

# HPD-INVARIANCE OF THE TATE CONJECTURE(S)

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ABSTRACT. We prove that the Tate conjecture (and its variants) is invariant under Homological Projective Duality. As an application, we obtain a proof, resp. an alternative proof, of the Tate conjecture (and of its variants) in the new case of linear sections of determinantal varieties, resp. in the old cases of Pfaffian cubic fourfolds and complete intersections of quadrics. In addition, we generalize the Tate conjecture (and its variants) from schemes to stacks and prove this generalized conjecture(s) for low-dimensional global orbifolds.

## 1. INTRODUCTION

Let  $k$  be a base field of characteristic  $p \geq 0$  and  $X$  a smooth projective  $k$ -scheme. Throughout the article, we will write  $\mathcal{Z}^*(X)_{\mathbb{Q}} := \bigoplus_{i=0}^{\dim(X)} \mathcal{Z}^i(X)_{\mathbb{Q}}$  for the graded  $\mathbb{Q}$ -vector space of algebraic cycles on  $X$  up to rational equivalence and  $\mathcal{Z}^*(X)_{\mathbb{Q}}/\sim_{\text{num}}$  for its quotient with respect to the numerical equivalence relation. Given a prime number  $l \neq p$ , consider the classical cycle class map

$$(1.1) \quad \mathcal{Z}^*(X)_{\mathbb{Q}_l} \longrightarrow H_{\text{et}}^{2*}(X_{k_s}, \mathbb{Q}_l(*))^{\text{Gal}(k_s/k)},$$

where  $k_s$  stands for a (chosen) separable closure of  $k$ ,  $\text{Gal}(k_s/k)$  for the absolute Galois group of  $k$ , and  $H_{\text{et}}^*(-, \mathbb{Q}_l) := (\lim_{\nu} H_{\text{et}}^*(-, \mathbb{Z}/l^{\nu})) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$  for  $l$ -adic cohomology. In the sixties, Tate [40, 41] conjectured the following:

Conjecture  $T^l(X)$ : *When  $k$  is finitely generated over its prime field, the above cycle class map (1.1) is surjective.*

Examples of fields which are finitely generated over their prime fields include finite fields, number fields, function fields, etc. In contrast, algebraically closed fields are *not* finitely generated over their prime fields. The Tate conjecture holds when  $\dim(X) \leq 1$ , when  $X$  is an abelian variety of dimension  $\leq 3$ , and also when  $X$  is a  $K3$ -surface; consult the surveys [4, 28, 44]. Besides these cases (and some other cases scattered in the literature), the Tate conjecture remains wide open. In Theorem 2.7, resp. in Theorems 2.13 and 2.15, below we will provide a proof, resp. an alternative proof, of the Tate conjecture in some new, resp. old, cases.

The above conjecture of Tate admits several variants. For example, in the sixties, Tate [40, 41] conjectured moreover the following:

Conjecture  $\overline{T}^l(X)$ : *When  $k$  is finitely generated over its prime field, the classical cycle class map is surjective*

$$(1.2) \quad \mathcal{Z}^*(X_{k_s})_{\mathbb{Q}_l} \longrightarrow \bigcup_{k'/k} H_{\text{et}}^{2*}(X_{k_s}, \mathbb{Q}_l(*))^{\text{Gal}(k_s/k')},$$

where the union runs over all the finite field extensions  $k'/k$  inside  $k_s$ .

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Conjecture  $\text{SS}^l(X)$ : *When  $k$  is finitely generated over its prime field, the (continuous)  $l$ -adic representations  $\{H_{\text{et}}^n(X_{k_s}, \mathbb{Q}_l) \mid 0 \leq n \leq 2\dim(X)\}$  of the absolute Galois group  $\text{Gal}(k_s/k)$  are semi-simple.*

It is well-known that  $\overline{\text{T}}^l(X) \Rightarrow \text{T}^l(X)$ . Moreover, when  $p = 0$ , we have the following implication  $\{\overline{\text{T}}^l(X) \mid X \in \text{SmProj}(k)\} \Rightarrow \{\text{SS}^l(X) \mid X \in \text{SmProj}(k)\}$ , where  $\text{SmProj}(k)$  stands for the category of smooth projective  $k$ -schemes; consult [29].

In the 2000's, Milne [28] formulated the following  $p$ -adic variant:

Conjecture  $\text{T}^p(X)$ : *When  $k$  is finite, the classical cycle class map is surjective*

$$(1.3) \quad \mathcal{Z}^*(X)_{\mathbb{Q}_p} \longrightarrow H_{\text{crys}}^{2*}(X)(*)^\phi,$$

where  $H_{\text{crys}}^*(-)$  stands for crystalline cohomology and  $\phi$  is the crystalline Frobenius.

Finally, in the eighties, Beilinson (see [15, Conj. 50]) conjectured the following:

Conjecture  $\text{B}(X)$ : *When  $k$  is finite, we have  $\mathcal{Z}^*(X)_{\mathbb{Q}} = \mathcal{Z}^*(X)_{\mathbb{Q}}/\sim_{\text{num}}$ .*

Note that combining conjecture  $\text{B}(X)$  with conjectures  $\text{T}^l(X)$  and  $\text{T}^p(X)$ , we conclude that the classical cycle class maps are bijective:

$$\mathcal{Z}^*(X)_{\mathbb{Q}_l} \xrightarrow{\sim} H_{\text{et}}^{2*}(X_{k_s}, \mathbb{Q}_l(*))^{\text{Gal}(k_s/k)} \quad \mathcal{Z}^*(X)_{\mathbb{Q}_p} \xrightarrow{\sim} H_{\text{crys}}^{2*}(X)(*)^\phi.$$

This provides an optimal description of algebraic cycles using  $l$ -adic and crystalline cohomology. The conjectures  $\text{C}(X)$ , with  $\text{C} \in \{\overline{\text{T}}^l, \text{SS}^l, \text{T}^p, \text{B}\}$ , also hold when  $\dim(X) \leq 1$  and when  $X$  is an abelian variety of dimension  $\leq 3$ ; consult [9, 15, 16, 46]. The conjectures  $\text{C}(X)$ , with  $\text{C} \in \{\overline{\text{T}}^l, \text{SS}^l, \text{T}^p\}$ , hold<sup>1</sup> moreover for  $K3$ -surfaces.

## 2. STATEMENT OF RESULTS

A *differential graded (=dg) category*  $\mathcal{A}$  is a category enriched over complexes of  $k$ -vector spaces; consult §3.1. As explained in §4, given a smooth proper dg category  $\mathcal{A}$  in the sense of Kontsevich, the Tate conjecture and its variants admit noncommutative analogues  $\text{C}_{\text{nc}}(\mathcal{A})$  with  $\text{C} \in \{\text{T}^l, \overline{\text{T}}^l, \text{SS}^l, \text{T}^p, \text{B}\}$ . Examples of smooth proper dg categories include finite dimensional  $k$ -algebras of finite global dimension  $A$  as well as the canonical dg enhancement  $\text{perf}_{\text{dg}}(X)$  of the category of perfect complexes  $\text{perf}(X)$  of every smooth proper  $k$ -scheme  $X$  (or, more generally, of every smooth proper algebraic stack  $\mathcal{X}$ ); consult [18, 26].

**Theorem 2.1.** *Given a smooth projective  $k$ -scheme  $X$ , we have the equivalences:*

$$\text{C}(X) \Leftrightarrow \text{C}_{\text{nc}}(\text{perf}_{\text{dg}}(X)) \quad \text{with} \quad \text{C} \in \{\text{T}^l, \overline{\text{T}}^l, \text{SS}^l, \text{T}^p, \text{B}\}.$$

Intuitively speaking, Theorem 2.1 shows that the Tate conjecture and its variants belong not only to the realm of algebraic geometry but also to the broad noncommutative setting of smooth proper dg categories in the sense of Kontsevich.

**HPD-invariance.** For surveys on Homological Projective Duality (=HPD), we invite the reader to consult [22, 42]. Let  $X$  be a smooth projective  $k$ -scheme equipped with a line bundle  $\mathcal{L}_X(1)$ ; we will write  $X \rightarrow \mathbb{P}(V)$  for the associated morphism, where  $V := H^0(X, \mathcal{L}_X(1))^\vee$ . Assume that the triangulated category  $\text{perf}(X)$  admits a Lefschetz decomposition  $\langle \mathbb{A}_0, \mathbb{A}_1(1), \dots, \mathbb{A}_{i-1}(i-1) \rangle$  with respect to  $\mathcal{L}_X(1)$  in the sense of [24, Def. 4.1]. Following [24, Def. 6.1], let  $Y$  be the HP-dual of  $X$ ,  $\mathcal{L}_Y(1)$  the HP-dual line bundle, and  $Y \rightarrow \mathbb{P}(V^\vee)$  the morphism

<sup>1</sup>When  $\dim(X) \leq 3$ , we have  $\text{T}^l(X) \Leftrightarrow \text{T}^p(X)$  (for every  $l \neq p$ ).

associated to  $\mathcal{L}_Y(1)$ . Given a linear subspace  $L \subset V^\vee$ , consider the linear sections  $X_L := X \times_{\mathbb{P}(V)} \mathbb{P}(L^\perp)$  and  $Y_L := Y \times_{\mathbb{P}(V^\vee)} \mathbb{P}(L)$ . As a first application of Theorem 2.1, we obtain the following result:

**Theorem 2.2** (HPD-invariance). *Assume the following:*

- (a) *The linear sections  $X_L$  and  $Y_L$  are smooth<sup>2</sup>.*
- (b) *We have  $\dim(X_L) = \dim(X) - \dim(L)$  and  $\dim(Y_L) = \dim(Y) - \dim(L^\perp)$ .*
- (c) *The conjectures  $C_{\text{nc}}(\mathbb{A}_0^{\text{dg}})$ , with  $C \in \{\mathbb{T}^l, \overline{\mathbb{T}}^l, \text{SS}^l, \mathbb{T}^p, \mathbb{B}\}$ , hold, where  $\mathbb{A}_0^{\text{dg}}$  stands for the dg enhancement of  $\mathbb{A}_0$  induced from  $\text{perf}_{\text{dg}}(X)$ .*

*Under the assumptions (a)-(c), we have the following equivalences:*

$$C(X_L) \Leftrightarrow C(Y_L) \quad \text{with} \quad C \in \{\mathbb{T}^l, \overline{\mathbb{T}}^l, \text{SS}^l, \mathbb{T}^p, \mathbb{B}\}.$$

*Remark 2.3* (Mild assumptions). Given a *generic* subspace  $L \subset V^\vee$ , the sections  $X_L$  and  $Y_L$  are smooth, and the equalities  $\dim(X_L) = \dim(X) - \dim(L)$  and  $\dim(Y_L) = \dim(Y) - \dim(L^\perp)$  hold. Moreover, the conjectures  $C_{\text{nc}}(\mathbb{A}_0^{\text{dg}})$ , with  $C \in \{\mathbb{T}^l, \overline{\mathbb{T}}^l, \text{SS}^l, \mathbb{T}^p, \mathbb{B}\}$ , hold whenever the triangulated category  $\mathbb{A}_0$  admits a full exceptional collection. This is the case in all the examples in the literature. For these reasons, the assumptions (a)-(c) of Theorem 2.2 are quite mild.

