the conditions for $(\mathfrak{g}, p)$ to be a bialgebra (definition 2).

Remark 2.4. $\mathfrak{L}_{S,0}$ is a Lie algebra if and only if $S \in (\Lambda^3\mathfrak{g})^{\mathrm{inv}}$. Furthermore $\mathfrak{L}_{S,0}$ is isomorphic to $\mathfrak{L}_{S',-\partial Q}$ for any $Q \in \Lambda^2\mathfrak{g}$ with $S' = S + 1/2[Q,Q]$.

$\mathfrak{L}_{0,\partial Q}$ is isomorphic to $\mathfrak{L}_{-1/2[Q,Q],0}$ and is a Lie algebra if and only if $[Q,Q] \in \Lambda^3\mathfrak{g}$ is invariant under $\mathfrak{g}$ and we get back the condition to have an exact Lie bialgebra (definition 3).

Remark 2.5. Observe that if $(\mathfrak{L}, \mathfrak{g})$ is a Manin pair for a nondegenerate symmetric invariant bilinear form, it is also a Manin pair for any nonzero multiple of that form. Choosing an isotropic subspace supplementary to $\mathfrak{g}$ in $\mathfrak{L}$, this amounts to say that if $\mathfrak{L}_{S,p}$ is a Lie algebra (i.e. $(\mathfrak{g}, p, S)$ is a quasi Lie bialgebra) then $\mathfrak{L}_{s^2S,sp}$ is also a Lie algebra (i.e. $(\mathfrak{g}, sp, s^2S)$ is a quasi Lie bialgebra) for any nonzero real number s and they are related by scaling on $\mathfrak{g}^*$:

$$\begin{aligned} &\text{if} \quad \varphi(X,\alpha) = (X, s\alpha) \\ &\text{then} \quad [\varphi(X,\alpha), \varphi(Y,\beta)]_{S,p} = \varphi([(X,\alpha),(Y,\beta)]_{s^2S,sp}). \end{aligned}$$

We shall allow these further isomorphisms of the Manin algebra $\mathfrak{L}$ corresponding to a Manin triple.

Definition 2.2. We shall say that two bialgebra structures on a given Lie algebra $\mathfrak{g}$ yield *isomorphic Manin algebras* $\mathfrak{L}$ and $\mathfrak{L}'$ if and only if there exists a map $\varphi : \mathfrak{L} \to \mathfrak{L}'$ which

- is an isomorphism of Lie algebras,
- maps $\mathfrak{g}$ to $\mathfrak{g}$,
- is a homothetic transformation from $\mathfrak{L}$ to $\mathfrak{L}'$, i.e. $\langle\langle\varphi(X),\varphi(Y)\rangle\rangle' = s\langle\langle X,Y\rangle\rangle, \quad \forall X,Y \in \mathfrak{L}$ for some nonzero real s.

Lemma 2.2. *Two Lie bialgebra structures on a given Lie algebra $\mathfrak{g}$, $(\mathfrak{g}, p)$ and $(\mathfrak{g}, p')$, yield isomorphic Manin algebras if and only if there are $Q \in \Lambda^2\mathfrak{g}$, A an automorphism of $\mathfrak{g}$ and s a nonzero real number such that*

$$\begin{cases} p' = sA(p - \partial Q); \\ 1/2[Q,Q](\alpha,\beta,\gamma) + \underset{\alpha\beta\gamma}{\sigma} Q({}^t p(\alpha,\beta),\gamma) = 0. \end{cases}$$

In particular, two exact Lie bialgebra structures on $\mathfrak{g}$, $(\mathfrak{g}, \partial Q)$ and $(\mathfrak{g}, \partial Q')$ yield isomorphic Manin algebras if and only if $[Q',Q'] = s^2 A[Q,Q]$ for some automorphism A of $\mathfrak{g}$ and some $s \neq 0 \in \mathbb{R}$.

Let $\mathfrak{g}$ be compact simple; since any 1-cocycle $p : \mathfrak{g} \to \Lambda^2\mathfrak{g}$ is exact, and since $(\Lambda^3\mathfrak{g})^{\mathrm{inv}}$ is 1-dimensional then any Lie bialgebra structure on $\mathfrak{g}$ is of the form $(\mathfrak{g}, \partial Q)$ with

$$[Q,Q] = \lambda\Omega \text{ where } \beta^{(3)}(X \wedge Y \wedge Z, \Omega) = \beta(X, [Y,Z])$$

for any X, Y, Z in $\mathfrak{g}$ ($\beta^{(3)}$ is the extension of β to $\Lambda^3\mathfrak{g}$).

Define $\alpha : \Lambda^2\mathfrak{g} \to \mathfrak{g}$, $X \wedge Y \mapsto [X,Y]$. Then $\beta^{(3)}([X \wedge Y, U \wedge V], \Omega) = 2\beta(\alpha(X \wedge Y), \alpha(U \wedge V))$. So $\beta^{(3)}([Q,Q],\Omega) = 2\beta(\alpha(Q),\alpha(Q))$. Hence:

Remark 2.6. For $\mathfrak{g}$ compact, since β is negative definite,

$$[Q,Q] = \lambda\Omega \quad \Rightarrow \quad \lambda \geq 0.$$

Using the possibility of scaling we mentioned above, one just has to consider 2 cases: $[Q,Q] = 0$, which yields for $\mathfrak{L}$ the Lie algebra of T^*G and one case $[Q,Q] = \lambda\Omega$ with $\lambda > 0$. Consider (as in Lu and Weinstein [**7**]) the Manin triple given by the Iwasawa decomposition of $\mathfrak{g}^{\mathbb{C}}$: ($\mathfrak{L} = \mathfrak{g}^{\mathbb{C}} = \mathfrak{g} + i\mathfrak{g} = \mathfrak{g} + \mathfrak{a} + \mathfrak{n}$, $\mathfrak{g}_1 = \mathfrak{g}$, $\mathfrak{g}_2 = \mathfrak{a} + \mathfrak{n}$) with the invariant symmetric bilinear form defined by the imaginary part of the Killing form. We get:

Lemma 2.3. *For $\mathfrak{g}$ compact simple, any bialgebra structure on $\mathfrak{g}$ yields a Manin triple whose corresponding Manin algebra $\mathfrak{L}$ is isomorphic to* $\mathrm{Lie}(T^*G)$ *or* $\mathfrak{g}^{\mathbb{C}}$.

