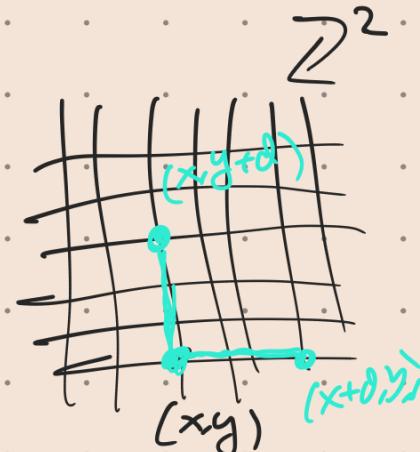


# Lecture 13 Geometric constructions in Extremal Combinatorics

## Corner-free sets

Def: Corner in  $\mathbb{Z}^2$ :  $(x, y)$     $d \neq 0$   
 $(x+d, y)$   
 $(x, y+d)$



Q: How large a corner-free set  $A \subseteq [N]^2$  can be?

Upper bound: max size corner-free in  $[N]^2 = f(N)$   
 $f(N) = o(N^2)$  (reg. lemma)

Lower bound

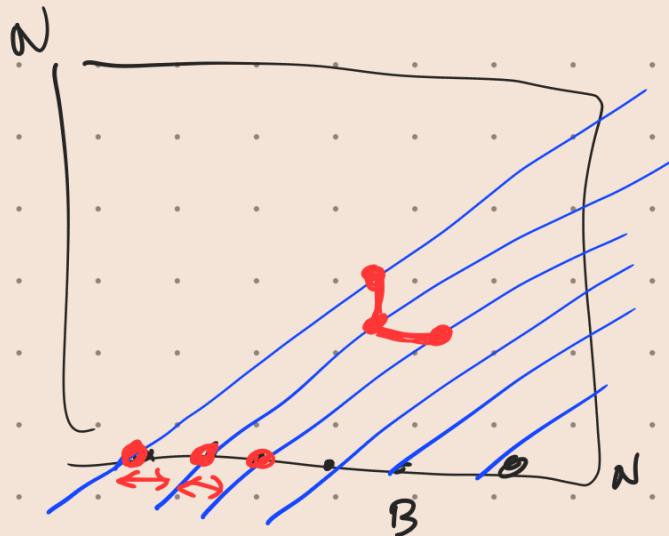
- Behrend 3-AP-free const  $C = 2\sqrt{2} \approx 2.828$
- Linial - Shraibman (21+)
- Green (21+)  $C = 2\sqrt{2 \log_2 \frac{4}{3}} \approx 1.822$

Rmk: Behrend:  $B \subseteq [N]$  3-AP-free  $|B| = \frac{N}{2^{(2.55+\epsilon)\sqrt{\log N}}}$   
 $A \subseteq [N]^2$

Take  $(x, y) \in A$

if  $x-y \in B$

$$|A| \leq |B| \cdot N$$



• Green's constr. for corner-free set base of expansion

$$N = q^d \quad x \in [N-1] \longrightarrow x = \sum_{i=0}^{d-1} x_i \cdot q^i$$

$$\pi: x \mapsto \pi(x) = (x_0, x_1, \dots, x_{d-1}) \in \mathbb{Z}^d$$

Idea

Geometric idea: Any line  $\cap$  any sphere at  $\leq$  two pts

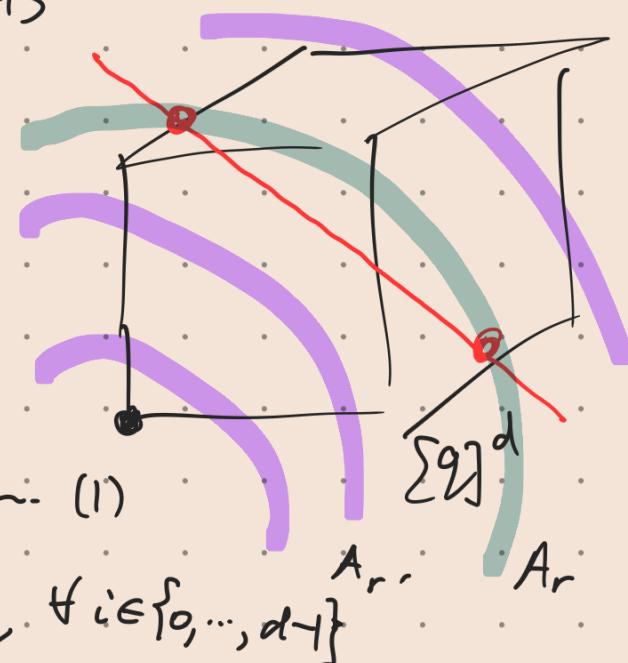
3 AP / corner  $\iff$  line  
 $[N]$        $[N]^2$        $[q]^d$