*Remark 2.4* (Generalization). Theorem 2.2 holds more generally when  $Y$  is singular. In this case we need to replace  $Y$  by a noncommutative resolution of singularities  $\text{perf}_{\text{dg}}(Y; \mathcal{F})$ , where  $\mathcal{F}$  is a certain sheaf of noncommutative algebras (consult [22, §2.4]), and conjecture  $C(Y)$  by its noncommutative analogue  $C_{\text{nc}}(\text{perf}_{\text{dg}}(Y; \mathcal{F}))$ .

To the best of the author's knowledge, Theorem 2.2 is new in the literature. In what follows, we illustrate its strength in three important examples:

**Example 1: Determinantal duality.** Let  $U_1$  and  $U_2$  be two  $k$ -vector spaces of dimensions  $d_1$  and  $d_2$ , with  $d_1 \leq d_2$ ,  $V := U_1 \otimes U_2$ , and  $0 < r < d_1$  an integer. Consider the determinantal variety  $\mathcal{Z}_{d_1, d_2}^r \subset \mathbb{P}(V)$  defined as the locus of those matrices  $U_2 \rightarrow U_1^\vee$  with  $\text{rank} \leq r$ . Recall that the determinantal varieties with  $r = 1$  are the classical Segre varieties. For example,  $\mathcal{Z}_{2,2}^1 \subset \mathbb{P}^3$  is the quadric hypersurface  $\{[v_0 : v_1 : v_2 : v_3] \mid v_0 v_3 - v_1 v_2 = 0\}$ . In contrast with the Segre varieties, the determinantal varieties  $\mathcal{Z}_{d_1, d_2}^r$ , with  $r \geq 2$ , are not smooth. The singular locus of  $\mathcal{Z}_{d_1, d_2}^r$  consists of those matrices  $U_2 \rightarrow U_1^\vee$  with  $\text{rank} < r$ , i.e. it agrees with the closed subvariety  $\mathcal{Z}_{d_1, d_2}^{r-1}$ . Nevertheless, it is well-known that  $\mathcal{Z}_{d_1, d_2}^r$  admits a canonical Springer resolution of singularities  $\mathcal{X}_{d_1, d_2}^r \rightarrow \mathcal{Z}_{d_1, d_2}^r$ , which comes equipped with a projection  $q: \mathcal{X}_{d_1, d_2}^r \rightarrow \text{Gr}(r, U_1)$  to the Grassmannian of  $r$ -dimensional subspaces in  $U_1$ . Following [5, §3.3], the category  $\text{perf}(X)$ , with  $X := \mathcal{X}_{d_1, d_2}^r$ , admits a Lefschetz decomposition  $\langle \mathbb{A}_0, \mathbb{A}_1(1), \dots, \mathbb{A}_{d_2 r - 1}(d_2 r - 1) \rangle$ , where  $\mathbb{A}_0 = \mathbb{A}_1 = \dots = \mathbb{A}_{d_2 r - 1} = q^*(\text{perf}(\text{Gr}(r, U_1))) \simeq \text{perf}(\text{Gr}(r, U_1))$ .

**Proposition 2.5.** *The conjectures  $C_{\text{nc}}(\mathbb{A}_0^{\text{dg}})$ , with  $C \in \{\mathbb{T}^l, \overline{\mathbb{T}}^l, \text{SS}^l, \mathbb{T}^p, \mathbb{B}\}$ , hold.*

Dually, consider the variety  $\mathcal{W}_{d_1, d_2}^r \subset \mathbb{P}(V^\vee)$  of those matrices  $U_2^\vee \rightarrow U_1$  with  $\text{corank} \geq r$ , and the associated resolution of singularities  $Y := \mathcal{Y}_{d_1, d_2}^r \rightarrow \mathcal{W}_{d_1, d_2}^r$ . As

<sup>2</sup>The linear section  $X_L$  is smooth if and only if the linear section  $Y_L$  is smooth; see [22, page 9].

proved<sup>3</sup> in [5, Prop. 3.4 and Thm. 3.5],  $X$  and  $Y$  are HP-dual to each other. Given a generic linear subspace  $L \subseteq V^\vee$ , consider the associated smooth linear sections  $X_L$  and  $Y_L$ ; note that whenever  $\mathbb{P}(L^\perp)$  does not intersect the singular locus of  $\mathcal{Z}_{d_1, d_2}^r$ , we have  $X_L = \mathbb{P}(L^\perp) \cap \mathcal{Z}_{d_1, d_2}^r$ . Theorem 2.2 yields the following result:

**Corollary 2.6.** *We have the following equivalences:*

$$C(X_L) \Leftrightarrow C(Y_L) \quad \text{with} \quad C \in \{\mathbb{T}^l, \overline{\mathbb{T}}^l, \text{SS}^l, \mathbb{T}^p, \mathbb{B}\}.$$

By construction,  $\dim(X) = r(d_1 + d_2 - r) - 1$  and  $\dim(Y) = r(d_1 - d_2 - r) + d_1 d_2 - 1$ . Consequently, we have  $\dim(X_L) = r(d_1 + d_2 - r) - 1 - \dim(L)$  and  $\dim(Y_L) = r(d_1 - d_2 - r) - 1 + \dim(L)$ . Since the Tate conjecture and its variants hold in dimensions  $\leq 1$ , we hence obtain from Corollary 2.6 the following result:

**Theorem 2.7** (Linear sections of determinantal varieties). *Let  $X_L$  and  $Y_L$  be as in Corollary 2.6 and  $C \in \{\mathbb{T}^l, \overline{\mathbb{T}}^l, \text{SS}^l, \mathbb{T}^p, \mathbb{B}\}$ .*

- (i) *When  $r(d_1 + d_2 - r) - 1 - \dim(L) \leq 1$ , the conjectures  $C(X_L)$  hold.*
- (ii) *When  $r(d_1 - d_2 - r) - 1 + \dim(L) \leq 1$ , the conjectures  $C(Y_L)$  hold.*

To the best of the author's knowledge, Theorem 2.7 is new in the literature. It proves the Tate conjecture and its variants in new cases. Here are two examples:

*Example 2.8* (Segre varieties). Let  $r = 1$ . Thanks to Theorem 2.7(i), the conjectures  $C(X_L)$ , with  $C \in \{\mathbb{T}^l, \overline{\mathbb{T}}^l, \text{SS}^l, \mathbb{T}^p, \mathbb{B}\}$ , hold when  $d_1 - d_2 - 2 + \dim(L) \leq 1$ . In all these cases,  $X_L$  is a linear section of the Segre variety  $\mathcal{Z}_{d_1, d_2}^1$  and its dimension is  $2(d_2 - \dim(L))$  or  $2(d_2 - \dim(L)) + 1$ . Therefore, by letting  $d_2 \rightarrow \infty$  and by keeping  $\dim(L)$  fixed, we obtain infinitely many new examples of smooth projective  $k$ -schemes  $X_L$ , of arbitrary dimension, satisfying the Tate conjecture and its variants.

*Subexample 2.9.* Let  $r = 1$ ,  $d_1 = 4$ , and  $d_2 = 2$ . In this particular case, the Segre variety  $\mathcal{Z}_{4, 2}^1 \subset \mathbb{P}^7$  agrees with the rational normal 4-fold scroll  $S_{1, 1, 1, 1}$ ; see [13, Ex. 8.27]. Choose a generic linear subspace  $L \subset V^\vee$  of dimension 1 such that the hyperplane  $\mathbb{P}(L^\perp) \subset \mathbb{P}^7$  does not contain any 3-plane of the ruling of  $S_{1, 1, 1, 1}$ . By combining Example 2.8 with [8, Prop. 2.5], we hence conclude that the rational normal 3-fold scroll  $X_L = S_{1, 1, 2}$  satisfies the Tate conjecture and its variants.

*Example 2.10* (Square matrices). Let  $d_1 = d_2 = d$ . Thanks to Theorem 2.7(i), the conjectures  $C(X_L)$ , with  $C \in \{\mathbb{T}^l, \overline{\mathbb{T}}^l, \text{SS}^l, \mathbb{T}^p, \mathbb{B}\}$ , hold when  $-r^2 - 1 + \dim(L) \leq 1$ . In all these cases  $X_L$  is of dimension  $2(dr - \dim(L))$  or  $2(dr - \dim(L)) + 1$ . Therefore, by letting  $d \rightarrow \infty$  and by keeping  $r$  and  $\dim(L)$  fixed, we obtain infinitely many new examples of smooth projective  $k$ -schemes  $X_L$ , of arbitrary dimension, satisfying the Tate conjecture and its variants.

