

Lecture 14 Complex Bollobás-Endrődi graphs.

Recall

$$2 \leq p < q$$

$$|G|=n;$$

- $\text{RT}_p(n, K_q, \varepsilon_n) = \max \{e(G) : \begin{array}{l} G \text{-free;} \\ \alpha_p(G) \leq \varepsilon_n \end{array}\}$

K_q -free;

$$\alpha_p(G) \leq \varepsilon_n$$

where $\alpha_p(G) = \max |U| : U \subseteq V(G) \text{ s.t. } G[U] \text{ is } K_p\text{-free}$

$$P_p(K_q) = \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{\text{RT}_p(n, K_q, \varepsilon_n)}{\binom{n}{2}}$$

Ramsey-Turán density

Szemerédi

- $P_2(K_4) \leq \frac{1}{4}$ ($\forall n\text{-ux graph } G : K_4\text{-free ; }$
 $\alpha(G) = o(n) \Rightarrow e(G) \in \left(\frac{1}{4} + o(1)\right) \binom{n}{2}$)

- Bollobás-Endrődi $P_2(K_4) \geq \frac{1}{4}$, i.e.

$\exists n\text{-ux Graph } G \text{ s.t. } - e(G) \geq \left(\frac{1}{8} - o(1)\right) n^2$

- K_4 -free

- $\alpha(G) = o(n)$.

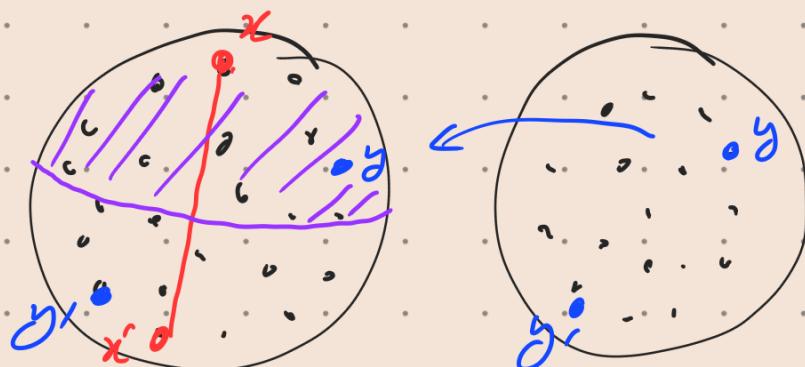
Construction

Inner: $x \sim x'$
 \Leftrightarrow almost antipodal

Cross: $x \sim y$

\Leftrightarrow if one is in the hemisphere centered at the other

(in the pic: $x \sim y, x \not\sim y'$)



$X = \text{high dim. Sphere}$

\Rightarrow every $x \sim$ half of us from the other side

$$\Rightarrow e(G) \geq n^2/8 + o(n^2)$$

- $\alpha(G) = o(n)$
- $\forall X \subseteq V(G), |X| = \Omega(n)$
- $\Rightarrow G[X]$ has an edge

Isoperimetric inequality

Given measure, spherical cap
minimises diameter

$$\text{diam}(X) \geq \text{diam}(C) ; \lambda(X) = \lambda(C) = \Omega(n)$$

upp bd on measure of sph. cap

$\Rightarrow \exists$ two pts almost antipodal in $X \Rightarrow$ form an edge.

- BE graph is K_4 -free

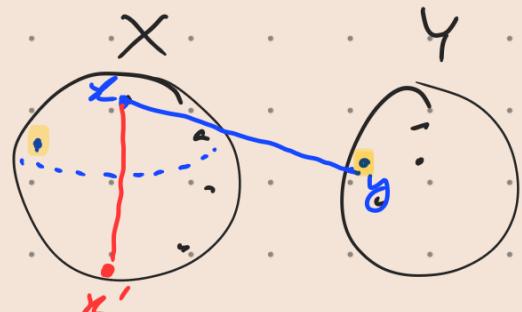
Details: • k dim. of high dim. sphere

$$M = \frac{\epsilon}{\sqrt{k}}$$

Lem: $\exists C > 0$ s.t. TPH. $\forall n \geq \left(\frac{C}{\delta}\right)^k$, can cut

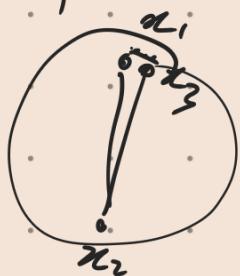
S^{k-1} into n pieces of the same measure, each w./ diameter $\leq \delta$.