Consider set  $A_r \subseteq [N-1]^2$

w/  $(x, y)$  s.t.

$$\{\| \pi(x) - \pi(y) \|_2^2 = r \sim (1)$$

$$\frac{q}{2} \leq x_i + y_i < \frac{3q}{2}, \quad \forall i \in \{0, \dots, d-1\}$$



Claim:  $A_r$  is corner-free.

PF: Supp.  $(x, y), (x+d, y), (x, y+d) \in A_r$

$$(1) \Rightarrow \| \pi(x) - \pi(y) \|_2^2 = \| \pi(x+d) - \pi(y) \|_2^2$$

$$= \| \pi(x) - \pi(y+d) \|_2^2 = r$$



Claim: (2):  $\pi(x+d) + \pi(y) = \pi(x) + \pi(y+d)$

Given (2), set  $a = \pi(x) - \pi(y)$

$$b = \pi(x+d) - \pi(x) \stackrel{(2)}{=} \pi(y+d) - \pi(y)$$

$$(1) \Rightarrow \|a\|_2^2 = \|a+b\|_2^2 = \|a-b\|_2^2 = r$$

$\Rightarrow$  3 pts  $a, a+b, a-b$  on a line  
all lie in the sphere of radius  $\sqrt{r}$

Pf of (2): Induct on  $i = 0, 1, \dots, d^r$  to show

$$(x+d)_i + y_i = x_i + (y+d)_i \quad i=0, 1, \dots$$

Sup. true for  $\forall i < j$ , fix  $j \geq 0$ .

- Write  $x_{\geq j} = \sum_{i \geq j} x_i q^{i-j}$  (x<sub>0</sub>, x<sub>1</sub>, ..., x<sub>j</sub>, x<sub>j+1</sub>, ...)

Ind. hyp: &  $(x+d) + y = x + (y+d)$

$$\Rightarrow (x+d)_{\geq j} + y_{\geq j} = x_{\geq j} + (y+d)_{\geq j}$$

$$\Rightarrow \underline{(x+d)_j + y_j = x_j + (y+d)_j \pmod{q}}$$

But both sides  $\in [\frac{q}{2}, \frac{3q}{2})$  by def. of Ar

$\Rightarrow (x+d)_j + y_j = x_j + (y+d)_j$  □

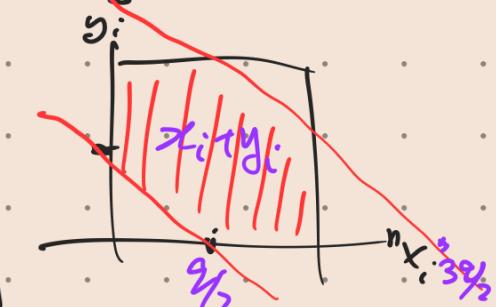
Size of Ar;  $N^2$  pts. total

$$\# \leq d \cdot q^2 \quad (x, y) \in A,$$

For each  $i \in [d]$ ,  $\frac{q}{2} \leq x_i + y_i < \frac{3q}{2}$

$$\text{Size } \approx \left( \frac{3}{4}q^2 + O(q) \right)^d$$

$$\Rightarrow |A_r| \geq \frac{1}{d q^2} \left( \frac{3}{4}q^2 + O(q) \right)^d$$



Rmk Behrend 3AP-free  $N = q^d$ ,  $[q^d - 1] = [q]^d$

$$x \rightarrow x_0, \dots, x_{d-1}$$

$A_r = x \in [q^{d-1}]$  s.t.  $\begin{cases} \| \pi(x) \|_2^2 = r \\ 0 \leq x_i < \frac{q}{2} \end{cases}$  if

Erdős & Rothschild question on  
edges in many triangles

Def: book is



book size = # triangles sharing the spine

Mantel  $\Rightarrow$   $< \frac{n^2}{4}$  edges  $\Rightarrow$  no  $\Delta$  guaranteed  
edge-density  $< \frac{1}{2}$  ( $\text{no book}$ )

Edwards / Khadziiwanou - Nikiforov



$\forall n \in \mathbb{N}$   $G$   
 $\geq \frac{n^2}{4} + t$  edges  $\Rightarrow \exists \frac{n^2}{6}$ -size book

Problem: What if every edge must be in a  $\Delta$ ?