*Subexample 2.11.* Let  $d_1 = d_2 = 3$  and  $r = 2$ . In this particular case, the determinantal variety  $\mathcal{Z}_{3, 3}^2 \subset \mathbb{P}^8$  has dimension 7 and its singular locus is the 4-dimensional Segre variety  $\mathcal{Z}_{3, 3}^1 \subset \mathcal{Z}_{3, 3}^2$ . Given a generic linear subspace  $L \subset V^\vee$  of dimension 5, the associated smooth linear section  $X_L$  is 2-dimensional and, thanks to

<sup>3</sup>In [5, Prop. 3.4 and Thm. 3.5] the authors worked over an algebraically closed field of characteristic zero. However, the same proof holds *mutatis mutandis* over any field  $k$ . Simply replace the reference [17] concerning the existence of a full strong exceptional collection on  $\text{perf}(\text{Gr}(r, U_1))$  by the reference [7, Thm. 1.3] concerning the existence of a tilting bundle on  $\text{perf}(\text{Gr}(r, U_1))$ . The author is grateful to Marcello Bernardara for an e-mail exchange on this issue.

Example 2.10, it satisfies the Tate conjecture and its variants. Note that since  $\text{codim}(L^\perp) = 5 > 4 = \dim(\mathcal{Z}_{3,3}^1)$ , the subspace  $\mathbb{P}(L^\perp) \subset \mathbb{P}^8$  does not intersect the singular locus  $\mathcal{Z}_{3,3}^1$  of  $\mathcal{Z}_{3,3}^2$ . Therefore, for all the above choices of  $L$ , the associated surface  $X_L$  is a linear section of the determinantal variety  $\mathcal{Z}_{3,3}^2$ .

**Example 2: Grassmannian-Pfaffian duality.** Assume that  $p = 0$ . Let  $W$  be a  $k$ -vector space of dimension 6 and  $X = \text{Gr}(2, W)$  the Grassmannian variety equipped with the Plücker embedding  $\text{Gr}(2, W) \rightarrow \mathbb{P}(\wedge^2(W))$ ,  $(w_1, w_2) \mapsto [w_1 \wedge w_2]$ . Following [25][22, §4.4], the category  $\text{perf}(X)$  admits a Lefschetz decomposition  $\langle \mathbb{A}_0, \dots, \mathbb{A}_5(5) \rangle$  with  $\mathbb{A}_0 = \mathbb{A}_1 = \mathbb{A}_2 = \langle \mathcal{O}_X, \mathcal{U}_X^\vee, S^2(\mathcal{U}_X^\vee) \rangle$  and  $\mathbb{A}_3 = \mathbb{A}_4 = \mathbb{A}_5 = \langle \mathcal{O}_X, \mathcal{U}_X^\vee \rangle$ , where  $\mathcal{U}_X^\vee$  stands for the dual of the tautological bundle on  $X$  and  $S^2(\mathcal{U}_X^\vee)$  for the symmetric power of  $\mathcal{U}_X^\vee$ . Following Remark 2.3, these full exceptional collections imply that the conjectures  $C_{\text{nc}}(\mathbb{A}_0^{\text{dg}})$ , with  $C \in \{\mathbb{T}^l, \overline{\mathbb{T}}^l, \text{SS}^l, \mathbb{T}^p, \mathbb{B}\}$ , hold. As proved in [25, Thm. 1][22, §4.4], the HP-dual  $Y$  of  $X$  is given by  $\text{perf}(\text{Pf}(4, W^\vee); \mathcal{F})$ , where  $\text{Pf}(4, W^\vee) \subset \mathbb{P}(\wedge^2(W^\vee))$  is the (singular) Pfaffian variety and  $\mathcal{F}$  is a certain sheaf of noncommutative algebras; consult Remark 2.4. Let  $L \subset \wedge^2(W^\vee)$  be a generic linear subspace. When  $\dim(L) \leq 6$ , the associated subspace  $\mathbb{P}(L) \subset \mathbb{P}(\wedge^2(W^\vee))$  does *not* intersect the singular locus of  $\text{Pf}(4, W^\vee)$ . Consequently, the linear section  $Y_L$  agrees with the smooth section  $\text{Pf}(4, W^\vee)_L := \text{Pf}(4, W^\vee) \cap \mathbb{P}(L)$ . Theorem 2.2 yields the following result:

**Corollary 2.12.** *When  $\dim(L) \leq 6$ , we have the following equivalences:*

$$C(X_L) \Leftrightarrow C(\text{Pf}(4, W^\vee)_L) \quad \text{with } C \in \{\mathbb{T}^l, \overline{\mathbb{T}}^l, \text{SS}^l\}.$$

By construction,  $\dim(X_L) = 8 - \dim(L)$  and  $\dim(Y_L) = \dim(L) - 2$ . Moreover, when  $\dim(L) = 6$ ,  $X_L$  is a  $K3$ -surface and  $Y_L$  a cubic fourfold; consult [25, §10]. Since the Tate conjecture and its variants hold for  $K3$ -surfaces, we hence obtain from Corollary 2.12 the following result:

**Theorem 2.13** (Pfaffian cubic fourfolds). *Let  $\text{Pf}(4, W^\vee)_L$  be as in Corollary 2.12. When  $\dim(L) = 6$ , the conjectures  $C(\text{Pf}(4, W^\vee)_L)$ , with  $C \in \{\mathbb{T}^l, \overline{\mathbb{T}}^l, \text{SS}^l\}$ , hold.*

An alternative (geometric) proof of Theorem 2.13, based on the Kuga-Satake correspondence, was obtained by André in the mid nineties, consult [1, Thm. 1.6.1].

**Example 3: Veronese-Clifford duality.** Let  $W$  be a  $k$ -vector space of dimension  $d$  and  $X$  the projective space  $\mathbb{P}(W)$  equipped with the double Veronese embedding  $\mathbb{P}(W) \rightarrow \mathbb{P}(S^2W)$ ,  $[w] \mapsto [w \otimes w]$ . Consider the Beilinson's full exceptional collection  $\text{perf}(X) = \langle \mathcal{O}_X(-1), \mathcal{O}_X, \mathcal{O}_X(1), \dots, \mathcal{O}_X(d-2) \rangle$  (see [3]) and set  $i := \lceil d/2 \rceil$  and

$$\mathbb{A}_0 = \mathbb{A}_1 = \dots = \mathbb{A}_{i-2} := \langle \mathcal{O}_X(-1), \mathcal{O}_X \rangle \quad \mathbb{A}_{i-1} := \begin{cases} \langle \mathcal{O}_X(-1), \mathcal{O}_X \rangle & \text{if } d = 2i \\ \langle \mathcal{O}_X(-1) \rangle & \text{if } d = 2i - 1. \end{cases}$$

Under these notations, the category  $\text{perf}(X)$  admits the Lefschetz decomposition  $\langle \mathbb{A}_0, \mathbb{A}_1(1), \dots, \mathbb{A}_{i-1}(i-1) \rangle$  with respect to the line bundle  $\mathcal{L}_X(1) = \mathcal{O}_X(2)$ . Following Remark 2.3, these full exceptional collections imply that the conjectures  $C_{\text{nc}}(\mathbb{A}_0^{\text{dg}})$ , with  $C \in \{\mathbb{T}^l, \overline{\mathbb{T}}^l, \text{SS}^l, \mathbb{T}^p, \mathbb{B}\}$ , hold.

Let  $\mathcal{H} := X \times_{\mathbb{P}(S^2W)} \mathcal{Q} \subset X \times \mathbb{P}(S^2(W^\vee))$  be the universal hyperplane section, where  $\mathcal{Q} \subset \mathbb{P}(S^2(W)) \times \mathbb{P}(S^2(W^\vee))$  stands for the incidence quadric. By construction, the projection  $q: \mathcal{H} \rightarrow \mathbb{P}(S^2(W^\vee))$  is a flat quadric fibration. As proved in [23, Thm. 5.4] (see also [2, Thm. 2.3.6]) the HP-dual  $Y$  of  $X$  is given

by  $\text{perf}_{\text{dg}}(\mathbb{P}(S^2(W^\vee)); \mathcal{C}l_0(q))$ , where  $\mathcal{C}l_0(q)$  stands for the sheaf of even Clifford algebras associated to  $q$ ; consult Remark 2.4. Let  $L \subset S^2(W^\vee)$  be a generic linear subspace. On the one hand,  $X_L$  corresponds to the smooth complete intersection of the  $\dim(L)$  quadric hypersurfaces in  $\mathbb{P}(W)$  parametrized by  $L$ . On the other hand,  $Y_L$  is given by  $\text{perf}_{\text{dg}}(\mathbb{P}(L); \mathcal{C}l_0(q)|_L)$ . Theorem 2.2 yields the following result:

**Corollary 2.14.** *We have the following equivalences:*

$$C(X_L) \Leftrightarrow C_{\text{nc}}(\text{perf}_{\text{dg}}(\mathbb{P}(L); \mathcal{C}l_0(q)|_L)) \quad \text{with } C \in \{\mathbb{T}^l, \overline{\mathbb{T}}^l, \text{SS}^l, \mathbb{T}^p, \mathbb{B}\}.$$

Recall that the space of quadrics  $\mathbb{P}(S^2(W^\vee))$  comes equipped with a canonical filtration  $\Delta_d \subset \cdots \subset \Delta_2 \subset \Delta_1 \subset \mathbb{P}(S^2(W^\vee))$ , where  $\Delta_i$  stands for the closed subscheme of those singular quadrics of corank  $\geq i$ .

**Theorem 2.15** (Intersection of two quadrics). *Let  $X_L$  be as in Corollary 2.14. Assume that  $\dim(L) = 2$  and that  $\mathbb{P}(L) \cap \Delta_2 = \emptyset$ .*

- (i) *When  $d$  is even, the conjectures  $C(X_L)$ , with  $C \in \{\mathbb{T}^l, \overline{\mathbb{T}}^l, \text{SS}^l, \mathbb{T}^p, \mathbb{B}\}$ , hold.*
- (ii) *When  $d$  is odd and  $k = \mathbb{F}_q$  is a finite field of characteristic  $p \geq 3$ , the conjectures  $C(X_L)$ , with  $C \in \{\mathbb{T}^l, \overline{\mathbb{T}}^l, \text{SS}^l, \mathbb{T}^p, \mathbb{B}\}$ , hold.*

The proof of Theorem 2.15 is based on the solution of the corresponding non-commutative conjectures of Corollary 2.14; consult §8 for details. In what concerns the Tate conjecture, an alternative (geometric) proof, based on the notion of variety of maximal planes, was obtained by Reid<sup>4</sup> in the early seventies; consult [31, Thms. 3.14 and 4.14]. Therein, Reid proved the Hodge conjecture but, as Bruno Kahn informed me, a similar proof works for the Tate conjecture. In what concerns the variants of the Tate conjecture, Theorem 2.15 is, to the best of the author's knowledge, new in the literature.