3. Exact Lie bialgebra structures

Let $\mathfrak{g}$ be a Lie algebra and let $Q \in \Lambda^2\mathfrak{g}$. Then $\mathfrak{g}^*$ is endowed with the bracket ${}^t\partial Q$:

$$\begin{aligned}\langle[\alpha,\beta], X\rangle &= \langle \alpha \wedge \beta, \partial Q(X)\rangle = (\partial Q(X))(\alpha,\beta) \\ &= \langle [X,Q], \alpha \wedge \beta\rangle\end{aligned}$$

and it satisfies Jacobi's identity if and only if $[Q,Q] \in (\Lambda^3\mathfrak{g})^{\mathrm{inv}}$. Introducing as before $\widehat{Q} : \mathfrak{g}^* \to \mathfrak{g}$ by

$$\langle \beta, \widehat{Q}(\alpha)\rangle = Q(\alpha,\beta)$$

we get

$$[\alpha,\beta] = \mathrm{ad}^*\, \widehat{Q}(\alpha)\beta - \mathrm{ad}^*\, \widehat{Q}(\beta)\alpha.$$

Properties.

1) $\widehat{Q}$ is a homomorphism of Lie algebras if and only if $[Q,Q] = 0$;
2) if $\mathfrak{g}_1 = \mathrm{Im}\,\widehat{Q}$ then $Q \in \Lambda^2\mathfrak{g}_1$;
3) if $[Q,Q] = 0$ and Q is nondegenerate, then $\mathfrak{g}^* \cong \mathfrak{g}$ and $\mathfrak{g}$ admits a 2-form F

   $$F(X,Y) = Q(\widehat{Q}^{-1}(X), \widehat{Q}^{-1}(Y)), \quad \text{such that} \quad \underset{\alpha\beta\gamma}{\sigma} F([X,Y],Z) = 0.$$

   In that case, if G is a connected Lie group with algebra $\mathfrak{g}$, G admits a left invariant symplectic structure.

Hence one gets:

LEMMA 3.1. *The study of solutions of* $[Q,Q]=0$ *(Yang–Baxter equation) on* $\mathfrak{g}$ *is equivalent to the study of subalgebras* $\mathfrak{g}_1$ *of* $\mathfrak{g}$ *corresponding to symplectic groups (i.e groups with an invariant symplectic structure)* [3]*. Precisely, if* Q *is a solution of Yang–Baxter equation on* $\mathfrak{g}$ *then* $\mathfrak{g}_1 = \operatorname{Im}\widehat{Q}$ *is the Lie algebra of a connected symplectic group* (G_1,ω) *where* $\omega_e(X,Y) = Q(\widehat{Q}^{-1}(X), \widehat{Q}^{-1}(Y)) \quad \forall X,Y \in \mathfrak{g}_1$*. Reciprocally if* $\mathfrak{g}_1$ *is a subalgebra of* $\mathfrak{g}$ *which is the Lie algebra of a symplectic group* (G_1,ω)*, then it defines a solution* $Q\in\Lambda^2(\mathfrak{g})$ *of Yang–baxter equation by* $Q(\alpha,\beta) = \omega_e((\pi(\alpha))^\#, (\pi(\beta))^\#)$ *where* $\pi : \mathfrak{g}^* \to {\mathfrak{g}_1}^*$ *is dual to the inclusion* $\mathfrak{g}_1 \subset \mathfrak{g}$ *and* $^\# : {\mathfrak{g}_1}^* \to \mathfrak{g}_1$ *is such that* $\omega_e(\gamma^\#, Y) = \langle\gamma, Y\rangle \quad (Y\in\mathfrak{g}_1, \gamma\in{\mathfrak{g}_1}^*)$*.*

Suppose, in what follows, that $\mathfrak{g}$ has a nondegenerate invariant symmetric bilinear form β. Then Q determines a linear map $\widetilde{Q} : \mathfrak{g}\to\mathfrak{g}$ defined by

$$\langle\alpha, \widetilde{Q}(X)\rangle = \beta(\widehat{Q}(\alpha), X)$$

or equivalently

$$\beta(\widetilde{Q}(Y), X) = \beta^{(2)}(Q, X\wedge Y) = Q(\widehat{\beta}^{-1}(X), \widehat{\beta}^{-1}(Y))$$

where $\widehat{\beta} : \mathfrak{g}^* \to \mathfrak{g}$ is such that $\beta(\widehat{\beta}(\alpha), X) = \alpha(X)$.

LEMMA 3.2. *If* $\mathfrak{g}$ *is (real or complex) semisimple, the linear map* $\rho : (S^2\mathfrak{g}^*)^{\mathrm{inv}} \to (\Lambda^3\mathfrak{g})^{\mathrm{inv}}$ *defined by* $\beta^{(3)}\langle\rho B, X\wedge Y\wedge Z\rangle = B([X,Y],Z)$ *for* $X,Y,Z\in\mathfrak{g}$ *is a linear isomorphism. [Again* $\beta^{(3)}(\rho B, X\wedge Y\wedge Z) = \rho B(\widehat{\beta}^{-1}(X), \widehat{\beta}^{-1}(Y), \widehat{\beta}^{-1}(Z))$*].*

Hence any bialgebra structure on $\mathfrak{g}$ *is defined by a* $Q\in\Lambda^2\mathfrak{g}$ *such that* $[Q,Q]\in(\Lambda^3\mathfrak{g})^{\mathrm{inv}}$ *so* $[Q,Q]=\rho B$ *where* $B\in(S^2\mathfrak{g}^*)^{\mathrm{inv}}$ *is of the form* $B(X,Y)=\beta(MX,Y)$*.*

The equations on the corresponding $\widetilde{Q}\in\operatorname{End}(\mathfrak{g})$ *are (using* $[Q,Q](\alpha,\beta,\gamma) = 2\underset{\alpha\beta\gamma}{\sigma}\langle\gamma, [\widehat{Q}(\alpha), \widehat{Q}(\beta)]\rangle$ *and* $\widehat{Q}(\widehat{\beta}^{-1}(X)) = -\widetilde{Q}(X)$*):*

$$\begin{aligned}&\beta(\widetilde{Q}X, Y) = -\beta(X, \widetilde{Q}Y)\\ &[\widetilde{Q}X, \widetilde{Q}Y] - \widetilde{Q}[\widetilde{Q}X, Y] - \widetilde{Q}[X, \widetilde{Q}Y] = 2M[X,Y]\end{aligned}$$

(Modified Yang–Baxter equation of coefficient M*).*

In general M can be quite complicated. If $\mathfrak{g}$ is complex simple, then M is a multiple of the identity. In these note we shall look at this case only, but where $\mathfrak{g}$ is any semisimple Lie algebra. Thus we consider the equations

$$\begin{cases}\beta(\widetilde{Q}X, Y) = -\beta(X, \widetilde{Q}Y);\\ [\widetilde{Q}X, \widetilde{Q}Y] - \widetilde{Q}[\widetilde{Q}X, Y] - \widetilde{Q}[X, \widetilde{Q}Y] = \lambda[X,Y]\end{cases} \qquad (*)$$

for $\widetilde{Q}\in\operatorname{End}(\mathfrak{g})$, $\mathfrak{g}$ a semisimple Lie algebra.