- $x \sim y \Leftrightarrow |x-y| < \sqrt{2} - \mu$
- $x \sim x' \Leftrightarrow |x-x'| > 2 - \mu$



Claim X, Y are Δ -free (high odd girth)

Pf: Supp. $x_1, x_2, x_3 = \Delta$



$$3 \cdot (2\pi\mu)^2 \leq \sum_{i,j \in \binom{[n]}{2}} |x_i - x_j|^2 = 2 \sum_{i \in [n]} |x_i|^2 - 2 \sum_{i,j \in \binom{[n]}{2}} \langle x_i, x_j \rangle$$

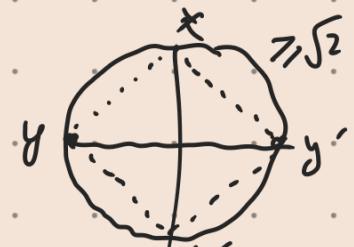
$$= 3 \sum_{i \in \{1\}} |x_i|^2 - |x_1 + x_2 + x_3|^2$$

$$\leq 3 \sum_{i \in \{3\}} |x_i|^2 = 3 \cdot 3 = 9$$

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Class K₄-free



pf: Supp. $x, x' \in X, y, y' \in Y = k_4$.

$$\begin{aligned}
 4(2 - 2\sqrt{2}\mu + O(\mu^2)) &= 4(2 - \mu)^2 > |x - y|^2 + |x - y'|^2 + |x' - y|^2 + |x' - y'|^2 \\
 8 - 8\sqrt{2}\mu + O(\mu^2) &= |x + x' - y - y'|^2 + |x - x'|^2 + |y - y'|^2 \\
 &> 2 \cdot (2 - \mu)^2 = 2(4 - 2\mu + \mu^2) = 8 - 4\mu + O(\mu^2)
 \end{aligned}$$

Two main problems (SOS conj)

$P=2$	$q=4$	$q=5$
		
$q=6$		$q=7$
		

$$\cdot P_3(K_5) = ?$$

$$\frac{1}{8} \leq P_3(K_5) \leq \frac{1}{6} \binom{n}{2} = \frac{n^2}{12} + o(n^2)$$

Thm (upp bd) If $n \rightarrow \infty$ graph w./ $\alpha_3(G) = o(n)$

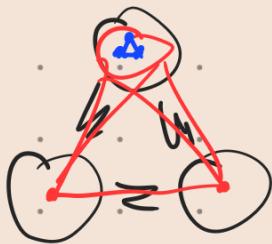
$$\& \text{ } K_5\text{-free} \Rightarrow e(G) \leq \frac{n^2}{12} + o(n^2)$$

PF: $G \xrightarrow{\text{reg lem}} R$ (reduced graph)

NTS { R is Δ -free

$$\frac{n^2}{\ell^2} \cdot \frac{r^2}{4} \cdot \frac{1}{3} = \frac{n^2}{\ell^2}$$

edge weight in R $\leq \frac{1}{3}$



Complex Bollobás - Erdős graphs (CBE)

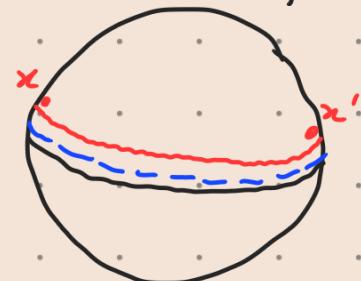
(based on L.-Reiher - Sharifzadeh - Staden)

Fix h dimension ($h \rightarrow \infty$)

$$S^{h-1}(\mathbb{C}) = \left\{ (z_1, \dots, z_h) \in \mathbb{C}^h : \sum |z_i|^2 = 1 \right\}$$

$$\langle x, y \rangle = \sum x_i \cdot \overline{y_i}$$

$$|x-y|^2 = |x|^2 + |y|^2 - 2 \operatorname{Re}(x, y)$$



isometry $(x_1 + y_1 i, x_2 + y_2 i, \dots, x_h + y_h i)$

$$\mapsto (x_1, y_1, x_2, y_2, \dots, x_h, y_h)$$

distance-preserving map.

$$\begin{array}{c} O \cong O \\ \uparrow \\ \text{density } 0 < p/q < 1 \end{array}$$

Construction (CBE)

$$1 \leq p < q$$

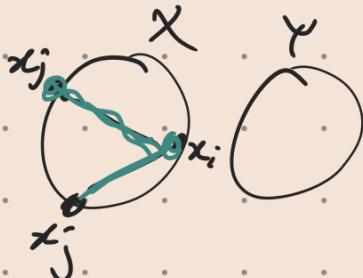
$$\text{Set } \omega = \cos \frac{2\pi}{q} + i \sin \frac{2\pi}{q} = e^{\frac{2\pi i}{q}}$$

$$\varepsilon \ll K^{-1} \ll q^{-1}, \quad h \rightarrow \infty$$

$$\mu = \frac{\varepsilon}{\sqrt{2h}}, \quad n \geq \left(\frac{\varepsilon}{\mu}\right)^{2h}$$

Partition $S^{h-1}(\mathbb{C})$ into $\frac{n}{2}$ equal pieces of same measure & diam $< \frac{1}{4}\mu$.