*Remark 2.16* (Intersection of even-dimensional quadrics). As explained in Remark 8.3 below, when  $d$  is even and  $\dim(L) \geq 2$ , i.e., when  $X_L$  is the intersection of  $\dim(L)$  even-dimensional quadric hypersurfaces, a proof similar to the one of Theorem 2.15 shows that the conjectures  $C(X_L)$ , with  $C \in \{\mathbb{T}^l, \overline{\mathbb{T}}^l, \text{SS}^l, \mathbb{T}^p, \mathbb{B}\}$ , are equivalent to the corresponding conjectures for the discriminant 2-fold cover of the projective space  $\mathbb{P}(L)$ . This result is also new in the literature.

**Tate conjecture(s) for stacks.** As a second application of Theorem 2.1, we obtain the following extension of the Tate conjecture and of its variants to the broad setting of smooth proper algebraic  $k$ -stacks:

$$C(\mathcal{X}) := C_{\text{nc}}(\text{perf}_{\text{dg}}(\mathcal{X})) \quad \text{with } C \in \{\mathbb{T}^l, \overline{\mathbb{T}}^l, \text{SS}^l, \mathbb{T}^p, \mathbb{B}\}.$$

The next results prove these extended conjectures in several cases:

**Theorem 2.17.** *Let  $G$  be a finite group of order  $n$ ,  $X$  a smooth projective  $k$ -scheme equipped with a  $G$ -action, and  $\mathcal{X} := [X/G]$  the associated global orbifold. When  $p \nmid n$ , we have the following implications*

$$(2.18) \quad \sum_{\sigma \subseteq G} C(X^\sigma \times \text{Spec}(k[\sigma])) \Rightarrow C(\mathcal{X}) \quad \text{with } C \in \{\mathbb{T}^l, \overline{\mathbb{T}}^l, \text{SS}^l, \mathbb{T}^p, \mathbb{B}\},$$

where  $\sigma$  is a cyclic subgroup of  $G$ . Moreover, when  $k$  contains the  $n^{\text{th}}$  roots of unity, the  $k$ -schemes  $X^\sigma \times \text{Spec}(k[\sigma])$  in (2.18) can be replaced by the  $k$ -schemes  $X^\sigma$ .

<sup>4</sup>Reid also assumed in *loc. cit.* that  $\mathbb{P}(L) \cap \Delta_2 = \emptyset$ ; consult [31, Def. 1.9].

**Corollary 2.19** (Low-dimensional orbifolds). *Assume that  $p \nmid n$ .*

- (i) *When  $\dim(X) \leq 1$ , the conjectures  $C(\mathcal{X})$ , with  $C \in \{\mathbb{T}^l, \overline{\mathbb{T}}^l, \text{SS}^l, \mathbb{T}^p, \mathbb{B}\}$ , hold.*
- (ii) *Assume that  $k$  contains the  $n^{\text{th}}$  roots of unity. When  $X$  is a connected surface, we have  $C(X) \Rightarrow C(\mathcal{X})$ , with  $C \in \{\mathbb{T}^l, \overline{\mathbb{T}}^l, \text{SS}^l, \mathbb{T}^p, \mathbb{B}\}$ . Consequently, when  $X$  is an abelian surface or a K3-surface, the conjectures  $C(\mathcal{X})$  hold.*
- (iii) *Assume that  $k$  contains the  $n^{\text{th}}$  roots of unity. When  $X$  is an abelian variety of dimension 3 and  $G$  acts by group homomorphisms<sup>5</sup>, the conjectures  $C(\mathcal{X})$ , with  $C \in \{\mathbb{T}^l, \overline{\mathbb{T}}^l, \text{SS}^l, \mathbb{T}^p, \mathbb{B}\}$ , hold.*

### 3. PRELIMINARIES

Throughout the article,  $k$  denotes a base field of characteristic  $p \geq 0$ .

**3.1. Dg categories.** For a survey on dg categories, we invite the reader to consult [18]. A *differential graded (=dg) category*  $\mathcal{A}$  is a category enriched over complexes of  $k$ -vector spaces. Let us write  $\text{dgc}at(k)$  for the category of (small) dg categories.

Let  $\mathcal{A}$  be a dg category. The opposite dg category  $\mathcal{A}^{\text{op}}$  has the same objects and  $\mathcal{A}^{\text{op}}(x, y) := \mathcal{A}(y, x)$ . A *right dg  $\mathcal{A}$ -module* is a dg functor  $M: \mathcal{A}^{\text{op}} \rightarrow \mathcal{C}_{\text{dg}}(k)$  with values in the dg category  $\mathcal{C}_{\text{dg}}(k)$  of complexes of  $k$ -vector spaces. Following [18, §3.2], the derived category  $\mathcal{D}(\mathcal{A})$  of  $\mathcal{A}$  is defined as the localization of the category of right dg  $\mathcal{A}$ -modules  $\mathcal{C}(\mathcal{A})$  with respect to the objectwise quasi-isomorphisms. Let us write  $\mathcal{D}_c(\mathcal{A})$  for the triangulated subcategory of compact objects.

A dg functor  $F: \mathcal{A} \rightarrow \mathcal{B}$  is called a *Morita equivalence* if it induces an equivalence on derived categories  $\mathcal{D}(\mathcal{A}) \simeq \mathcal{D}(\mathcal{B})$ ; see [18, §4.6]. As explained in [36, §1.6], the category  $\text{dgc}at(k)$  admits a Quillen model structure whose weak equivalences are the Morita equivalences. Let us denote by  $\text{Hmo}(k)$  the associated homotopy category.

The *tensor product*  $\mathcal{A} \otimes \mathcal{B}$  of dg categories is defined as follows: the set of objects is  $\text{obj}(\mathcal{A}) \times \text{obj}(\mathcal{B})$  and  $(\mathcal{A} \otimes \mathcal{B})((x, w), (y, z)) := \mathcal{A}(x, y) \otimes \mathcal{B}(w, z)$ . As explained in [18, §2.3], this construction gives rise to a symmetric monoidal structure  $- \otimes -$  on  $\text{dgc}at(k)$  which descends to the homotopy category  $\text{Hmo}(k)$ .

A *dg  $\mathcal{A}$ - $\mathcal{B}$ -bimodule* is a dg functor  $B: \mathcal{A} \otimes \mathcal{B}^{\text{op}} \rightarrow \mathcal{C}_{\text{dg}}(k)$  or, equivalently, a right dg  $(\mathcal{A}^{\text{op}} \otimes \mathcal{B})$ -module. A standard example is the dg  $\mathcal{A}$ - $\mathcal{B}$ -bimodule

$$(3.1) \quad {}_F\mathcal{B}: \mathcal{A} \otimes \mathcal{B}^{\text{op}} \longrightarrow \mathcal{C}_{\text{dg}}(k) \quad (x, z) \mapsto \mathcal{B}(z, F(x))$$

associated to a dg functor  $F: \mathcal{A} \rightarrow \mathcal{B}$ . Following Kontsevich [19, 20, 21], a dg category  $\mathcal{A}$  is called *smooth* if the dg  $\mathcal{A}$ - $\mathcal{A}$ -bimodule  ${}_{\text{id}}\mathcal{A}$  belongs to the category  $\mathcal{D}_c(\mathcal{A}^{\text{op}} \otimes \mathcal{A})$  and *proper* if  $\sum_n \dim H^n \mathcal{A}(x, y) < \infty$  for any pair of objects  $(x, y)$ .

**3.2. Additive invariants.** Given dg categories  $\mathcal{A}$  and  $\mathcal{B}$  and a dg  $\mathcal{A}$ - $\mathcal{B}$ -bimodule  $B$ , consider the following dg category  $T(\mathcal{A}, \mathcal{B}; B)$ : the set of objects is  $\text{obj}(\mathcal{A}) \amalg \text{obj}(\mathcal{B})$ ; the complexes of  $k$ -vector spaces of morphisms  $T(\mathcal{A}, \mathcal{B}; B)(x, y)$  are equal to  $\mathcal{A}(x, y)$  when  $x, y \in \mathcal{A}$ , to  $\mathcal{B}(x, y)$  when  $x, y \in \mathcal{B}$ , to  $B(x, y)$  when  $x \in \mathcal{A}$  and  $y \in \mathcal{B}$ , and to 0 when  $x \in \mathcal{B}$  and  $y \in \mathcal{A}$ ; and the composition law is induced by the composition law of  $\mathcal{A}$  and  $\mathcal{B}$  and by the dg  $\mathcal{A}$ - $\mathcal{B}$ -bimodule structure of  $B$ . By construction, we have canonical dg functors  $\iota_{\mathcal{A}}: \mathcal{A} \rightarrow T(\mathcal{A}, \mathcal{B}; B)$  and  $\iota_{\mathcal{B}}: \mathcal{B} \rightarrow T(\mathcal{A}, \mathcal{B}; B)$ .

Recall from [36, Def. 2.1] that a functor  $E: \text{dgc}at(k) \rightarrow \mathcal{D}$ , with values in an additive category, is called an *additive invariant* if it satisfies the following conditions:

- (i) It sends the Morita equivalences to isomorphisms.

<sup>5</sup>For example, in the case where  $G = \mathbb{Z}/2$ , we can consider the canonical involution  $x \mapsto -x$ .