A. The complex case.

This problem was solved, for $\mathfrak{g}$ simple and $\lambda\neq 0$, by Belavin and Drinfeld [**1**]. We follow their approach with small modifications, but we also allow $\mathfrak{g}$ to be semisimple throughout.

[3] These have been partially studied by Lichnerowicz, Medina, Revoy.

For each complex number μ let $\mathfrak{g}_\mu$ denote the corresponding generalized eigenspace of $\widetilde{Q}$

$$\mathfrak{g}_\mu = \{X \in \mathfrak{g} \mid (\widetilde{Q}-\mu)^k X = 0 \text{ for some positive integer } k\}.$$

The second equation in (*) can be rewritten, for any $\mu, \rho \in \mathbb{C}$ as

$$\begin{aligned}[(\widetilde{Q}-\mu)X,(\widetilde{Q}-\rho)Y] - (\widetilde{Q}-\rho)[(\widetilde{Q}-\mu)X,Y] - (\widetilde{Q}-\mu)[X,(\widetilde{Q}-\rho)Y]\\ = (\lambda-\mu\rho)[X,Y] + (\rho+\mu)\widetilde{Q}[X,Y]\end{aligned} \qquad (**)$$

so that one easily deduces:

1) *If $\mu \neq -\rho$ $[\mathfrak{g}_\mu, \mathfrak{g}_\rho] \subset \mathfrak{g}_\sigma$, for $\sigma = \frac{\rho\mu-\lambda}{\rho+\mu}$, and $\mathfrak{g}_\rho, \mathfrak{g}_\mu$ are β-orthogonal;*
2) *$[\mathfrak{g}_\mu, \mathfrak{g}_{-\mu}] = 0$ if $\mu^2 \neq -\lambda$.*

Let $a^2 = -\lambda$, then we conclude that

i) *$\mathfrak{g}_a$ and $\mathfrak{g}_{-a}$ are subalgebras of $\mathfrak{g}$ which are isotropic with respect to β;*
ii) *$\mathfrak{g}' = \sum_{\mu \neq \pm a} \mathfrak{g}_\mu$ is a subalgebra;*
iii) *$\mathfrak{g}_{\pm a} + \mathfrak{g}'$ are subalgebras in which $\mathfrak{g}_{\pm a}$ are ideals.*

Let $Q^\pm = \widetilde{Q} \pm a$. Then $Q^\pm$ is invertible on $\mathfrak{g}' + \mathfrak{g}_{\pm a}$. It follows from (**) that

$$Q^+[Q^-X, Q^-Y] = Q^-[Q^+X, Q^+Y] \ \forall X, Y \in \mathfrak{g}.$$

Thus, since $Q^\pm$ are invertible on $\mathfrak{g}'$, $\psi = Q^+|_{\mathfrak{g}'} \circ (Q^-|_{\mathfrak{g}'})^{-1}$ is an automorphism of $\mathfrak{g}'$ without 1 as an eigenvalue (if $\psi Z = Z$ then $Q^+Z = Q^-Z \Rightarrow 2aZ = 0 \Rightarrow Z = 0$).

Lemma 3.3 (Belavin–Drinfeld [**1**]). *If ψ is an automorphism of a finite dimensional semisimple Lie algebra then it has 1 as an eigenvalue. If ψ is an automorphism of a Lie algebra without 1 as an eigenvalue then the Lie algebra is solvable.*

Lemma 3.4 (Cartan [**2**]). *If ρ is a faithful representation of a semisimple Lie algebra $\mathfrak{g}$ on a finite dimensional vector space V, the trace form ρ is non-degenerate on $\mathfrak{g}$. From this it follows that any subalgebra of a semisimple Lie algebra which is isotropic with respect to the Killing form is solvable.*

Corollary. *$\mathfrak{g}_a$, $\mathfrak{g}_{-a}$, $\mathfrak{g}'$, $\mathfrak{g}_a+\mathfrak{g}'$, $\mathfrak{g}_{-a}+\mathfrak{g}'$ are all solvable and $\mathfrak{g} = \mathfrak{g}_a+\mathfrak{g}'+\mathfrak{g}_{-a}$.*

Proof $\mathfrak{g}_a$ and $\mathfrak{g}_{-a}$ are isotropic for β whilst $\mathfrak{g}'$ has an automorphism without 1 as an eigenvalue. Since $\mathfrak{g}_{\pm a}$ are ideals in $\mathfrak{g}_{\pm a} + \mathfrak{g}'$, the latter are also solvable.

Since each of $\mathfrak{g}_{\pm a} + \mathfrak{g}'$ is solvable, it is contained in some Borel subalgebra $\mathfrak{b}_\pm$ of $\mathfrak{g}$. Since $\mathfrak{b}_+ + \mathfrak{b}_-$ contains $\mathfrak{g}_a$, $\mathfrak{g}'$ and $\mathfrak{g}_{-a}$ we have $\mathfrak{b}_+ + \mathfrak{b}_- = \mathfrak{g}$ so $\mathfrak{h} = \mathfrak{b}_+ \cap \mathfrak{b}_-$ is a Cartan subalgebra of $\mathfrak{g}$. If $\mathfrak{n}_\pm$ is the nilradical of $\mathfrak{b}_\pm$ we have $\mathfrak{b}_\pm = \mathfrak{h} + \mathfrak{n}_\pm$ and $\mathfrak{n}_\pm$ is the Killing form-orthogonal of $\mathfrak{b}_+$ (observe that $\mathfrak{n}_\pm$ consist of all elements X in $\mathfrak{b}_\pm$ such that ad X is nilpotent as an endomorphism of $\mathfrak{g}$). But $\mathfrak{g}_a + \mathfrak{g}'$ has Killing form orthogonal $\mathfrak{g}_a$ so $\mathfrak{g}_a + \mathfrak{g}' \subset \mathfrak{b}_+$ implies $\mathfrak{n}_+ \subset \mathfrak{g}_a$. Since the only Borel containing $\mathfrak{n}_+$ is $\mathfrak{b}_+$, it follows that $\mathfrak{b}_+$ is the unique Borel containing $\mathfrak{g}_a + \mathfrak{g}'$. Likewise $\mathfrak{b}_-$ is the unique Borel containing $\mathfrak{g}_{-a} + \mathfrak{g}'$. Also $\mathfrak{h} = \mathfrak{b}_+ \cap \mathfrak{b}_- \supset \mathfrak{g}'$ is uniquely determined by $\widetilde{Q}$.

