

Lecture 8 Pf of Sz.R.L

- i) RL : Ramsey-Turán upper bd \rightsquigarrow weighted Turán prob.
 - ii) bd deg H has linear Ramsey #.
 - iii) Spectral pf of RL
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$RT(n, K_4, o(n)) = \max$ # edges in $\text{an } n\text{-vc K}_4\text{-free graph } G$
with $\Delta(G) = o(n)$

$$\text{Thm (Sz 76)} \quad RT(n; K_4, o(n)) \leq \frac{n^2}{8} + o(n^2)$$

PF : Let G be as above $0 < \varepsilon < \delta < 1$

$$G \xrightarrow{\text{RL}} R \text{ reduced graph } R(\varepsilon, \delta)$$

Claim: Suffices to show $\begin{cases} 1) R \text{ is } \Delta\text{-free} \\ 2) R \text{ has no chubby edge} \\ ij \in E(R), d(V_i, V_j) \geq \frac{1}{2} + o(1) \end{cases}$

If true, 2) $\Rightarrow \forall ij \in E(R) \Rightarrow d(V_i, V_j) \leq \frac{1}{2} + o(1)$

$$\begin{matrix} u_r \\ v_r \\ \vdots \\ 0 \\ \vdots \\ \leq \frac{1}{2} + o(1) \end{matrix}$$

$$1) \Rightarrow \text{Turán Thm } e(R) \leq \frac{r^2}{4}$$

$$e(G) \leq e(R) \cdot \frac{n^2}{r^2} \left(\frac{1}{2} + o(1) \right) + \underbrace{(e(G) - e(R))}_{o(n^2)}$$

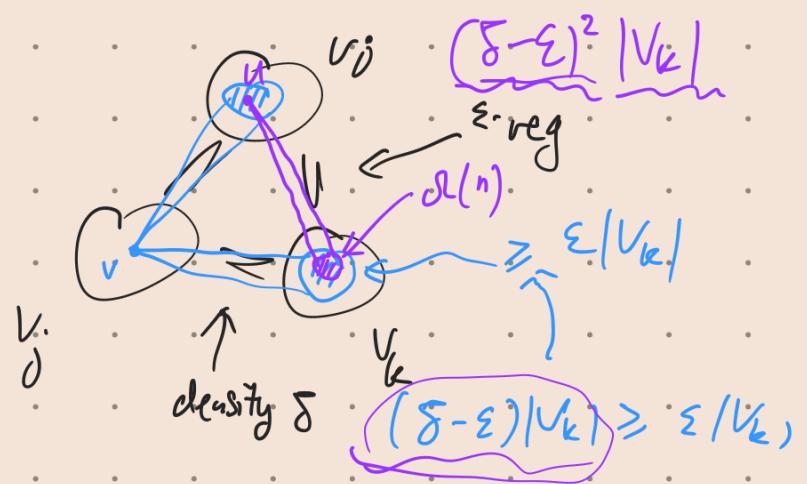
$$\begin{aligned} &= \frac{r^2}{4} \cdot \frac{n^2}{r^2} \left(\frac{1}{2} + o(1) \right) + o(n^2) \\ &= \frac{n^2}{8} + o(n^2) \end{aligned} \quad \left. \begin{array}{l} \text{edges cleaned} \\ \text{is sparse pairs} \\ \text{is long pairs} \\ V_i, i \in [r] \end{array} \right\}$$