(ii) Given  $\mathcal{A}$ ,  $\mathcal{B}$  and  $B$ , as above, the dg functors  $\iota_{\mathcal{A}}$  and  $\iota_{\mathcal{B}}$  induce an isomorphism

$$E(\mathcal{A}) \oplus E(\mathcal{B}) \xrightarrow{\simeq} E(T(\mathcal{A}, \mathcal{B}; B)).$$

Let us write  $\text{rep}(\mathcal{A}, \mathcal{B})$  for the full triangulated subcategory of  $\mathcal{D}(\mathcal{A}^{\text{op}} \otimes \mathcal{B})$  consisting of those dg  $\mathcal{A}$ - $\mathcal{B}$ -modules  $B$  such that for every object  $x \in \mathcal{A}$  the associated right dg  $\mathcal{B}$ -module  $B(x, -)$  belongs to  $\mathcal{D}_c(\mathcal{B})$ . As explained in [36, §1.6.3], there is a natural bijection between  $\text{Hom}_{\text{Hmo}(k)}(\mathcal{A}, \mathcal{B})$  and the set of isomorphism classes of the category  $\text{rep}(\mathcal{A}, \mathcal{B})$ . Under this bijection, the composition law of  $\text{Hmo}(k)$  corresponds to the (derived) tensor product of bimodules. Therefore, since the dg  $\mathcal{A}$ - $\mathcal{B}$ -bimodules (3.1) belong to  $\text{rep}(\mathcal{A}, \mathcal{B})$ , we have the following functor:

$$(3.2) \quad \text{dgc}at(k) \longrightarrow \text{Hmo}(k) \quad \mathcal{A} \mapsto \mathcal{A} \quad (\mathcal{A} \xrightarrow{F} \mathcal{B}) \mapsto {}_F\mathcal{B}.$$

The *additivization* of  $\text{Hmo}(k)$  is the additive category  $\text{Hmo}_0(k)$  with the same objects as  $\text{Hmo}(k)$  and with abelian groups of morphisms  $\text{Hom}_{\text{Hmo}_0(k)}(\mathcal{A}, \mathcal{B})$  given by the Grothendieck group  $K_0\text{rep}(\mathcal{A}, \mathcal{B})$  of the triangulated category  $\text{rep}(\mathcal{A}, \mathcal{B})$ . As explained in [36, §2.3], the following composition is the *universal* additive invariant:

$$(3.3) \quad U: \text{dgc}at(k) \xrightarrow{(3.2)} \text{Hmo}(k) \longrightarrow \text{Hmo}_0(k) \quad \mathcal{A} \mapsto \mathcal{A} \quad (\mathcal{A} \xrightarrow{F} \mathcal{B}) \mapsto [{}_F\mathcal{B}].$$

**3.3. Noncommutative motives.** For a book on noncommutative motives, we invite the reader to consult [36]. Recall from [36, §4.1] that the category of *noncommutative Chow motives*  $\text{NChow}(k)_{\mathbb{Q}}$  (with  $\mathbb{Q}$ -coefficients) is defined as the idempotent completion of the full subcategory of  $\text{Hmo}_0(k)_{\mathbb{Q}}$  consisting of those objects  $U(\mathcal{A})_{\mathbb{Q}}$  with  $\mathcal{A}$  a smooth proper dg category. This category  $\text{NChow}(k)_{\mathbb{Q}}$  is  $\mathbb{Q}$ -linear, additive, and rigid symmetric monoidal. Moreover, we have natural isomorphisms:

$$(3.4) \quad \text{Hom}_{\text{NChow}(k)_{\mathbb{Q}}}(U(\mathcal{A})_{\mathbb{Q}}, U(\mathcal{B})_{\mathbb{Q}}) := K_0(\text{rep}(\mathcal{A}^{\text{op}} \otimes \mathcal{B}))_{\mathbb{Q}} \simeq K_0(\mathcal{A}^{\text{op}} \otimes \mathcal{B})_{\mathbb{Q}}.$$

Given a  $\mathbb{Q}$ -linear, additive, rigid symmetric monoidal category  $(\mathcal{C}, \otimes, \mathbf{1})$ , its  $\mathcal{N}$ -ideal is defined as follows ( $\text{tr}(g \circ f)$  stands for the categorical trace of  $g \circ f$ ):

$$\mathcal{N}(a, b) := \{f \in \text{Hom}_{\mathcal{C}}(a, b) \mid \forall g \in \text{Hom}_{\mathcal{C}}(b, a) \text{ we have } \text{tr}(g \circ f) = 0\}.$$

Under these notations, recall from [36, §4.6] that the category of *noncommutative numerical motives*  $\text{NNum}(k)_{\mathbb{Q}}$  (with  $\mathbb{Q}$ -coefficients) is defined as the idempotent completion of the quotient category  $\text{NChow}(k)_{\mathbb{Q}}/\mathcal{N}$ .

#### 4. NONCOMMUTATIVE CONJECTURES

Let  $\mathcal{A}$  be a smooth proper ( $k$ -linear) dg category.

**Noncommutative conjecture**  $\text{T}_{\text{nc}}^l(\mathcal{A})$ . Let  $l \neq p$  be a prime number, Following Thomason [43], consider the  $l$ -adic étale  $K$ -theory groups

$$(4.1) \quad K_n^{\hat{l}, \text{et}}(\mathcal{A} \otimes_k k_s) := \pi_n(\text{holim}_{\nu} L_{KU} \mathbb{K}(\mathcal{A} \otimes_k k_s; \mathbb{Z}/l^{\nu})) \quad n \in \mathbb{Z},$$

where  $\mathbb{K}(\mathcal{A} \otimes_k k_s; \mathbb{Z}/l^{\nu})$  stands for the (non-connective) algebraic  $K$ -theory spectrum with  $\mathbb{Z}/l^{\nu}$ -coefficients of the dg category  $\mathcal{A} \otimes_k k_s$  and  $L_{KU} \mathbb{K}(\mathcal{A} \otimes_k k_s; \mathbb{Z}/l^{\nu})$  for the Bousfield localization of  $\mathbb{K}(\mathcal{A} \otimes_k k_s; \mathbb{Z}/l^{\nu})$  with respect to complex topological  $K$ -theory  $KU$ . Thanks to the work of Suslin [32, 33] and Gabber [12], we have a  $\mathbb{Z}$ -graded ring isomorphism  $K_*^{\hat{l}, \text{et}}(k_s) \simeq \mathbb{Z}_l[t, t^{-1}]$ , where  $t$  is of degree 2. Note that this implies that the above  $K$ -theory groups (4.1) are  $\mathbb{Z}_l$ -modules and that  $K_n^{\hat{l}, \text{et}}(\mathcal{A} \otimes_k k_s) \simeq K_{n+2}^{\hat{l}, \text{et}}(\mathcal{A} \otimes_k k_s)$  for every  $n \in \mathbb{Z}$ . Note also that, by construction,

the absolute Galois group  $\text{Gal}(k_s/k)$  acts on the above  $\mathbb{Z}_l$ -modules (4.1) and that we have a canonical  $\mathbb{Q}_l$ -linear homomorphism:

$$(4.2) \quad K_0(\mathcal{A})_{\mathbb{Q}_l} \longrightarrow K_0^{\hat{l}, \text{et}}(\mathcal{A} \otimes_k k_s)_{1/l}^{\text{Gal}(k_s/k)}.$$

Conjecture  $\overline{T}_{\text{nc}}^l(\mathcal{A})$ : *When  $k$  is finitely generated over its prime field, the above homomorphism (4.2) is surjective.*

**Noncommutative conjecture  $\overline{T}_{\text{nc}}^l(\mathcal{A})$ .** Let  $l \neq p$  be a prime number. Similarly to (4.2), we have a canonical  $\mathbb{Q}_l$ -linear homomorphism

$$(4.3) \quad K_0(\mathcal{A} \otimes_k k')_{\mathbb{Q}_l} \longrightarrow K_0^{\hat{l}, \text{et}}(\mathcal{A} \otimes_k k_s)_{1/l}^{\text{Gal}(k_s/k')}$$

for every finite field extension  $k'/k$  inside  $k_s$ . Moreover, since  $k_s = \bigcup_{k'/k} k'$ , where the union runs over all the finite field extensions  $k'/k$  inside  $k_s$ , and the functors  $\mathcal{A} \otimes_k -$  and  $K_0(-)_{\mathbb{Q}_l}$  preserve filtered colimits, the  $\mathbb{Q}_l$ -linearized Grothendieck group  $K_0(\mathcal{A} \otimes_k k_s)_{\mathbb{Q}_l}$  identifies with  $\text{colim}_{k'/k} K_0(\mathcal{A} \otimes_k k')_{\mathbb{Q}_l}$ . Consequently, we obtain a canonical  $\mathbb{Q}_l$ -linear homomorphism:

$$(4.4) \quad K_0(\mathcal{A} \otimes_k k_s)_{\mathbb{Q}_l} \longrightarrow \bigcup_{k'/k} K_0^{\hat{l}, \text{et}}(\mathcal{A} \otimes_k k_s)_{1/l}^{\text{Gal}(k_s/k')}.$$

Conjecture  $\overline{T}_{\text{nc}}^l(\mathcal{A})$ : *When  $k$  is finitely generated over its prime field, the above homomorphism (4.4) is surjective.*

**Lemma 4.5.** *We have the implication  $\overline{T}_{\text{nc}}^l(\mathcal{A}) \Rightarrow T_{\text{nc}}^l(\mathcal{A})$ .*

*Proof.* Let  $\beta$  be an element of  $K_0^{\hat{l}, \text{et}}(\mathcal{A} \otimes_k k_s)_{1/l}$  which is fixed under the  $\text{Gal}(k_s/k)$ -action. We need to construct an element  $\alpha$  of  $K_0(\mathcal{A})_{\mathbb{Q}_l}$  which is mapped to  $\beta$  by the homomorphism (4.2). Since the conjecture  $\overline{T}_{\text{nc}}^l(\mathcal{A})$  holds, there exists a finite field extension  $k'/k$  inside  $k_s$  (which we can assume without loss of generality to be Galois<sup>6</sup>) and an element  $\alpha'$  of  $K_0(\mathcal{A} \otimes_k k')_{\mathbb{Q}_l}$  which is mapped to  $\beta$  by the homomorphism (4.3). Recall from [27, §7] that we have well-defined  $\mathbb{Q}_l$ -homomorphisms

$$- \otimes_k k' : K_0(\mathcal{A})_{\mathbb{Q}_l} \longrightarrow K_0(\mathcal{A} \otimes_k k')_{\mathbb{Q}_l} \quad \text{res}_{k'/k} : K_0(\mathcal{A} \otimes_k k')_{\mathbb{Q}_l} \longrightarrow K_0(\mathcal{A})_{\mathbb{Q}_l}$$

such that  $\text{res}_{k'/k}(\alpha') \otimes_k k' = \sum_{\sigma \in \text{Gal}(k'/k)} \sigma(\alpha')$ . Let us take  $\alpha := \frac{1}{[k':k]} \text{res}_{k'/k}(\alpha')$ . Since  $\beta$  is fixed by the  $\text{Gal}(k_s/k)$ -action and (4.3) is  $\text{Gal}(k'/k)$ -equivariant, we hence conclude that  $\alpha$  is mapped to  $\beta$  by the homomorphism (4.2).  $\square$

**Noncommutative conjecture  $\text{SS}_{\text{nc}}^l(\mathcal{A})$ .** Recall from above that, up to isomorphism, we have two (continuous)  $l$ -adic representations

$$(4.6) \quad K_0^{\hat{l}, \text{et}}(\mathcal{A} \otimes_k k_s)_{1/l} \quad K_1^{\hat{l}, \text{et}}(\mathcal{A} \otimes_k k_s)_{1/l}$$

of the absolute Galois group  $\text{Gal}(k_s/k)$ .