Def :  $b(n, p) =$  largest book size in any  
 $n$ -ver G w./  $\geq p \binom{n}{2}$  edges & every edge  
in  $\geq 2$   $\Delta$ .

Ques Erdős-Rothschild:

$\forall p > 0$ ,  $b(n, p) \geq n^{O(1)}$  ?

Rmk: Removal lemma  $\Rightarrow b(n, p) \geq e^{c(\log^* n)}$

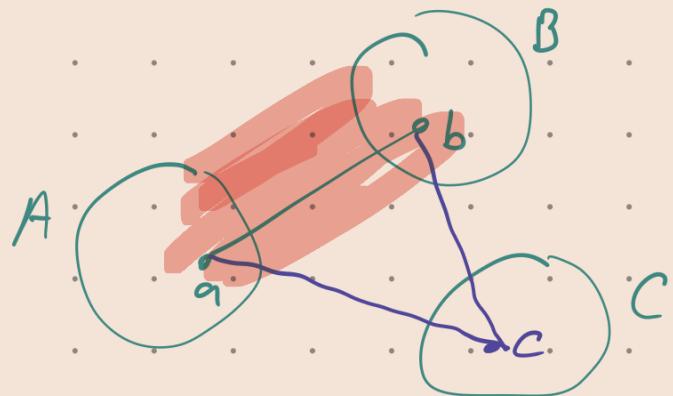
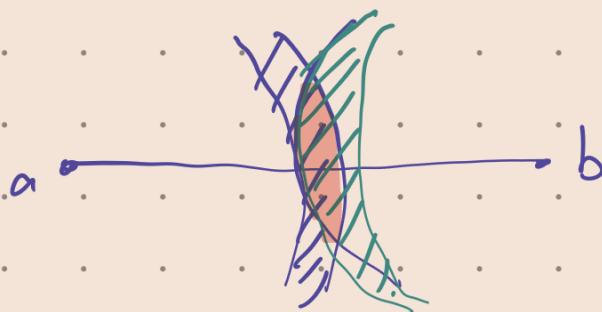
Thm (Fox-Loh 12) (not true up to edge density  $\frac{1}{\epsilon} - o(1)$ )

$\exists G$  w/  $\frac{n-6x}{n^2} \left(1 - e^{-(\log n)^{1/6}}\right)$  s.t.  
if edge in  $\geq 1 \Delta$  &  $b(G) \leq n^{\frac{14}{\log \log n}}$

Idea: in high-dim grid  $[n]^d$ , almost all pairs of pts are at some typical distance apart.

$$A = B = C = [r]^d$$

$a \sim b$  iff  $\text{dist}(a, b) = \text{typical}$



Setup:  $r \in \mathbb{N}$ ,  $d = r^5$ ,  $n = r^d$

$$A = B = C = [r]^d$$

$$(\text{typical dist}) \quad \mu = \frac{r^2 - 1}{6} \cdot d$$

$$a \in A \sim b \in B \iff \|a - b\|_2^2 = \mu \pm d$$

$$a \in A \sim c \in C \iff \|a - c\|_2^2 = \frac{\mu}{4} \pm 2d$$

( $\{B, C\} \cong \{A, C\}$ )

Lem 1:  $[A, B]$  almost complete around  
 (almost all pairs of pts in  $[r]^d$  are at typical dist)

If two unif chosen  $X, Y \sim [r]^d$

$$\mathbb{P} \left( \|X - Y\|_2^2 = \mu \pm d \right) \geq 1 - 2e^{-\frac{d}{2r^4}}$$