Let Δ be the set of roots of $\mathfrak{g}$ relative to the Cartan subalgebra $\mathfrak{h}$. If $\alpha \in \Delta$ denote by $\mathfrak{g}^\alpha$ the corresponding root space and choose $\mathfrak{b}_+$ to determine the positive roots Δ^+ ($\mathfrak{b}_+ = \mathfrak{h} + \oplus_{\alpha\in\Delta^+}\mathfrak{g}^\alpha$). Then, since $\mathfrak{b}_+ + \mathfrak{b}_- = \mathfrak{g}$, $\mathfrak{n}_-$ corresponds to the negative roots ($\mathfrak{n}_- = \oplus_{\alpha\in\Delta^+}\mathfrak{g}^{-\alpha}$).

Lemma 3.5 (cf [**1**]). *$\widetilde{Q}(\mathfrak{b}_\pm) \subset \mathfrak{b}_\pm$, $\widetilde{Q}(\mathfrak{n}_\pm) \subset \mathfrak{n}_\pm$, $\widetilde{Q}(\mathfrak{h}) \subset \mathfrak{h}$.*

Proof Consider $\mathfrak{l} = \{X \in \mathfrak{g} \mid [X, \mathfrak{g}_a] \subset \mathfrak{g}_a\}$; then $\mathfrak{n}_+ \subset \mathfrak{g}_a$ so $[\mathfrak{l}, \mathfrak{n}_+] \subset \mathfrak{b}_+$. But the maximal algebra which satisfies $[\mathfrak{l}, \mathfrak{n}_+] \subset \mathfrak{b}_+$ is $\mathfrak{b}_+$ itself, so $\mathfrak{l} \subset \mathfrak{b}_+$. However $\mathfrak{g}_a \subset \mathfrak{b}_+$ so $[\mathfrak{b}_+, \mathfrak{g}_a] \subset [\mathfrak{b}_+, \mathfrak{b}_+] \subset \mathfrak{n}_+ \subset \mathfrak{g}_a$ so $\mathfrak{b}_+ \subset \mathfrak{l}$. Thus $\mathfrak{l} = \mathfrak{b}_+$.

Now, if $Y \in \mathfrak{g}_a$, we have $\widetilde{Q}Y \in \mathfrak{g}_a$ so if $X \in \mathfrak{l}$, (**) implies (taking $\mu = \rho = a$)

$$(\widetilde{Q} - a)[(\widetilde{Q} - a)X, Y] - [(\widetilde{Q} - a)X, (\widetilde{Q} - a)Y] \in \mathfrak{g}_a$$

and a simple induction gives

$$(\widetilde{Q} - a)^k[(\widetilde{Q} - a)X, Y] - [(\widetilde{Q} - a)X, (\widetilde{Q} - a)^kY] \in \mathfrak{g}_a.$$

Since $Y \in \mathfrak{g}_a$ there is a k so that $(\widetilde{Q} - a)^kY = 0$ and then $[(\widetilde{Q} - a)X, Y] \in \mathfrak{g}_a$. Thus $[\widetilde{Q}X, Y] \in \mathfrak{g}_a$ and we see $\widetilde{Q}\mathfrak{l} \subset \mathfrak{l}$. This shows $\widetilde{Q}\mathfrak{b}_+ \subset \mathfrak{b}_+$, and since $\widetilde{Q}$ is skew-symmetric, this implies $\widetilde{Q}\mathfrak{n}_+ \subset \mathfrak{n}_+$.

A similar argument using $\mathfrak{g}_{-a}$ gives $\widetilde{Q}\mathfrak{b}_- \subset \mathfrak{b}_-$ and $\widetilde{Q}\mathfrak{n}_- \subset \mathfrak{n}_-$. Finally $\widetilde{Q}\mathfrak{h} = \widetilde{Q}(\mathfrak{b}_+ \cap \mathfrak{b}_-) \subset \mathfrak{b}_+ \cap \mathfrak{b}_- = \mathfrak{h}$.

Now let $\mathfrak{c}_\pm = \operatorname{Im} Q^\pm$. From (**) we see that both $\mathfrak{c}_\pm$ are subalgebras of $\mathfrak{g}$ and $\mathfrak{g}_{\pm a} + \mathfrak{g}' \subset \mathfrak{c}_\pm$ so $\mathfrak{n}_\pm \subset \mathfrak{c}_\pm$. The proposition in the appendix implies that $\mathfrak{h} + \mathfrak{c}_\pm$ is a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}_\pm$.

Hence, there exist two subsets Γ_+ and Γ_- of the simple roots Φ in Δ^+ so that

$$\mathfrak{c}^+ = \operatorname{Im}(\widetilde{Q} + a) = \mathfrak{n}_{\Gamma_+} \oplus \sum_{\alpha\in\widehat{\Gamma}_+} (\mathfrak{g}^\alpha + \mathfrak{g}^{-\alpha} + [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]) \oplus V^+$$
$$\mathfrak{c}^- = \operatorname{Im}(\widetilde{Q} - a) = \mathfrak{n}_{\Gamma_-} \oplus \sum_{\alpha\in\widehat{\Gamma}_-} (\mathfrak{g}^\alpha + \mathfrak{g}^{-\alpha} + [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]) \oplus V^-$$

where

- $\widehat{\Gamma}_+$ (resp. $\widehat{\Gamma}_-$) is the set of positive roots which can be written as integer combinations of the simple roots in Γ_+ (resp Γ_-)
- $\mathfrak{n}_{\Gamma_+} = \sum_{\alpha\in\Delta^+\setminus\widehat{\Gamma}_+} \mathfrak{g}^\alpha, \qquad \mathfrak{n}_{\Gamma_-} = \sum_{\alpha\in\Delta^+\setminus\widehat{\Gamma}_-} \mathfrak{g}^{-\alpha}$
- $V^\pm$ is a subspace of $\mathfrak{h}$ in $(\sum_{\alpha\in\widehat{\Gamma}_\pm} H_\alpha)^{\perp_\beta}$ such that $(V^\pm)^\perp \subset V^\pm$.