Conjecture  $\text{SS}_{\text{nc}}^l(\mathcal{A})$ : *When  $k$  is finitely generated over its prime field, the above  $l$ -adic representations (4.6) are semi-simple.*

<sup>6</sup>If the field extension  $k'/k$  is not Galois, take the normal closure of  $k'$  inside  $k_s$ .

**Noncommutative conjecture**  $TP^p(\mathcal{A})$ . Let  $k := \mathbb{F}_q$  be a finite field of characteristic  $p > 0$ ,  $W(k)$  the associated ring of  $p$ -typical Witt vectors,  $K := W(k)_{1/p}$  the fraction field of  $W(k)$ , and  $\sigma: K \xrightarrow{\sim} K$  the automorphism induced by the Frobenius map  $\lambda \mapsto \lambda^p$  on  $k$ . Recall that the associated field extension  $K/\mathbb{Q}_p$  is finite.

Consider the topological Hochschild homology  $THH(\mathcal{A})$  of the dg category  $\mathcal{A}$ . The canonical  $S^1$ -action on  $THH(\mathcal{A})$  gives rise to the spectrum of homotopy orbits  $THH(\mathcal{A})_{hS^1}$ , to the spectrum of homotopy fixed-points  $TC^-(\mathcal{A}) := THH(\mathcal{A})^{hS^1}$ , and also to the Tate construction  $TP(\mathcal{A}) := THH(\mathcal{A})^{tS^1}$ . As explained in [30, Cor. I.4.3], these spectra are related by the following cofiber sequence

$$(4.7) \quad \Sigma THH(\mathcal{A})_{hS^1} \xrightarrow{N} THH(\mathcal{A})^{hS^1} \xrightarrow{\text{can}} THH(\mathcal{A})^{tS^1},$$

where  $N$  is the norm map. It is well-known that the abelian groups  $THH_*(\mathcal{A})$  are  $k$ -linear. Hence, the spectrum  $\Sigma THH(\mathcal{A})_{hS^1}$  becomes trivial after inverting  $p$ . Consequently, the cofiber sequence (4.7) leads to a canonical isomorphism:

$$(4.8) \quad \text{can}: TC_0^-(\mathcal{A})_{1/p} \xrightarrow{\cong} TP_0(\mathcal{A})_{1/p}.$$

Recall that  $TP_0(\mathcal{A})_{1/p}$  is a finitely-generated module over  $TP_0(k)_{1/p} \simeq K$ , i.e., a finite-dimensional  $K$ -vector space. Since  $\mathcal{A}$  is smooth and proper, it is also well-known that the spectrum  $THH(\mathcal{A})$  is bounded below and  $p$ -complete. Making use of [30, Lem. II 4.2], we hence obtain moreover a ‘‘cyclotomic Frobenius’’:

$$(4.9) \quad \varphi_p: TC_0^-(\mathcal{A})_{1/p} \longrightarrow TP_0(\mathcal{A})_{1/p}.$$

Let us write  $\varphi := \varphi_p \circ \text{can}^{-1}$  for the associated endomorphism of  $TP_0(\mathcal{A})_{1/p}$ . This endomorphism  $\varphi$  is *not*  $K$ -linear but only  $\sigma$ -semilinear.

Now, recall from [34, Prop. 4.2] that the assignment  $\mathcal{A} \mapsto TP_0(\mathcal{A})_{1/p}$  gives rise to a  $\mathbb{Q}_p$ -linear functor from  $\text{NChow}(k)_{\mathbb{Q}_p}$  to the category of  $K$ -vector spaces  $\text{Vect}(K)$ . By composing it with the forgetful functor from  $K$ -vector spaces to  $\mathbb{Q}_p$ -vector spaces, we hence obtain the following  $\mathbb{Q}_p$ -linear functor:

$$(4.10) \quad TP_0(-)_{1/p}: \text{NChow}(k)_{\mathbb{Q}_p} \longrightarrow \text{Vect}(\mathbb{Q}_p).$$

This leads, in particular, to the induced  $\mathbb{Q}_p$ -linear homomorphism:

$$(4.11) \quad \begin{array}{c} K_0(\mathcal{A})_{\mathbb{Q}_p} \simeq \text{Hom}_{\text{NChow}(k)_{\mathbb{Q}_p}}(U(k)_{\mathbb{Q}_p}, U(\mathcal{A})_{\mathbb{Q}_p}) \\ \downarrow \\ TP_0(\mathcal{A})_{1/p} \simeq \text{Hom}_{\text{Vect}(\mathbb{Q}_p)}(TP_0(k)_{1/p}, TP_0(\mathcal{A})_{1/p}). \end{array}$$

**Lemma 4.12.** *The preceding homomorphism (4.11) take values in the  $\mathbb{Q}_p$ -linear subspace  $TP_0(\mathcal{A})_{1/p}^\varphi$  of those elements that are fixed by  $\varphi$ .*

*Proof.* On the one hand, the  $\mathbb{Q}_p$ -linear endomorphisms  $\varphi: TP_0(\mathcal{A})_{1/p} \rightarrow TP_0(\mathcal{A})_{1/p}$  (parametrized by the smooth proper dg categories  $\mathcal{A}$ ) give rise to a natural transformation from (4.10) to itself. On the other hand, thanks to the enriched Yoneda lemma, the  $\mathbb{Q}_p$ -linear natural transformations from the following functor

$$K_0(-)_{\mathbb{Q}_p} \simeq \text{Hom}_{\text{NChow}(k)_{\mathbb{Q}_p}}(U(k)_{\mathbb{Q}_p}, -): \text{NChow}(k)_{\mathbb{Q}_p} \longrightarrow \text{Vect}(\mathbb{Q}_p)$$

to the above functor (4.10) are in one-to-one correspondence with the elements of the  $\mathbb{Q}_p$ -vector space  $TP_0(k)_{1/p} \simeq K$ . Under this bijection, the unit element  $1 \in K$  corresponds to the above homomorphism (4.11). Therefore, in order to prove

Lemma 4.12, it suffices to show that the endomorphism  $\varphi: TP_0(k)_{1/p} \rightarrow TP_0(k)_{1/p}$  sends 1 to 1. This follows from the following explicit descriptions

$$\begin{aligned} \text{can}: W(k)[u, v]/(uv - p) &\longrightarrow W(k)[\delta, \delta^{-1}] & u &\mapsto p\delta & v &\mapsto \delta^{-1} \\ \varphi_p: W(k)[u, v]/(uv - p) &\longrightarrow W(k)[\delta, \delta^{-1}] & u &\mapsto \delta & v &\mapsto p\delta^{-1} \end{aligned}$$

of the homomorphisms  $\text{can}, \varphi_p: TC_*^-(k) \rightarrow TP_*(k)$ , where the variables  $u$  and  $\delta$  have degree 2 and the variable  $v$  has degree  $-2$ ; consult [6, Props. 6.2-6.3].  $\square$

Thanks to Lemma 4.12, we have an induced  $\mathbb{Q}_p$ -linear homomorphism:

$$(4.13) \quad K_0(\mathcal{A})_{\mathbb{Q}_p} \longrightarrow TP_0(\mathcal{A})_{1/p}^\varphi.$$

Conjecture  $T_{\text{nc}}^p(\mathcal{A})$ : *The above homomorphism (4.13) is surjective.*

**Noncommutative conjecture  $B_{\text{nc}}(\mathcal{A})$ .** Recall from [36, §4.7] that the Grothendieck group  $K_0(\mathcal{A}) := K_0(\mathcal{D}_c(\mathcal{A}))$  of the dg category  $\mathcal{A}$  is equipped with the Euler pairing  $\chi: K_0(\mathcal{A}) \times K_0(\mathcal{A}) \rightarrow \mathbb{Z}$  defined as  $([M], [N]) \mapsto \sum_n (-1)^n \dim \text{Hom}_{\mathcal{D}_c(\mathcal{A})}(M, N[-n])$ . This bilinear pairing is not symmetric neither skew-symmetric. Nevertheless, as proved in [36, Prop. 4.24], the left and right kernels of  $\chi$  agree. Consequently, we have a well-defined numerical Grothendieck group  $K_0(\mathcal{A})/\sim_{\text{num}} := K_0(\mathcal{A})/\text{Ker}(\chi)$ . In what follows, we will write  $K_0(\mathcal{A})_{\mathbb{Q}}/\sim_{\text{num}}$  for the associated  $\mathbb{Q}$ -vector space  $(K_0(\mathcal{A})/\sim_{\text{num}})_{\mathbb{Q}}$ , which is isomorphic to the quotient  $K_0(\mathcal{A})_{\mathbb{Q}}/\text{Ker}(\chi_{\mathbb{Q}})$ .