Suppose $\Delta \in R$ $\underline{i j k}$

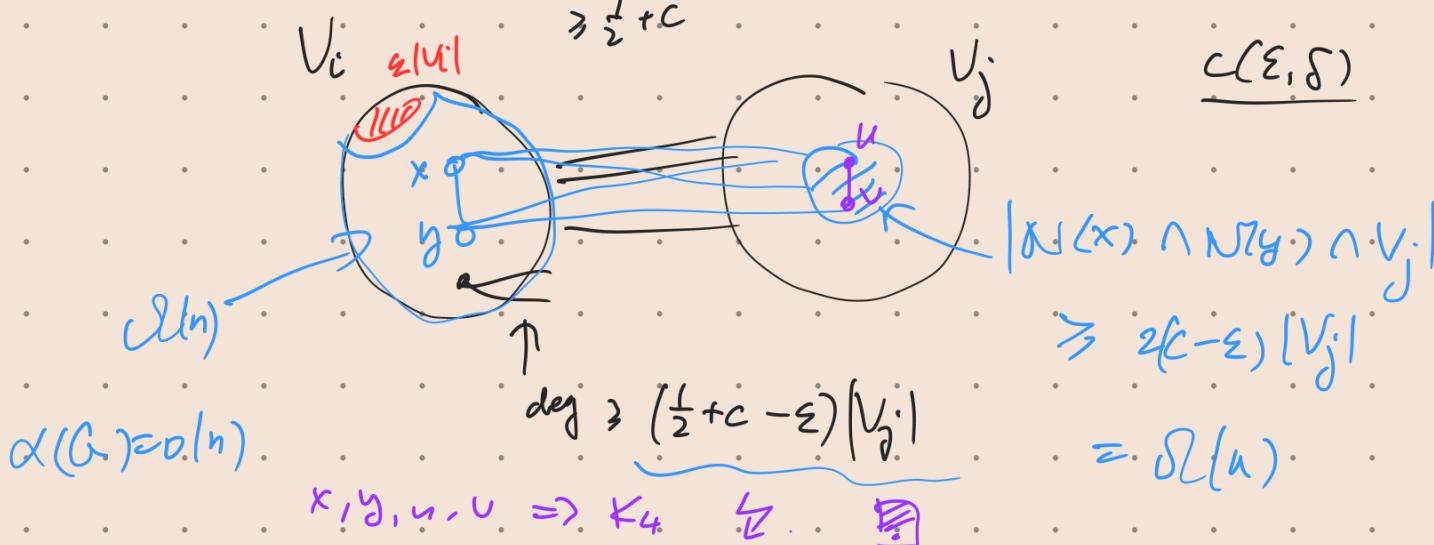
$\Rightarrow \alpha(G)$

$$\alpha(G) = o(n)$$

\Rightarrow purple point has
an edge + u, v
 $\Rightarrow K_4$



- Suppose $\exists i, j \in E(F)$ s.t. $d(V_i, V_j) \geq \frac{1}{2} + c$



Chvátal - Rödl - Szemerédi - Trotter theorem

Def. G graph. $r(G, G) = \min N$ s.t.

$\nexists 2\text{-edge-colouring of } K_N \Rightarrow \exists$ monochromatic G

$G = K_n$, $2^{\frac{n}{2}} \leq r(K_n, K_n) < 4^n$
exponential in $|G|$

Thm (CRSzT) $\forall d \in \mathbb{N}, \exists G$ w/ $\Delta(G) \leq d$
 $\Rightarrow r(G, G) = O_d(n)$ ($r(G, G) \leq C(d) \cdot n$)

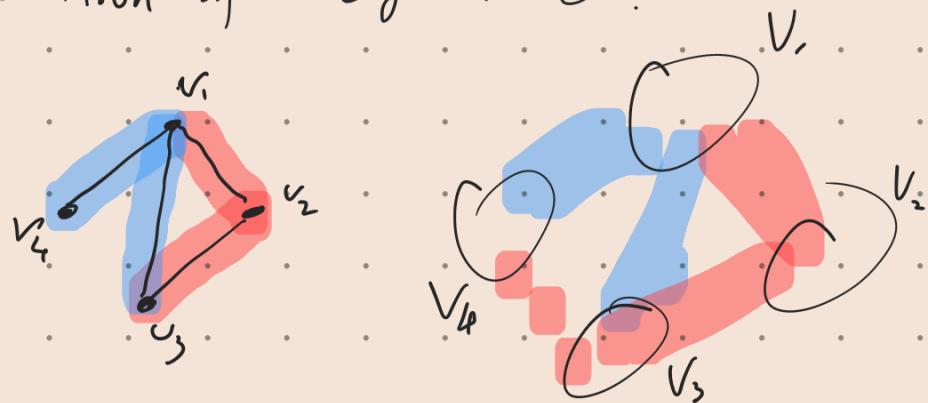
Recall Thm Brook's Thm: If H w/ $\Delta(H) \leq d$,
 $\chi(H) \leq d+1$.

