

Lecture 11 Fractional Helly &

Kruskal - Katona

Thm 1.1 (Frac. Helly) Let $\alpha \in (0, 1)$, \mathcal{F} col^o of conv. sets in \mathbb{R}^d

If $|\mathcal{F}| \geq \alpha \binom{[d]}{d+1}$ $(d+1)$ -tuples $A \in \binom{\mathcal{F}}{d+1}$ satisfy $\cap A \neq \emptyset$, then

$$\exists \mathcal{F}' \subseteq \mathcal{F} \quad \text{s.t.} \quad \begin{cases} \cap \mathcal{F}' \neq \emptyset \\ |\mathcal{F}'| \geq \beta |\mathcal{F}| \end{cases} \quad \text{where } \beta \geq 1 - (1-\alpha)^{1/d+1}$$

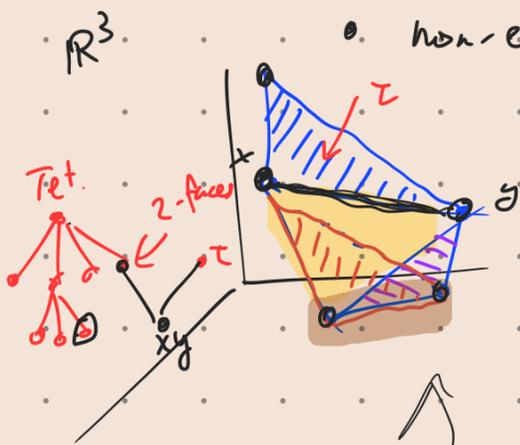
$|\mathcal{F}| = n$

Upper bound Thm by Alon - Kalai

Recall A **simplicial complex** K is a set of simplices

s.t.
 • Every face of a simplex in K also in K

• non-empty intersection of $\sigma_1, \sigma_2 \in K$ is a face of both σ_1, σ_2 .



(xy is not free)

Notation. $f_j(K) = \#$ j -dim. faces of K .

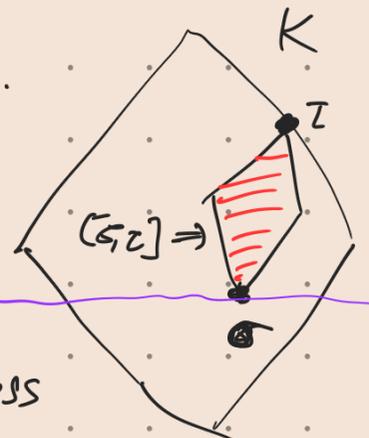
$$f_0(K) = 5$$

Def A face $\sigma \in K$ is a **free face** if it is

contained in a unique maximal face τ .

Given σ free, $\tau \supseteq \sigma$ is the unique max

write $[\sigma, \tau] = \{ \text{all faces } \supseteq \sigma \}$



An **elementary (a, b) -collapse** is the process

of removing $[\sigma, \tau]$ for some free $|\sigma| = a, |\tau| = b$

Def A simplicial complex is d -collapsible if all faces of size $\geq d$ can be removed after some seq of elementary collapses, in which, each collapse is of type (d, d') for some $d' \geq d$.

Ex: a tree is 1-collapsible. (a free face of size = leaf)

i.e. $K \rightarrow K_1 = K \setminus [\sigma_1, \tau_1] \rightarrow K_2 = K_1 \setminus [\sigma_2, \tau_2] \rightarrow \dots \rightarrow K_t$, where $f_{d-1}(K_t) = 0$

Thm (AK 85) Let K be a d -collapsible sim. complex on

n vxs. If $\dim K < d+r$, i.e. $f_{d+r}(K) = 0$,

$\Rightarrow \forall d \leq j \leq d+r-1$,

$$f_j(K) \leq \sum_{i=j+1-d}^r \binom{n-i}{d} \binom{i-1}{j-d}$$



In particular,

$$f_d(K) \leq \binom{n}{d+1} - \binom{n-r}{d+1} \dots (*)$$

Ex Derive frac. Helly from (*). Take $r = \beta n - d$

Apply the above to nerve complex

Def: The nerve complex of \mathcal{F} (colⁿ conv. sets in \mathbb{R}^d)

vxs = elem^{ts} $F \in \mathcal{F}$

face = tuple of sets $\{F_1, \dots, F_i\} \in \mathcal{F}$ that has non-empty intersection.