Conjecture  $B_{\text{nc}}(\mathcal{A})$ : *When  $k$  is finite, we have  $K_0(\mathcal{A})_{\mathbb{Q}} = K_0(\mathcal{A})_{\mathbb{Q}}/\sim_{\text{num}}$ .*

## 5. PROOF OF THEOREM 2.1

We start by proving the equivalence  $T^l(X) \Leftrightarrow T_{\text{nc}}^l(\text{perf}_{\text{dg}}(X))$ . On the one hand, recall from [11, §18.3] that since  $\mathbb{Q} \subset \mathbb{Q}_l$  we have a natural isomorphism between the  $\mathbb{Q}_l$ -linearized Grothendieck group  $K_0(\text{perf}_{\text{dg}}(X))_{\mathbb{Q}_l} \simeq K_0(X)_{\mathbb{Q}_l}$  and the direct sum  $\bigoplus_{i=0}^{\dim(X)} \mathcal{Z}^i(X)_{\mathbb{Q}_l}$ . On the other hand, it follows from the canonical Morita equivalence  $\text{perf}_{\text{dg}}(X) \otimes_k k_s \rightarrow \text{perf}_{\text{dg}}(X_{k_s})$  and from the work of Thomason [43, Thm. 4.1] and Soulé [45, §3.3.2] that we have a natural isomorphism between the  $\mathbb{Z}[1/l]$ -linearized  $l$ -adic étale  $K$ -theory group  $K_0^{l, \text{et}}(\text{perf}_{\text{dg}}(X) \otimes_k k_s)_{1/l}$  and the direct sum  $\bigoplus_{i=0}^{\dim(X)} H_{\text{et}}^{2i}(X_{k_s}, \mathbb{Q}_l(i))$ . Moreover, as explained by Friedlander in [10, §5], under the preceding isomorphisms the homomorphism (4.2) (with  $\mathcal{A} = \text{perf}_{\text{dg}}(X)$ ) corresponds to the graded homomorphism (1.1). Consequently, (1.1) is surjective if and only if (4.2) is surjective.

We now prove the equivalence  $\overline{T}^l(X) \Leftrightarrow \overline{T}_{\text{nc}}^l(\text{perf}_{\text{dg}}(X))$ . Recall from §4 that the canonical  $\mathbb{Q}_l$ -linear homomorphism (4.4) (with  $\mathcal{A} = \text{perf}_{\text{dg}}(X)$ ) may be described as

$$\text{colim}_{k'/k} \left( K_0(\text{perf}_{\text{dg}}(X) \otimes_k k')_{\mathbb{Q}_l} \xrightarrow{(4.3)} K_0^{l, \text{et}}(\text{perf}_{\text{dg}}(X) \otimes_k k_s)_{1/l}^{\text{Gal}(k_s/k')} \right),$$

where  $k'/k$  is a finite field extension inside  $k_s$ . Similarly, the canonical  $\mathbb{Q}_l$ -linear homomorphism (1.2) may be described as the following filtered colimit:

$$\text{colim}_{k'/k} \left( \mathcal{Z}^*(X_{k'})_{\mathbb{Q}_l} \longrightarrow H_{\text{et}}^{2*}(X_{k_s}, \mathbb{Q}_l(*))_{\text{Gal}(k_s/k')} \right).$$

Consequently, the proof is now similar to the proof of  $T^l(X) \Leftrightarrow T_{\text{nc}}^l(\text{perf}_{\text{dg}}(X))$  with  $k$  replaced by a finite field extension  $k'$  of  $k$  inside  $k_s$ .

We now prove the equivalence  $\text{SS}^l(X) \Leftrightarrow \text{SS}_{\text{nc}}^l(\text{perf}_{\text{dg}}(X))$ . As above, by combining the canonical Morita equivalence  $\text{perf}_{\text{dg}}(X) \otimes_k k_s \rightarrow \text{perf}_{\text{dg}}(X_{k_s})$  with the work

of Thomason [43, Thm. 4.1] and Soulé [45, §3.3.2], we obtain natural isomorphisms between the  $l$ -adic (continuous) representations

$$K_0^{\hat{l}, \text{et}}(\text{perf}_{\text{dg}}(X) \otimes_k k_s)_{1/l} \quad K_1^{\hat{l}, \text{et}}(\text{perf}_{\text{dg}}(X) \otimes_k k_s)_{1/l}$$

of the absolute Galois group  $\text{Gal}(k_s/k)$  and the direct sums  $\bigoplus_{i=0}^{\dim(X)} H_{\text{et}}^{2i}(X_{k_s}, \mathbb{Q}_l(i))$  and  $\bigoplus_{i=0}^{\dim(X)} H_{\text{et}}^{2i+1}(X_{k_s}, \mathbb{Q}_l(i))$ , respectively. Note that these direct sums are semi-simple if and only if the  $l$ -adic representations  $\{H_{\text{et}}^{2i}(X_{k_s}, \mathbb{Q}_l) \mid 0 \leq i \leq \dim(X)\}$  and  $\{H_{\text{et}}^{2i+1}(X_{k_s}, \mathbb{Q}_l) \mid 0 \leq i \leq \dim(X)\}$  are semi-simple. Therefore, we conclude that the  $l$ -adic representations (4.6) (with  $\mathcal{A} = \text{perf}_{\text{dg}}(X)$ ) are semi-simple if and only if the  $l$ -adic representations  $\{H_{\text{et}}^n(X_{k_s}, \mathbb{Q}_l) \mid 0 \leq n \leq 2\dim(X)\}$  are semi-simple.

We now prove the equivalence  $\text{T}^p(X) \Leftrightarrow \text{T}_{\text{nc}}^p(\text{perf}_{\text{dg}}(X))$ . Note first that we have the following equality of graded  $\mathbb{Q}_l$ -vector spaces  $H_{\text{crys}}^{2*}(X)(*)^\phi = H_{\text{crys}}^{2*}(X)^{\frac{1}{p^*}\phi}$ , where  $\phi$  stands for the crystalline Frobenius. Therefore, the conjecture  $\text{T}^p(X)$  may be re-formulated as the surjectivity of the classical cycle class map:

$$(5.1) \quad \mathcal{Z}^*(X)_{\mathbb{Q}_p} \longrightarrow H_{\text{crys}}^{2*}(X)^{\frac{1}{p^*}\phi}.$$

On the one hand, recall from [11, §18.3] that since  $\mathbb{Q} \subset \mathbb{Q}_l$  we have a natural isomorphism between the  $\mathbb{Q}_p$ -linearized Grothendieck group  $K_0(\text{perf}_{\text{dg}}(X))_{\mathbb{Q}_p} \simeq K_0(X)_{\mathbb{Q}_p}$  and the direct sum  $\bigoplus_{i=0}^{\dim(X)} \mathcal{Z}^i(X)_{\mathbb{Q}_p}$ . On the other hand, recall from [35, Thm. 5.2] that we have a natural isomorphism between the  $\mathbb{Z}[1/p]$ -linearized topological periodic cyclic homology group  $TP_0(\text{perf}_{\text{dg}}(X))_{1/p}$  and the direct sum  $\bigoplus_{i=0}^{\dim(X)} H_{\text{crys}}^{2i}(X)$ , which identifies  $\varphi$  with  $\frac{1}{p^*}\phi$ . Under the preceding isomorphisms, the homomorphism (4.13) (with  $\mathcal{A} = \text{perf}_{\text{dg}}(X)$ ) corresponds to the graded homomorphism (5.1). Consequently, (5.1) is surjective if and only if (4.13) is surjective.

Finally, we prove the equivalence  $\text{B}(X) \Leftrightarrow \text{B}_{\text{nc}}(\text{perf}_{\text{dg}}(X))$ . Note first that since  $\mathcal{D}_c(\text{perf}_{\text{dg}}(X)) \simeq \text{perf}(X)$ , the Euler pairing  $\chi: K_0(X) \times K_0(X) \rightarrow \mathbb{Z}$  may be re-written as  $([\mathcal{F}], [\mathcal{G}]) \mapsto \sum_n (-1)^n \dim \text{Hom}_{\text{perf}(X)}(\mathcal{F}, \mathcal{G}[-n])$ . Recall from [11, §19] that a graded algebraic cycle  $\alpha \in \mathcal{Z}^*(X)_{\mathbb{Q}}$  is *numerically equivalent to zero* if  $\int_X \beta \cdot \alpha = 0$  for every  $\beta \in \mathcal{Z}^*(X)_{\mathbb{Q}}$ . Recall also that we have the isomorphism

$$(5.2) \quad \tau: K_0(X)_{\mathbb{Q}} \xrightarrow{\simeq} \bigoplus_{i=0}^{\dim(X)} \mathcal{Z}^i(X)_{\mathbb{Q}} \quad [\mathcal{F}] \mapsto \text{ch}(\mathcal{F}) \cdot \sqrt{\text{Td}_X},$$

where  $\text{ch}(\mathcal{F})$  stands for the Chern character of  $\mathcal{F}$  and  $\sqrt{\text{Td}_X}$  for the square root of the Todd class; see [11, §18.3]. Given any two perfect complexes  $\mathcal{F}, \mathcal{G} \in \text{perf}(X)$ , the Hirzebruch-Riemann-Roch theorem (see [11, Cor. 18.3.1]) yields the equality  $\text{Eu}(\pi_*(\mathcal{F}^\vee \otimes_{\mathcal{O}_X} \mathcal{G})) = \int_X \tau([\mathcal{F}^\vee]) \cdot \tau([\mathcal{G}])$ , where  $\text{Eu}$  denotes the Euler characteristic and  $\pi: X \rightarrow \text{Spec}(k)$  is the structural morphism of  $X$ . Since  $\mathcal{F}^\vee \otimes_{\mathcal{O}_X} \mathcal{G} \simeq \underline{\text{Hom}}(\mathcal{F}, \mathcal{G})$ , where  $\underline{\text{Hom}}(-, -)$  stands for the internal Hom of the rigid symmetric monoidal category  $\text{perf}(X)$ , we hence conclude that  $\text{Eu}(\pi_*(\underline{\text{Hom}}(\mathcal{F}, \mathcal{G})))$  agrees with  $\chi([\mathcal{F}], [\mathcal{G}])$ . This implies that (5.2) descends to the numerical quotients:

$$(5.3) \quad \begin{array}{ccc} K_0(X)_{\mathbb{Q}} & \xrightarrow[\simeq]{\tau} & \bigoplus_{i=0}^{\dim(X)} \mathcal{Z}^i(X)_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ K_0(X)_{\mathbb{Q}}/\sim_{\text{num}} & \xrightarrow[\tau]{\simeq} & \bigoplus_{i=0}^{\dim(X)} \mathcal{Z}^i(X)_{\mathbb{Q}}/\sim_{\text{num}}. \end{array}$$

Consequently, the proof follows now from the fact that the conjecture  $B(X)$ , resp.  $B_{\text{nc}}(\text{perf}_{\text{dg}}(X))$ , is equivalent to the injectivity of the vertical graded homomorphism on the right-hand side of (5.3), resp. to the injectivity of the vertical homomorphism on the left-hand side of (5.3).