Then

$$\operatorname{Ker}(\widetilde{Q} + a) = \operatorname{Im}(\widetilde{Q} - a)^\perp = \mathfrak{n}_{\Gamma_-} + (V^-)^\perp$$
$$\operatorname{Ker}(\widetilde{Q} - a) = \operatorname{Im}(\widetilde{Q} + a)^\perp = \mathfrak{n}_{\Gamma_+} + (V^+)^\perp$$

We have, as mentioned before,

$$(\widetilde{Q} + a)[(\widetilde{Q} - a)X, (\widetilde{Q} - a)Y] = (\widetilde{Q} - a)[(\widetilde{Q} + a)X, (\widetilde{Q} + a)Y]$$

so $(\widetilde{Q}-a)(\widetilde{Q}+a)^{-1}$ induces a Lie algebra isomorphism

$$\theta \;:\; \mathfrak{c}^+/(\mathfrak{c}^+)^\perp = \sum_{\alpha\in\widehat{\Gamma}_+}(\mathfrak{g}^\alpha+\mathfrak{g}^{-\alpha}+[\mathfrak{g}^\alpha,\mathfrak{g}^{-\alpha}]) + V^+/(V^+)^\perp$$
$$\to \mathfrak{c}^-/(\mathfrak{c}^-)^\perp = \sum_{\alpha\in\widehat{\Gamma}_-}(\mathfrak{g}^\alpha+\mathfrak{g}^{-\alpha}+[\mathfrak{g}^\alpha,\mathfrak{g}^{-\alpha}]) + V^-/(V^-)^\perp.$$

This induces a map $\tau : \Gamma_+ \to \Gamma_-$ (which are the simple roots of these reductive algebras) such that

$$(\tau(\alpha),\tau(\beta)) = (\alpha,\beta), \qquad \forall\alpha,\beta\in\Gamma_+. \tag{1}$$

Choosing compatible Weyl bases, one has:

$$\left.\begin{aligned}\theta(H_\alpha) &= H_{\tau(\alpha)}\\ \theta(E_\alpha) &= E_{\tau(\alpha)}\end{aligned}\right\}\ \forall\alpha\in\widehat{\Gamma}_+.$$

Observe that $(\widetilde{Q}+a) : \mathfrak{n}^+ \to \mathfrak{n}^+$ must be a bijection. Indeed $\widetilde{Q}$, hence $(\widetilde{Q}+a)$ and $(\widetilde{Q}-a)$, maps $\mathfrak{n}^+$ into $\mathfrak{n}^+$, $\mathfrak{n}^-$ into $\mathfrak{n}^-$ and $\mathfrak{h}$ into $\mathfrak{h}$ and $\mathrm{Ker}(\widetilde{Q}+a)\cap\mathfrak{n}^+ = \{0\}$, $\mathrm{Im}(\widetilde{Q}+a)\supset\mathfrak{n}^+$ and, in terms of τ, one has

$$\psi = (\widetilde{Q}-a)(\widetilde{Q}+a)^{-1} \;:\; \mathfrak{n}^+\to\mathfrak{n}^+ : \begin{aligned} E_\alpha &\mapsto E_{\tau(\alpha)} && \forall\alpha\in\widehat{\Gamma}_+;\\ E_\gamma &\mapsto 0 && \forall\gamma\in\phi^+\backslash\widehat{\Gamma}_+.\end{aligned}$$

Thus, on $\mathfrak{n}^+, (1-\psi)\widetilde{Q} = a(\psi+1)$. Also $(1-\psi) = 2a(\widetilde{Q}+a)^{-1}$ is an invertible map on $\mathfrak{n}^+$.

LEMMA 3.6 (BELAVIN-DRINFELD [**1**]). *$(1-\psi)$ is invertible on $\mathfrak{n}^+$ if and only if, for any $\alpha\in\Gamma_+$, there is a positive integer k such that*

$$\alpha,\tau(\alpha),\ldots,\tau^{k-1}(\alpha)\in\Gamma_+ \text{ and } \tau^k(\alpha)\notin\Gamma_+. \tag{2}$$

Then $\widetilde{Q}$ on $\mathfrak{n}^+$ is given from τ satisfying (1) and (2) by $\widetilde{Q} = a(1-\psi)^{-1}(\psi+1) = a(1+\psi+\psi^2+\ldots+\psi^k+\ldots)(1+\psi)$ so that

$$\begin{cases}\widetilde{Q}(E_\gamma) = aE_\gamma, & \forall\gamma\in\phi^+\backslash\widehat{\Gamma}_+;\\ \widetilde{Q}(E_\alpha) = a(E_\alpha + 2\sum_{\beta>\alpha}E_\beta), & \forall\alpha\in\widehat{\Gamma}_+\end{cases}$$

where one writes $\beta>\alpha$ if $\alpha,\tau(\alpha),\ldots,\tau^{k-1}(\alpha)\in\widehat{\Gamma}_+$ and $\tau^k(\alpha)=\beta$ for some integer $k\geq 1$.

Finally $\widetilde{Q}$ is then completely determined on $\mathfrak{n}^-$ by

$$\beta(\widetilde{Q}X,Y) = -\beta(X,\widetilde{Q}Y)$$

since $(\mathfrak{n}^+)^\perp = \mathfrak{n}^-$. Observe that $\widetilde{Q}|_\mathfrak{h} \;:\; \mathfrak{h}\to\mathfrak{h}$ must satisfy

(i) $\beta(\widetilde{Q}X,Y) = -\beta(X,\widetilde{Q}Y)$, $\forall X,Y\in h$;

(ii) $(\widetilde{Q}-a)H_\alpha = (\widetilde{Q}+a)H_{\tau(\alpha)}$, $\forall\alpha\in\Gamma_+$.