Pf: Fix a seq

$$K \rightarrow K_1 = K \setminus [\sigma_1, \tau_1] \rightarrow K_2 = K_1 \setminus [\sigma_2, \tau_2] \rightarrow \dots \rightarrow K_t, \text{ where } f_{d-i}(K_t) = 0$$

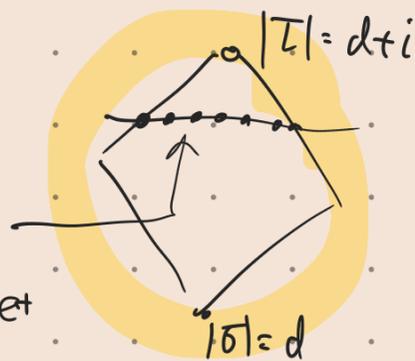
where $|\sigma_i| = d$, $|\tau_i| \geq d$

• For each $i \geq 0$, write $h_i = \# \text{ ele}^g (d, d+i)$ -collapses.

• Note in each such $(d, d+i)$ -collapse, $\#$ j -dim faces deleted

$$d+1 \leq j \leq d+i-1, \quad = \binom{i}{j+1-d}$$

Out of $|\tau \setminus \sigma| = i$ choose $j+1-d$ elements to add to σ to get a j -dim face.



• Recall that $f_{d+r}(K) = 0$ for $0 \leq i \leq r$,

write $H_i = \sum_{i'=i}^r h_{i'}$ (# of $(d, \geq d+i)$ -collapse)

Then for each $d \leq j \leq d+r-1$,

$$\begin{aligned} f_j(K) &= \sum_{i=j+1-d}^r h_i \binom{i}{j+1-d} = \sum (H_i - H_{i+1}) \binom{i}{j+1-d} \\ &= \sum_{i=j+1-d}^r H_i \cdot \binom{i-1}{j-d} \end{aligned}$$

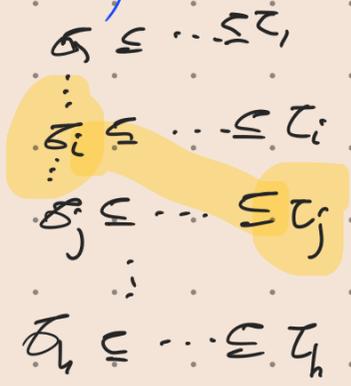
• Left to show $\forall i \geq 0$,

$$H_i \leq \binom{n-i}{d}$$

\hookrightarrow # of (d, d') -collapses, $d' \geq d+i$.

Let $[\sigma_k, \tau_k]$, $k \in [h]$, be the collapses corresponding to H_i . ($|\tau_k| \geq d + i$)

- $\forall i \in [h], \sigma_i \in \tau_i \Rightarrow A_i \cap B_i = \emptyset$
- $\forall i < j, \sigma_i \notin \tau_j \Rightarrow A_i \cap B_j = \emptyset$



Let $A_i = \sigma_i$, $B_i = [n] \setminus \tau_i$

$$|A_i| = d, \quad |B_i| \leq n - d - i$$

Skewed set-pair inequality $\Rightarrow h = H_i \leq \binom{d+n-d-i}{d} \leq \binom{n-i}{d}$

Kruskal - Katona

- Erdős-Rademacher Δ minimisation (4.5) \leftarrow notoriously difficult

Given n -set & m edges \Rightarrow $\min \Delta$?

\uparrow
max ?

much easier \rightarrow

consequence of Kruskal - Katona thm.

Ex Solve Δ -max. problem using K.K.

Def $\mathcal{F} \subseteq \binom{[n]}{k}$ its shadow $\partial \mathcal{F} \subseteq \binom{[n]}{k-1}$

consisting of all $(k-1)$ -tuples contained in some edge of \mathcal{F} .

$$\partial \mathcal{F} = \left\{ F \in \binom{[n]}{k-1} : F \subseteq G, G \in \mathcal{F} \right\}$$

• K.K. studies size of shadow.

Q: $n \geq k$, $0 \leq m \leq \binom{n}{k}$ What is the min. size shadow of an n -x k -unif \mathcal{F} with m edges?