PF:  $X = (X_1, \dots, X_d)$

$Y = (Y_1, \dots, Y_d)$

$$\|X - Y\|_2^2 = \sum_i (X_i - Y_i)^2$$

Fix  $i \in [d]$ ,  $\mathbb{E}(X_i - Y_i)^2 = \mathbb{E}(X_i^2 + Y_i^2 - 2X_i Y_i)$

$$\begin{aligned} X_i, Y_i &\stackrel{\text{indep}}{\sim} [r] \\ &= \mathbb{E}X_i^2 + \mathbb{E}Y_i^2 - 2\mathbb{E}X_i \mathbb{E}Y_i \end{aligned}$$

$$= 2(\mathbb{E}X_i^2 - (\mathbb{E}X_i)^2)$$

$$\mathbb{E}X_i = \frac{r+1}{2}$$

$$\mathbb{E}X_i^2 = \frac{1}{r} \sum_{i=1}^r i^2 = \frac{(r+1)(2r+1)}{6}$$

$$\Rightarrow \mathbb{E}(X_i - Y_i)^2 = \frac{r^2 - 1}{6}$$

$$\Rightarrow \mathbb{E}\|X - Y\|_2^2 = \sum_{i=1}^d \mathbb{E}(X_i - Y_i)^2 = \mu$$

Use concentration of measure to conclude.

Def:  $X(\omega)$  r.v. on d-dim prob. sp.  $\Omega = \prod_{i=1}^d \mathcal{D}_i$

C-Lipschitz if  $X(\omega)$  changes by  $\leq C$

When one coord. of  $\omega$  is changed.

Hoeffding - Azuma Ineq  $X$  C-Lip. r.v. on

d-dim,  $t \geq 0$

$$-\frac{t^2}{2C^2d}$$

$$\mathbb{P}(|X - \mathbb{E}X| > t) \leq 2e^{-\frac{t^2}{2C^2d}}$$

$\|X - Y\|_2^2$  r.v. on dim-d,  $r^2$ -Lip. 

Left to show

$$n = r^d = r^s = d^{d^s}$$

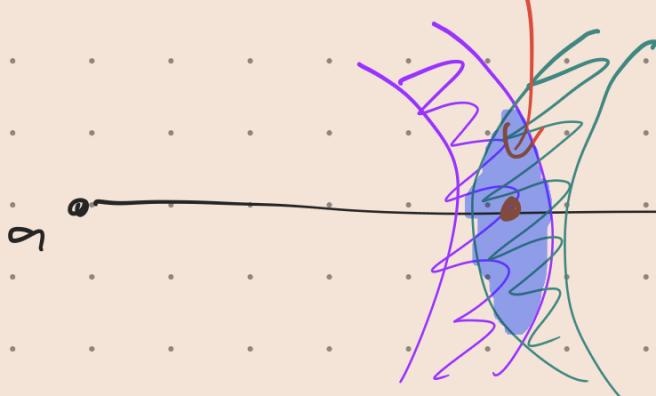
1) If ab edge  $\in [A, B]$  is in  
btr  $[1, 15d]$   $\Delta^s$ .

2) bootstrap edge-density to  $\frac{1}{2} \pm \epsilon$   
blow up each  $x_i$  in  $A \cup B$  to  
 $2^d$  copies

Pf of 1)

Lower  $\frac{6d}{\text{lattice}}$

$$\geq \frac{1}{8} \cdot 8^d$$



$$\|a - b\|_2^2 = n \pm d$$

$$\|a - c\|_2^2 = \frac{n}{4} \pm 2d$$

Vpp bd: NTS (blue area small)

Take arb.  $\|a-b\|_2^2 = \text{and} \dots \quad (*)$

Set  $x_i = b_i - a_i$

bd # half-lattice pt  $c = a_i + \frac{x_i}{2} + \frac{w_i}{2}$  where  $w_i \in \mathbb{Z}$

If  $c \sim \frac{a}{b} \Rightarrow$

$$\|c-a\|_2^2 = \|c-b\|_2^2 = \frac{1}{4} \pm 2d \quad (**)$$

$$\hookrightarrow \sum_i \left( \frac{x_i}{2} + \frac{w_i}{2} \right)^2 = \frac{\|b-a\|_2^2}{4} + \frac{1}{4} \sum w_i^2 + \frac{1}{2} \sum w_i x_i$$

$$\|c-b\|_2^2 = \sum_i (-)^2 = \underline{\hspace{10em}} - \frac{1}{2} \sum w_i x_i$$

$$\Rightarrow \|c-a\|_2^2 + \|c-b\|_2^2 = \frac{\|b-a\|_2^2}{2} + \frac{1}{2} \sum w_i^2$$

(\*)  $\Rightarrow \sum w_i^2 \leq 9d$

$$\#C = \#W \text{ s.t. } \sum w_i^2 \leq 9d$$

$$= \text{vol} \left( B_{\sqrt{3}d}^d \right) < 15^d. \quad \boxed{\phantom{0}}$$