(Indeed $\theta(H_\alpha) = H_{\tau(\alpha)}$ so $(\widetilde{Q}-a)x = H_{\tau(\alpha)}+y$ for an x such that $(\widetilde{Q}+a)x = H_\alpha$ and a y in $\mathrm{Ker}(\widetilde{Q}+a)$).

Hence we get:

Theorem 3.1 (Belavin–Drinfeld [**1**]). *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let $Q \in \Lambda^2\mathfrak{g}$ satisfy*

$$\beta^{(3)}([Q,Q], X\wedge Y\wedge Z) = \beta(\frac{\lambda}{2}[X,Y], Z).$$

Then, there exist a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, a system of positive roots Δ^+ of $(\mathfrak{g},\mathfrak{h})$, two subsets Γ_+ and Γ_- of the set Φ of simple roots corresponding to Δ^+ and a map $\tau : \Gamma_+ \to \Gamma_-$ satisfying

- $\langle\tau(\alpha),\tau(\beta)\rangle = \langle\alpha,\beta\rangle$, $\forall\alpha,\beta\in\Gamma_+$;
- $\forall\alpha\in\Gamma_+$, *there exists a positive integer k such that $\tau^\ell(\alpha)\in\Gamma_+$, $\forall\ell<k$ and $\tau^k(\alpha)\notin\Gamma_+$ such that, for a choice of Weyl basis E_α in $\mathfrak{g}^\alpha$ with $\beta(E_\alpha,E_{-\alpha})=1$:*

$$Q = Q_0 + a\Big(\sum_{\alpha\in\Delta^+} E_{-\alpha}\wedge E_\alpha + 2\sum\nolimits_{\alpha\in\widehat{\Gamma}_+,\alpha<\beta} E_{-\beta}\wedge E_\alpha\Big)$$

  where $a^2=-\lambda$ and $Q_0\in\Lambda^2\mathfrak{h}$ is determined by $Q(\alpha,\beta)$, $\forall\alpha,\beta\in\Phi$ and those must verify:
- $Q(\tau(\alpha),\beta) = Q(\alpha,\beta) - a(\langle\alpha,\beta\rangle + \langle\tau(\alpha),\beta\rangle)$, $\forall\alpha\in\Gamma_+$, $\forall\beta\in\Phi$.

Observe — as in Belavin–Drinfeld — that, reciprocally, any Q described above gives a solution of the problem. Indeed, any $\widetilde{Q}\in\mathrm{End}(\mathfrak{g})$ which has the following properties:

- $\beta(\widetilde{Q}X,Y) = -\beta(X,\widetilde{Q}Y)$;
- $\mathrm{Im}(\widetilde{Q}\pm a) = \mathfrak{c}^\pm$ are subalgebras such that $\mathfrak{c}^\pm\supset(\mathfrak{c}^\pm)^\perp$, $(a^2=-\lambda)$;
- $(\widetilde{Q}-a)(\widetilde{Q}+a)^{-1}$ induces a Lie algebra isomorphism

$$\theta : \mathfrak{c}^+/(\mathfrak{c}^+)^\perp \to \mathfrak{c}^-/(\mathfrak{c}^-)^\perp$$

satisfies

$$(\widetilde{Q}-a)[(\widetilde{Q}+a)X,(\widetilde{Q}+a)Y] = (\widetilde{Q}+a)[(\widetilde{Q}-a)X,(\widetilde{Q}-a)Y]$$

hence is a solution of (*).

B. The real case.

a. Let us first consider the case $\lambda=0$ when $\mathfrak{g}$ is compact. Observe that if a Lie algebra $\mathfrak{g}$ has an invertible derivation then it is solvable.

Corollary 3.1. *If $Q\in\Lambda^2\mathfrak{g}$ is of maximal rank and $[Q,Q]=0$ then $\mathfrak{g}$ cannot be semisimple since $\widetilde{Q}^{-1}$ would be a derivation.*

Corollary 3.2. *If $\mathfrak{g}$ is compact and $Q\in\Lambda^2\mathfrak{g}$ satisfies $[Q,Q]=0$, the image of $\widehat{Q}$ is an abelian subalgebra.*

Indeed, $\mathfrak{g}_1 = \operatorname{Im}\widehat{Q}$ is compact so has an nondegenerate invariant bilinear form but then the corresponding $\widetilde{Q}^{-1}$ is an invertible derivation of $\mathfrak{g}_1$ so $\mathfrak{g}_1$ is solvable, hence abelian.

Thus the solutions of $[Q,Q]=0$ in the compact case are precisely the elements of the second exterior powers of abelian subalgebras.

b. Consider now the case $\lambda \neq 0$ when $\mathfrak{g}$ is compact. As mentioned before, this implies $\lambda > 0$.

We use our study of the complex case for $\mathfrak{g}^{\mathbb{C}}$ (the complex linear extension of $\widetilde{Q}$ is clearly a solution of (*) on $\mathfrak{g}^{\mathbb{C}}$). Observe that a is purely imaginary. $\mathfrak{g}_{-a} = \bar{\mathfrak{g}}_a$ (where $^-$ denotes the conjugation of $\mathfrak{g}^{\mathbb{C}}$ relative to $\mathfrak{g}$) and the $\mathfrak{g}_\mu$ are eigenspaces. Hence we get:

Proposition. *If $\mathfrak{g}$ is a compact semisimple Lie algebra and if $\widetilde{Q} \in \operatorname{End}(\mathfrak{g})$ is a solution of*

$$\left.\begin{aligned} &\beta(\widetilde{Q}X,Y) = -\beta(X,\widetilde{Q}Y)\\ &[\widetilde{Q}X,\widetilde{Q}Y] - \widetilde{Q}[\widetilde{Q}X,Y] - \widetilde{Q}[X,\widetilde{Q}Y] = \lambda[X,Y] \end{aligned}\right\} \tag{$*$}$$

with $\lambda \neq 0$, then $\lambda > 0$, there exist a maximal toral subalgebra $\mathfrak{t}$ of $\mathfrak{g}$, the corresponding root space decomposition $\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \sum_{\alpha\in\Delta} \mathfrak{g}^\alpha$ and a choice of a system of positive roots Δ^+ so that

$$\left\{\begin{aligned} &\left.\begin{aligned} &\widetilde{Q}|_{\mathfrak{g}^\alpha} = i\sqrt{-\lambda}\ \operatorname{Id}|_{\mathfrak{g}^\alpha},\\ &\widetilde{Q}|_{\mathfrak{g}^{-\alpha}} = -i\sqrt{-\lambda}\ \operatorname{Id}|_{\mathfrak{g}^{-\alpha}}, \end{aligned}\right\} \forall\alpha\in\Delta^+;\\ &\widetilde{Q}(\mathfrak{t}^{\mathbb{C}}) \subset \mathfrak{t}^{\mathbb{C}}. \end{aligned}\right.$$

The corresponding $Q \in \Lambda^2\mathfrak{g}$ is of the form

$$Q = R_0 - \frac{\sqrt{-\lambda}}{2} \sum_{\alpha\in\Delta^+} i(E_\alpha - E_{-\alpha}) \wedge (E_\alpha + E_{-\alpha})$$

where $E_\alpha \in \mathfrak{g}^\alpha$, $\bar{E}_\alpha = E_{-\alpha}$, $B(E_\alpha, E_{-\alpha}) = -1$ and $R_0 \in \Lambda^2\mathfrak{t}$.