Colex

Try $k=3$, 3-unif hypergraph \mathcal{F} , ($\partial\mathcal{F}$ a graph)

• Fix $n_3 \in \mathcal{N}$ s.t. $\binom{n_3}{3} \leq m < \binom{n_3+1}{3}$

Fix n_2 s.t. Take $|V_3| = n_3$

• $\binom{n_2}{2} \leq m - \binom{n_3}{3}$ edges left $< \binom{n_2+1}{2}$

Take all triples $\{v, e\}$, $e \in \binom{V_2}{2}$

$|V_2| \leq V_3$

\parallel
 n_2

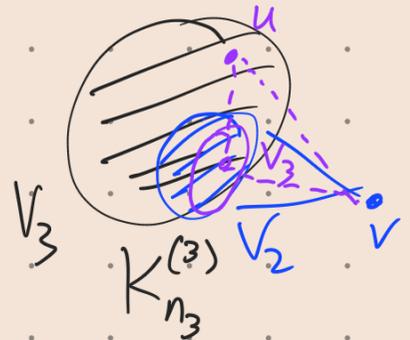
size of $V_1 \leq V_2$

• $m - \binom{n_3}{3} - \binom{n_2}{2} < \binom{n_2+1}{2} - \binom{n_2}{2} = n_2$

Take all triples $\{u, v, z\}$, $z \in V_1$

$$|\partial\mathcal{F}| = \binom{n_3}{2} + \binom{n_2}{1} + \binom{n_1}{0}$$

$$m = \binom{n_3}{3} + \binom{n_2}{2} + \binom{n_1}{1}$$



This is colexicographic ordering of finite sets of \mathcal{N} .

Thm (Kruskal-Katona 63) [Colex min. shadow.]

$\forall \mathcal{F}$ k -unif hypergraph with

$$m = \binom{n_k}{k} + \binom{n_{k-1}}{k-1} + \dots + \binom{n_s}{s}, \text{ then}$$

$$|\partial \mathcal{F}| \geq \binom{n_k}{k-1} + \binom{n_{k-1}}{k-2} + \dots + \binom{n_s}{s-1}$$

————— Shift / Compression —————

Def: $\mathcal{F} \subseteq \binom{[n]}{k}$, $2 \leq i \leq n$, $\forall F \in \mathcal{F}$ define

$$S_i(F) = \begin{cases} F \setminus \{i\} \cup \{1\}, & \text{if } i \in F \ \& \ 1 \notin F, \text{ and} \\ & F \setminus \{i\} \cup \{1\} \notin \mathcal{F} \\ F & , \text{ otherwise.} \end{cases}$$

Write $S_i(\mathcal{F}) = \{S_i(F) : F \in \mathcal{F}\}$

\mathcal{F} is *compressed* if $\forall 2 \leq i \leq n$, $S_i(\mathcal{F}) = \mathcal{F}$.

Fact 2.5 $\mathcal{F} \subseteq \binom{[n]}{k}$, $2 \leq i \leq n$

Map $\mathcal{F} \rightarrow S_i(\mathcal{F})$ is injective $\Rightarrow |\mathcal{F}| = |S_i(\mathcal{F})|$

Fact 2.6 $\partial(S_i(\mathcal{F})) \subseteq S_i(\partial \mathcal{F})$

$$\Rightarrow |\partial(S_i(\mathcal{F}))| \leq |S_i(\partial \mathcal{F})| \stackrel{2.5}{=} |\partial \mathcal{F}|$$

i.e. shifting does not increase shadow size.

Notation Given $\mathcal{F} \subseteq \binom{[n]}{k}$, write

$$\begin{cases} \mathcal{F}_1 = \{F \in \mathcal{F} : 1 \in F\} \\ \mathcal{F}_1^c = \mathcal{F} \setminus \mathcal{F}_1 = \{F \in \mathcal{F} : 1 \notin F\} \\ \mathcal{L}_1 = \{F \setminus \{1\} : F \in \mathcal{F}_1\} \end{cases}$$

Fact 2.7 If \mathcal{F} is compressed, then

$$(i) \quad \partial \mathcal{F}_1^c \subseteq \mathcal{L}_1$$

$$(ii) \quad \partial \mathcal{F} = \mathcal{L}_1 \cup \{E \cup \{1\} : E \in \partial \mathcal{L}_1\}$$

$$\Rightarrow |\partial \mathcal{F}| = |\mathcal{L}_1| + |\partial \mathcal{L}_1|$$

Pf. (K.K.) double induction on k then m .