($k=2$)

- For a k -edge-coloured graph G ; a partition $V(G) = V_1 \cup \dots \cup V_r$ is ε -regular if
 - $|V_i| - |V_j| | \leq 1$
 - for all but $\leq \varepsilon \binom{r}{2}$ pairs $i, j \in \binom{[r]}{2}$, (V_i, V_j) is ε -regular in **every** colour.

Lem (Multi-ed R.L.) $\forall \varepsilon > 0$, $\forall k \in \mathbb{N}$, $\forall m$, $\exists M = M(\varepsilon, k, m)$ s.t. \forall k -edge-col. G with $n \geq m$ vxs admits an ε -reg partition $V_1 \cup \dots \cup V_r$, $m \leq r \leq M$.

Given a reg partition of edge-col. G



Pf (CRS₂T) $r(G, G) = O(n)$ $\begin{cases} \Delta(G) = d \\ |G| = n \end{cases}$ $\begin{matrix} \text{upper bd} \\ \# \text{parts} \end{matrix}$

Take $C = \frac{2M}{(\frac{1}{2} - \varepsilon)^d}$, where $M = M(\varepsilon, m, 2)$ ↗ M.R.L.

$$r(G, G) \leq C \cdot n$$

$$m \geq 5r(K_{d+1}, K_{d+1})$$

Fix an arbitrary 2-edge-col. of K_{cn}

NT embed monoX copy of G in K_{Cn} .

K_{Cn} 



Note that R is almost complete

$$e(R) \geq (1-\varepsilon) \binom{r}{2} = \left(1 - \frac{1}{m}\right) \binom{r}{2}$$

by choosing $m = \frac{1}{\varepsilon}$, $\varepsilon = \frac{1}{m}$

$$\frac{m}{3} > r(K_{dR}, K_{dR})$$

by Turán's thm $\Rightarrow R \supseteq K_{m/3}$

$\Rightarrow \exists$ monoX. K_{dR} in R

$$\Delta(G) \leq d$$

\exists homo.

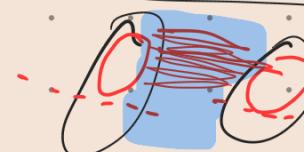
$$G \rightarrow K_{dR}$$

$\begin{matrix} \nearrow \\ K_{Cn} \end{matrix} \quad \begin{matrix} \searrow \\ \text{embedding} \\ R \end{matrix}$



Lem G n -vx, $\varepsilon > 0$. Then there exists

a partition $V = V_1 \cup \dots \cup V_M$, $M \leq M(\varepsilon)$



s.t. apart from an exceptional set

$$\Delta \subseteq \binom{[M]}{2} \text{ with}$$

$$\sum_{ij \in \Delta} |V_i||V_j| = O(\varepsilon |V|^2)$$



we have $\forall i, j \notin \Delta$, $A \subseteq V_i$, $B \subseteq V_j$ that

$$|e(A, B) - d_{i,j} |A||B| | = O(\varepsilon |V_i||V_j|)$$

$$e(A, B) = \mathbb{1}_A^* T \mathbb{1}_B = \mathbb{1}_A^* T_1 \mathbb{1}_B + \mathbb{1}_A^* T_2 \mathbb{1}_B + \mathbb{1}_A^* T_3 \mathbb{1}_B$$

Idea $T = \text{adj matrix of } G$ sym. real
 $= \sum_{i=1}^n \lambda_i u_i u_i^*$ (Eigenvalue decoupl.)