Say $x_1, \dots, x_{n_2}, y_1, \dots, y_{n_2}$.



$$G = (V, E) \quad V = X \cup Y$$

$x_i x_j \in E \Leftrightarrow \exists k \in [q-1]$ s.t.

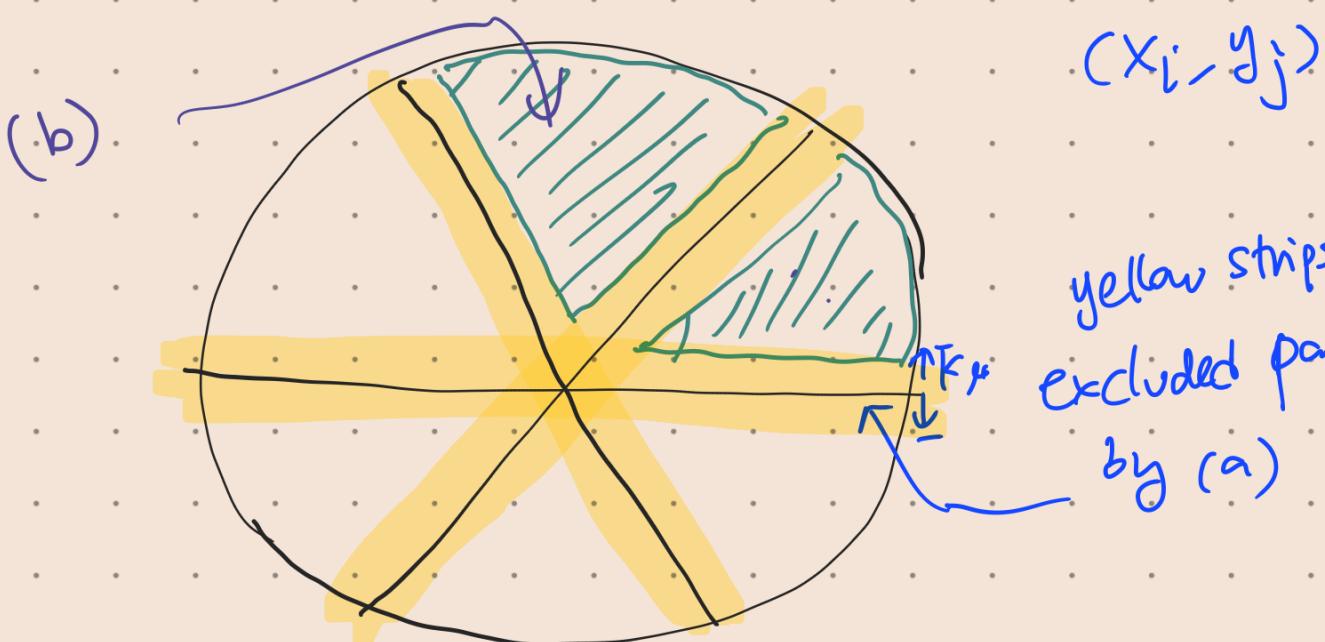
$$|x_i - \omega^k x_j| < \sqrt{\mu} \quad (\text{same for } Y)$$

$x_i y_j \in E \Leftrightarrow \begin{cases} \text{a)} \forall k \in [q] \Rightarrow |\operatorname{Im}(\omega^k(x_i - y_j))| \geq \frac{1}{2}\mu \\ \text{b)} \exists \alpha \in [0, \frac{2\pi p}{q}] \text{ s.t.} \end{cases}$

$$e^{-i\alpha}(x_i - y_j) \in [0, 1]$$

Picture

when $p=1, q=3$



(b) \Rightarrow every $x_k \sim \frac{p}{q}$ -fraction of $x_k s$ from the other side

$\Rightarrow \frac{P}{g}$ cross density.

$$P=1 \quad g=3 \Rightarrow \frac{1}{3} \cdot \frac{n}{2} \frac{n-1}{2} = \frac{n^2}{12} \text{ edges}$$

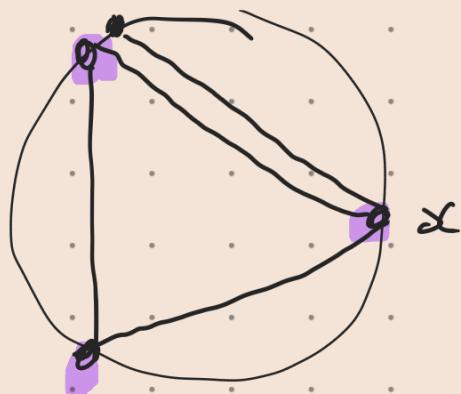
$$\Rightarrow P_3(K_5) = \frac{1}{6}$$

$$P=1 \quad g=3$$

Claim: X, Y is K_4 -free.