(Indeed $\mathfrak{b}_- = \overline{\mathfrak{b}_+}$, $\mathfrak{b}\cap\bar{\mathfrak{b}} = \mathfrak{t}^{\mathbb{C}}$ so $\mathfrak{b} = \mathfrak{t}^{\mathbb{C}} + \sum_{\alpha\in\Delta^+}\mathfrak{g}^\alpha$ for a choice of positive root system Δ^+. Then $\mathfrak{n} = \sum_{\alpha\in\Delta^+}\mathfrak{g}^\alpha \subset \mathfrak{g}_{i\sqrt{-\lambda}}$ so $\widetilde{Q}|_{\mathfrak{g}^\alpha} = i\sqrt{-\lambda}\operatorname{Id}|_{\mathfrak{g}^\alpha}$ for $\alpha\in\Delta^+$. Furthermore as $\widetilde{Q}\mathfrak{b} \subset \mathfrak{b}$ and $\widetilde{Q}\bar{\mathfrak{b}} \subset \bar{\mathfrak{b}}$, one has $\widetilde{Q}(\mathfrak{t}^{\mathbb{C}}) \subset \mathfrak{t}^{\mathbb{C}}$).

Combining this result with the corollary 2 of point **a**, we get the classification of all bialgebra structures on a compact simple Lie algebra $\mathfrak{g}$:

Theorem 3.2 ((Soibelman [**8**])). *Let $\mathfrak{g}$ be a compact simple Lie algebra. Any bialgebra structure $(\mathfrak{g},p)$ on $\mathfrak{g}$ is given by $p = \partial Q$ where $Q \in \Lambda^2\mathfrak{g}$ is of the form*

$$Q = R_0 + r\sum_{\alpha\in\Delta^+} i(E_\alpha - E_{-\alpha}) \wedge (E_\alpha + E_{-\alpha})$$

where $R_0 \in \Lambda^2\mathfrak{t}$ for some maximal toral subalgebra $\mathfrak{t}$ of $\mathfrak{g}$, where $r\in\mathbb{R}$ and where the E_α are defined as before.

c. Consider now the case where $\lambda < 0$ and $\mathfrak{g}$ is real semisimple. We want to find any $\widetilde{Q} \in \mathrm{End}(\mathfrak{g})$ which is a solution of (*).

We use again our study of the complex case (the complex linear extension of $\widetilde{Q}$ is again clearly a solution of (*) on $\mathfrak{g}^{\mathbb{C}}$). Then a is real so that $\mathfrak{g}_a, \mathfrak{g}_{-a}$ and $\mathfrak{g}'$ are complexifications of real subalgebras of $\mathfrak{g}$, which we denote $\mathfrak{g}_a^{\mathbb{R}}$, $\mathfrak{g}_{-a}^{\mathbb{R}}$, $\mathfrak{g}'^{\mathbb{R}}$. The Borel $\mathfrak{b}_+$ containing $\mathfrak{g}_a+\mathfrak{g}'$ is unique so $\mathfrak{b}_+ = \bar{\mathfrak{b}}_+$ (where $^-$ denotes the conjugation of $\mathfrak{g}^{\mathbb{C}}$ relative to $\mathfrak{g}$) and similarly for $\mathfrak{b}_-$. Hence $\mathfrak{h} = \mathfrak{b}_+ \cap \mathfrak{b}_-$ is the complexification of a Cartan Lie subalgebra $\mathfrak{h}^{\mathbb{R}}$ of $\mathfrak{g}$ and $\mathfrak{b}_\pm$ are the complexification of solvable subalgebras $\mathfrak{b}_\pm^{\mathbb{R}}$ of $\mathfrak{g}$.

Take a Cartan decomposition of $\mathfrak{g}$, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ so that $\mathfrak{h}^{\mathbb{R}} = \mathfrak{t} + \mathfrak{a}$, $\mathfrak{t} \subset \mathfrak{k}$, $\mathfrak{a} \subset \mathfrak{p}$. Denote by Δ^+ the set of roots of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h})$ so that the corresponding root spaces are in $\mathfrak{b}_+$. Denote by α' the restriction of the root α to $\mathfrak{h}^{\mathbb{R}}$.

Since $\overline{\mathfrak{g}^{\alpha}} = \mathfrak{g}^{\beta}$ where $\beta' = \overline{\alpha'}$, α' cannot have purely imaginary values. This shows that no root α is such that $\alpha'|_{\mathfrak{a}} = 0$. Hence the centralizer of $\mathfrak{a}$ is abelian and $\mathfrak{a}$ is maximal abelian in $\mathfrak{p}$. Then $\mathfrak{b}_+^{\mathbb{R}}$ is the minimal parabolic and it has to be solvable or equivalently $\mathfrak{m}$ (= centralizer of $\mathfrak{a}$ in $\mathfrak{k}$) is abelian.

It is now a simple task to check the list of real forms in Helgason [**6**] and see when $\mathfrak{m}$ is abelian. This is obviously the case if $\mathfrak{g}$ is split over $\mathbb{R}$ or complex.

THEOREM 3.3. *If $\lambda < 0$ then $\mathfrak{g}$ must be a sum of simple ideals which are either split, complex or one of the following cases (using the notation in Helgason):*
(i) *$SU(p,p), SU(p,p+1)$;*
(ii) *$SO(p,p+2)$;*
(iii) *EII.*

For each case we have a solution given by

$$\widetilde{Q}(x) = \begin{cases} ax & x \in \mathfrak{n}^{\mathbb{R}}; \\ 0 & x \in \mathfrak{h}^{\mathbb{R}}; \\ -ax & x \in \mathfrak{n}^{\mathbb{R}}. \end{cases}$$

The only thing remaining is to check that $\widetilde{Q}$ satisfies (*) but this is an easy calculation.