Partition T into main term (λ_i large)
error term (λ_i small)

Will partition $U(G)$ s.t. every 'heavy' eigenvector
does NOT fluctuate much in each part. \square

Pf: $T = \sum_{i=1}^n \lambda_i u_i u_i^*$ $|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_n|$

Split T

$$T = T_1 + T_2 + T_3$$

$T_{i,j}^k = \# \text{ length } k\text{-walk from } u_i \text{ to } u_j \text{ in } G$

$$\sum_{i=1}^n |\lambda_i|^2 = \text{tr}(T^2) = \sum_i d_i = 2e(G) \leq n^2$$

(*) $i |\lambda_i|^2 \leq \sum |\lambda_i|^2 \leq n^2 \Rightarrow |\lambda_i| \leq \frac{n}{\sqrt{\epsilon}}$

Take $F = F(\epsilon) : \mathbb{N} \rightarrow \mathbb{N}$ w/ $F(i) \geq i$

By averaging, $\exists J \leq F^{\frac{1}{\epsilon^3}}(1)$ s.t. $F^2(1) = F_0(F(1))$

$$\sum_{i \in [J, F(J)]} |\lambda_i|^2 \leq \epsilon^3 n^2$$

$$\underbrace{F \circ F \circ \dots \circ F}_{\geq \epsilon^3}(1)$$

$$T_1 = \sum_{i \leq J} \lambda_i u_i u_i^*$$

$$T_2 = \sum_{J < i \leq F(J)} \lambda_i u_i u_i^*$$

$$T_3 = \sum_{i > F(J)} \lambda_i u_i u_i^*$$



(i) Partition for the structured part T_1 .

(ii) bad error terms $T_2 \& T_3$

For (i) Want to construct a partition of $V(G)$ s.t.

$T_1 = \sum_{i \leq J} \lambda_i u_i^* u_i$ is approximately constant on most parts.

For each $i \leq J$, partition $V(G)$ into $O_{J,\varepsilon}(1)$ parts

in which u_i only fluctuates by $O\left(\frac{\varepsilon^{3/2}}{J} n^{-1/2}\right)$

apart from an exceptional set $\leq \frac{\varepsilon n}{J}$ where $|u_i|$ too large



Take $u = u_i$, $u(j) \in j^{\text{th}}$ coord. of u

$$\|u\|_2^2 = \sum_{j \in [n]} u_j^2 = 1$$

large coord. $\leq \frac{\varepsilon n}{J}$

$$\frac{\varepsilon n}{J} \cdot \left(\sqrt{\frac{J}{\varepsilon}} n^{-1/2} \right)^2 > 1$$

For the rest of the coordinates part. it into

$$\frac{\sqrt{J/\varepsilon} n^{-1/2}}{\frac{\varepsilon^{3/2}}{J} n^{-1/2}} = \# \text{ parts} = O\left(\frac{1}{J\varepsilon}\right)$$