Claim $\chi_3(G[X])$

$$\chi_3(G[Y]) \approx o(n)$$



Follows from Isoperimetric ineq &
Concentration of measure.

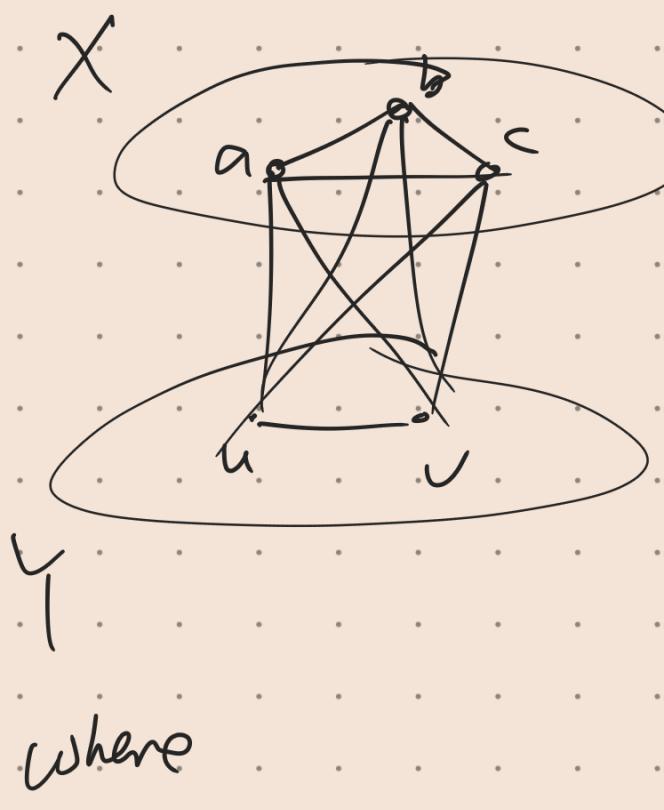
Claim: K_5 -free

Inner edges

$$\Rightarrow |a - b \cdot \omega| < \sqrt{\mu}$$

$$|a - c \cdot \omega^2| < \sqrt{\mu}$$

$$|u - v \cdot \omega| < \sqrt{\mu}$$



$$\omega = e^{2\pi i/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad (\bar{c}\omega + \omega^2 = 0)$$

$$\Rightarrow a + \underline{a \cdot \omega + a \cdot \omega^2} = 0$$

$$\begin{aligned} |\underline{a+b+c}| &= |(b-a\omega^2) + (c-a\omega)| \\ &\leq |b-a\omega^2| + |c-a\omega| \\ &= |\cancel{\omega^2(b\omega-a)}| + |\cancel{\omega(c\omega^2-a)}| \end{aligned}$$

$$\leq 2\sqrt{\mu}$$

$$\left| \frac{a+b+c}{3} \cdot (u-v\omega) \right| \stackrel{C-S}{\leq} \left| \frac{a+b+c}{3} \right| \cdot |u-v\omega|$$

$$\leq \frac{2}{3}\sqrt{\mu} \cdot \sqrt{\mu} = \frac{2}{3}\mu$$

Six cross edges.

$$\Rightarrow (a,u), (a,v), (b,u), (b,v)$$

$$(c,u), (c,v)$$

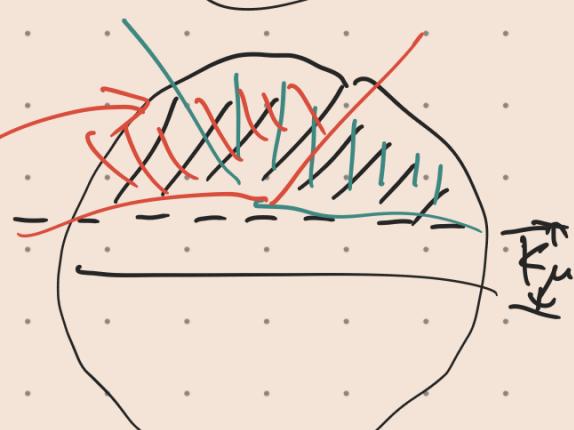


C



$$\left(\frac{a+b+c}{3}, -v\omega \right)$$

$$\Rightarrow \left(\frac{a+b+c}{3}, \underline{u-v\omega} \right) \in$$



$$K\mu \leq \left| \left(\frac{a+b+c}{3}, u-vw \right) \right| \stackrel{(*)}{\leq} \frac{2}{3}\mu$$