Observe that any solution of this problem is now given — as in the complex case — in terms of subset Γ_+, Γ_- and a map τ which have to be compatible with the conjugation of $\mathfrak{g}^{\mathbb{C}}$ relative to $\mathfrak{g}$.

4. Appendix

We prove here the result (used in §3) that any subalgebra of a semisimple Lie algebra which contains the nilradical of a Borel is normalized by the Borel and hence is essentially a parabolic subalgebra. First we establish some notation.

Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra, Δ the set of roots, Δ_+ a positive root system, Φ the set of simple roots which we enumerate as $\{\alpha_1, \ldots, \alpha_\ell\}$. Put

$$\mathfrak{n} = \sum_{\alpha\in\Delta_+} \mathfrak{g}_\alpha, \qquad \mathfrak{n}^- = \sum_{\alpha\in\Delta_+} \mathfrak{g}_{-\alpha}, \qquad \mathfrak{b} = \mathfrak{h} + \mathfrak{n}.$$

Any positive root α can be written as a positive integer combination $\sum_{c=1}^{\ell} n_i\alpha_i$ of simple roots and $\sum_{c=1}^{\ell} n_i$ is called the **height** $n(\alpha)$ of α. If $\alpha+\beta$ is a root then $n(\alpha+\beta)=n(\alpha)+n(\beta)$.

PROPOSITION. *If $\mathfrak{c}\subset\mathfrak{g}$ is a subalgebra with $\mathfrak{n}\subset\mathfrak{c}$ then $[\mathfrak{h},\mathfrak{c}]\subset\mathfrak{c}$ and $\mathfrak{h}+\mathfrak{c}$ is a parabolic subalgebra of $\mathfrak{g}$.*

Proof If $[\mathfrak{h},\mathfrak{c}]\subset\mathfrak{c}$, $\mathfrak{h}+\mathfrak{c}$ is a subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$ so is parabolic.

To establish $[\mathfrak{h},\mathfrak{c}]\subset\mathfrak{c}$ it is enough to show that $\mathfrak{c}$ is the direct sum of $\mathfrak{h}\cap\mathfrak{c}$ and a sum of root spaces. Since $\mathfrak{g}^\alpha\subset\mathfrak{c}$ for all $\alpha\in\Delta_+$ we only need to show that for any element ξ in $\mathfrak{c}\cap(\mathfrak{h}+\mathfrak{n}_-)$ if we write it as $\xi_0+\sum_{\alpha\in\Delta_+}\xi_{-\alpha}$ then $\xi_0\in\mathfrak{c}$ and $\xi_{-\alpha}\in\mathfrak{c}$ for all $\alpha\in\Delta_+$.

Let us say that an element ξ of $\mathfrak{h}+\mathfrak{n}_-$ has height k if $\xi_0+\sum_{\alpha\in\Delta_+}\xi_{-\alpha}$ and $n(\alpha)\le k$ for all α with $\xi_{-\alpha}\ne 0$ and equality for at least one α. If we show that, for an element ξ with height $k\ge 1$ and any α of height k with $\xi_{-\alpha}\ne 0$, we have $\mathfrak{g}^{-\alpha}\subset\mathfrak{c}$ then a decreasing induction on k gives the result. Suppose we have such a ξ and α. Then α can be expressed as a sum (with repetitions) of simple roots so that each partial sum is also a root (see, e.g. Helgason [**6**] p.460). Thus if we pick any simple root α_{i_0} occurring in this expression, there are simple roots $\alpha_{i_1},\dots,\alpha_{i_r}$ and a root β with $\beta,\beta+\alpha_{i_0},\beta+\alpha_{i_0}+\alpha_{i_1},\dots,\beta+\alpha_i+\alpha_{i_0}+\cdots+\alpha_{i_r}=\alpha$ all roots. Then $[[\dots[\mathfrak{g}^{-\alpha},\mathfrak{g}^{\alpha_{i_r}}]\dots,\mathfrak{g}^{\alpha_{i_1}}],\mathfrak{g}^{\beta}]=\mathfrak{g}^{-\alpha_{i_0}}$ so that $[[\dots[\xi,E_{\alpha_{i_r}}]\dots,E_{\alpha_{i_1}}],E_\beta]$ is an element η of height 1 with a non-zero component in $\mathfrak{g}^{-\alpha_{i_0}}$ (after removing terms in $\mathfrak{n}$). So $\eta=\eta_0+\displaystyle\sum_{i=1}^{\ell}\eta_{-\alpha_i}\in\mathfrak{c}$ where $\eta_{-\alpha_{i_0}}\ne 0$. Bracketing with $E_{\alpha_{i_0}}$ (since the difference of two simple roots is never a root), we conclude $[\mathfrak{g}^{-\alpha_{i_0}},\mathfrak{g}^{\alpha_{i_0}}]=\mathbb{C}H_{\alpha_{i_0}}\subset\mathfrak{c}$.

Thus for each simple root α_i with $\eta_{-\alpha_i}\ne 0$ we deduce $H_{\alpha_{i_0}}\in\mathfrak{c}$. Bracketing η with any such H_{α_j} we have

$$-\sum_{i=1}^{\ell}\alpha_i(H_{\alpha_j})\eta_{-\alpha_i}\in\mathfrak{c}.$$

The set of simple roots where $\eta_{-\alpha_i}\ne 0$ form the simple roots of a semisimple subalgebra of $\mathfrak{g}$ with the span of the corresponding H_{α_i} as Cartan subalgebra. In this Cartan we can choose a dual basis to the set of α_i with $\eta_{-\alpha_i}\ne 0$ and so conclude that each $\mathfrak{g}^{-\alpha_i}\subset\mathfrak{c}$ if $\eta_{-\alpha_i}\ne 0$. Thus we now know that if α has the same height as ξ and $\xi_{-\alpha}\ne 0$ then for every simple root α_i occurring in α we have $\mathfrak{g}^{-\alpha_i}\subset\mathfrak{c}$. Since $\mathfrak{c}$ is an algebra, it follows $\mathfrak{g}^{-\alpha}\subset\mathfrak{c}$. This completes the proof.

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