↑
fluctuation

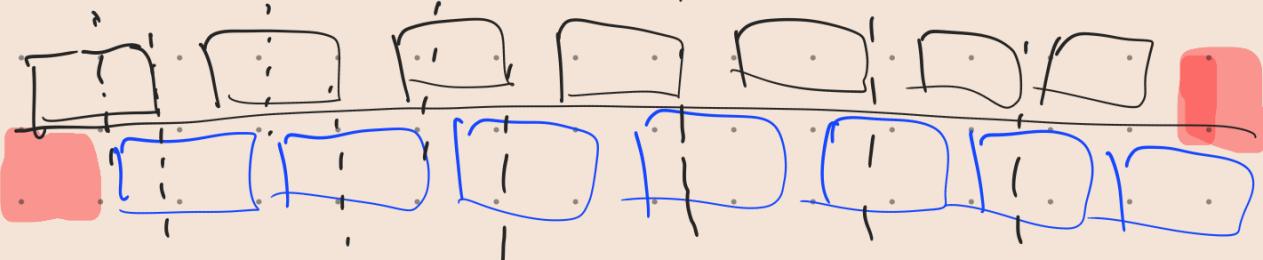


Combine all J partition together

$$\Rightarrow V(G) = V_1 \cup \dots \cup V_M \cup V_M$$



$$u_i = u$$



$0 \circ 0 \circ \dots \circ \underset{v_i}{\circ} \dots 0 \circ 0 \circ 0 \circ \circ \circ 0 \circ \underset{v_j}{\circ} \dots 0 \circ \dots$ v_M

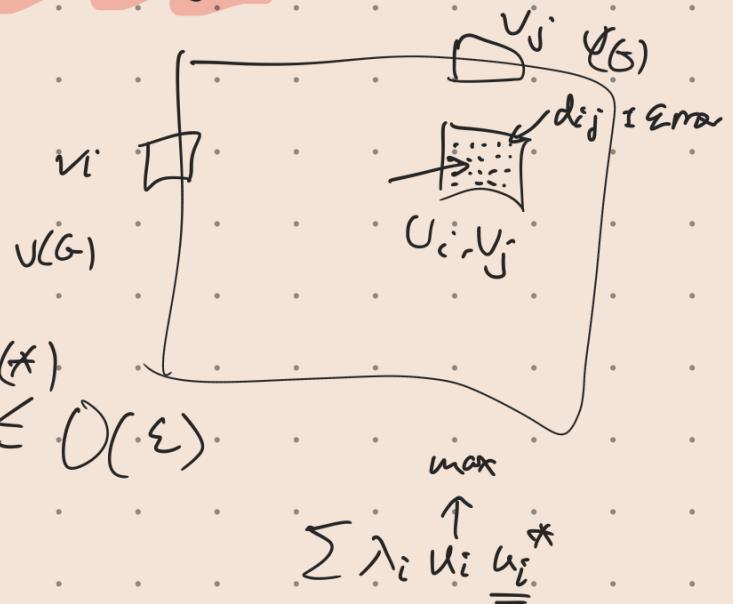
$$1_A^* T_1 1_B = d_{ij} |A| |B| + O(\epsilon |V_i| |V_j|) \quad T_1 = \sum_{i \in J} \lambda_i u_i u_i^*$$

$$|u_i| \leq \frac{n}{\sqrt{\epsilon}} \quad (\star)$$

$$V_i, V_j - \text{entry of } T_1 = \sum \lambda_i u_i u_i^*$$

fluctuates at most

$$\sum_{i \in J} \lambda_i \underbrace{\sqrt{\frac{J}{\epsilon}} n^{-\frac{1}{2}}}_{\text{max value}} \cdot \underbrace{O\left(\frac{\epsilon^{3/2}}{J} n^{-1/2}\right)}_{\text{fluctuation}} \stackrel{(\star)}{\leq} O(\epsilon)$$



Left to show $\forall k \in \{2, 3\}$

$$1_A^* T_k 1_B = O(\epsilon |V_i| |V_j|)$$

$$e(A, B) = \sum_{k \in \{2, 3\}} 1_A^* T_k 1_B = d_{ij} |A| |B| + O(\epsilon |V_i| |V_j|)$$

$$1_A^* T_2 1_B = O(\epsilon |V_i| |V_j|)$$

$$\text{b/c} \quad \sum_{i \in (J, F(J))} |\lambda_{ii}|^2 \leq \epsilon^3 n^2$$

$$1_A^* T_3 1_B \leq O\left(\frac{n^2}{\sqrt{P(J)}}\right) \quad F: N \rightarrow N$$

$$= O(\epsilon |V_G| |U_{ij}|)$$