## 6. PROOF OF THEOREM 2.2

By definition of the Lefschetz decomposition  $\langle \mathbb{A}_0, \mathbb{A}_1(1), \dots, \mathbb{A}_{i-1}(i-1) \rangle$ , we have a chain of admissible triangulated subcategories  $\mathbb{A}_{i-1} \subseteq \dots \subseteq \mathbb{A}_1 \subseteq \mathbb{A}_0$  with  $\mathbb{A}_r(r) := \mathbb{A}_r \otimes \mathcal{L}_X(r)$ . Note that  $\mathbb{A}_r(r) \simeq \mathbb{A}_r$ . Let  $\mathfrak{a}_r$  be the right orthogonal complement to  $\mathbb{A}_{r+1}$  in  $\mathbb{A}_r$ ; these are called the *primitive subcategories* in [24, §4]. By construction, we have the following semi-orthogonal decompositions:

$$(6.1) \quad \mathbb{A}_r = \langle \mathfrak{a}_r, \mathfrak{a}_{r+1}, \dots, \mathfrak{a}_{i-1} \rangle \quad 0 \leq r \leq i-1.$$

As proved in [24, Thm. 6.3] (see also [2, Thm. 2.3.4]), the category  $\text{perf}(Y)$  admits a HP-dual Lefschetz decomposition  $\langle \mathbb{B}_{j-1}(1-j), \mathbb{B}_{j-2}(2-j), \dots, \mathbb{B}_0 \rangle$  with respect to  $\mathcal{L}_Y(1)$ ; as above, we have a chain of admissible triangulated subcategories  $\mathbb{B}_{j-1} \subseteq \mathbb{B}_{j-2} \subseteq \dots \subseteq \mathbb{B}_0$ . Moreover, the primitive subcategories coincide (via a Fourier-Mukai type functor) with those of  $\text{perf}(X)$  and we have the following semi-orthogonal decompositions:

$$(6.2) \quad \mathbb{B}_r = \langle \mathfrak{a}_0, \mathfrak{a}_1, \dots, \mathfrak{a}_{\dim(V)-r-2} \rangle \quad 0 \leq r \leq j-1.$$

Furthermore, the assumptions (a)-(b) of Theorem 2.2 imply the existence of the following semi-orthogonal decompositions

$$(6.3) \quad \text{perf}(X_L) = \langle \mathbb{C}_L, \mathbb{A}_{\dim(V)}(1), \dots, \mathbb{A}_{i-1}(i - \dim(V)) \rangle$$

$$(6.4) \quad \text{perf}(Y_L) = \langle \mathbb{B}_{j-1}(\dim(L^\perp) - j), \dots, \mathbb{B}_{\dim(L^\perp)}(-1), \mathbb{C}_L \rangle,$$

where  $\mathbb{C}_L$  is a common (triangulated) category. Let us denote by  $\mathbb{C}_L^{\text{dg}}$ ,  $\mathbb{A}_r^{\text{dg}}$ , and  $\mathfrak{a}_r^{\text{dg}}$ , the dg enhancement of  $\mathbb{C}_L$ ,  $\mathbb{A}_r$ , and  $\mathfrak{a}_r$ , induced from the dg category  $\text{perf}_{\text{dg}}(X_L)$ . Similarly, let us denote by  $\mathbb{C}_L^{\text{dg}'}$  and  $\mathbb{B}_r^{\text{dg}}$  the dg enhancement of  $\mathbb{C}_L$  and  $\mathbb{B}_r$  induced from the dg category  $\text{perf}_{\text{dg}}(Y_L)$ . Note that thanks to assumption (a) of Theorem 2.2 all the above dg categories are smooth and proper.

We start by proving the equivalence  $T^l(X_L) \Leftrightarrow T^l(Y_L)$ . As explained in [36, Prop. 2.2], since the functor (3.3) is an additive invariant, the above semi-orthogonal decomposition (6.3) gives rise to the following direct sum decomposition

$$(6.5) \quad U(\text{perf}_{\text{dg}}(X_L))_{\mathbb{Q}_l} \simeq U(\mathbb{C}_L^{\text{dg}})_{\mathbb{Q}_l} \oplus U(\mathbb{A}_{\dim(V)}^{\text{dg}})_{\mathbb{Q}_l} \oplus \dots \oplus U(\mathbb{A}_{i-1}^{\text{dg}})_{\mathbb{Q}_l}$$

in the  $\mathbb{Q}_l$ -linearized category  $\text{NChow}(k)_{\mathbb{Q}_l}$ . Consider the following functors

$$(6.6) \quad K_n^{\hat{l}, \text{et}}(- \otimes_k k_s)_{1/l} : \text{dgc}at(k) \longrightarrow \text{Vect}(\mathbb{Q}_l) \quad n \in \mathbb{Z}$$

with values in the category of  $\mathbb{Q}_l$ -vector spaces.

**Proposition 6.7.** *The above functors (6.6) are additive invariants.*

*Proof.* Let  $F: \mathcal{A} \rightarrow \mathcal{B}$  be a Morita equivalence. As proved in [27, Prop. 7.1], the induced dg functor  $F \otimes_k k_s: \mathcal{A} \otimes_k k_s \rightarrow \mathcal{B} \otimes_k k_s$  is also a Morita equivalence. Therefore, since (non-connective) algebraic  $K$ -theory with  $\mathbb{Z}/l^\nu$ -coefficients sends Morita equivalences to equivalences of spectra (consult [36, §2.2.2]), we conclude that the above functors (6.6) sends Morita equivalences to isomorphisms.

Let  $\mathcal{A}$  and  $\mathcal{B}$  be two dg categories and  $B$  a dg  $\mathcal{A}$ - $\mathcal{B}$ -bimodule. Following §3.2, we need to show that the dg functors  $\iota_{\mathcal{A}}$  and  $\iota_{\mathcal{B}}$  induce an isomorphism:

$$(6.8) \quad K_n^{\hat{i}, \text{et}}(\mathcal{A} \otimes_k k_s)_{1/l} \oplus K_n^{\hat{i}, \text{et}}(\mathcal{B} \otimes_k k_s)_{1/l} \longrightarrow K_n^{\hat{i}, \text{et}}(T(\mathcal{A}, \mathcal{B}; B))_{1/l}.$$

Consider the dg categories  $\mathcal{A} \otimes_k k_s$  and  $\mathcal{B} \otimes_k k_s$  and the dg bimodule  $B \otimes_k k_s$ . Since algebraic  $K$ -theory with  $\mathbb{Z}/l^\nu$ -coefficients is an additive invariant, the dg functors  $\iota_{\mathcal{A} \otimes_k k_s}$  and  $\iota_{\mathcal{B} \otimes_k k_s}$  induce an equivalence of spectra between the wedge sum  $\mathbb{K}(\mathcal{A} \otimes_k k_s; \mathbb{Z}/l^\nu) \vee \mathbb{K}(\mathcal{B} \otimes_k k_s; \mathbb{Z}/l^\nu)$  and  $\mathbb{K}(T(\mathcal{A} \otimes_k k_s, \mathcal{B} \otimes_k k_s; B \otimes_k k_s); \mathbb{Z}/l^\nu)$ . Therefore, using the fact that the dg category  $T(\mathcal{A} \otimes_k k_s, \mathcal{B} \otimes_k k_s; B \otimes_k k_s)$  is equal to the dg category  $T(\mathcal{A}, \mathcal{B}; B) \otimes_k k_s$ , we conclude from the definition of the above functors (6.6) that (6.8) is indeed an isomorphism.  $\square$

Proposition 6.7 yields, in particular, the following  $\mathbb{Q}_l$ -linear functor:

$$(6.9) \quad K_0^{\hat{i}, \text{et}}(- \otimes_k k_s)_{1/l} : \text{NChow}(k)_{\mathbb{Q}_l} \longrightarrow \text{Vect}(\mathbb{Q}_l).$$

Making use of the functor (6.9), the canonical  $\mathbb{Q}_l$ -linear homomorphism (4.2) (with  $\mathcal{A} = \text{perf}_{\text{dg}}(X_L)$ ) may be described as the induced homomorphism

$$(6.10) \quad \begin{array}{c} \text{Hom}_{\text{NChow}(k)_{\mathbb{Q}_l}}(U(k)_{\mathbb{Q}_l}, U(\text{perf}_{\text{dg}}(X_L))_{\mathbb{Q}_l}) \\ \downarrow \\ \text{Hom}_{\text{Vect}(\mathbb{Q}_l)}(K_0^{\hat{i}, \text{et}}(k_s)_{1/l}, K_0^{\hat{i}, \text{et}}(\text{perf}_{\text{dg}}(X_L) \otimes_k k_s)_{1/l}); \end{array}$$

note that (6.10) takes values in  $K_0^{\hat{i}, \text{et}}(\text{perf}_{\text{dg}}(X_L) \otimes_k k_s)_{1/l}^{\text{Gal}(k_s/k)}$ . Therefore, thanks to the above direct sum decomposition (6.5), the induced homomorphism (6.10) identifies with the (diagonal)  $\mathbb{Q}_l$ -linear homomorphism:

$$\begin{array}{c} K_0(\mathbb{C}_L^{\text{dg}})_{\mathbb{Q}_l} \oplus \bigoplus_{r=\dim(V)}^{i-1} K_0(\mathbb{A}_r^{\text{dg}})_{\mathbb{Q}_l} \\ \downarrow \\ K_0^{\hat{i}, \text{et}}(\mathbb{C}_L^{\text{dg}} \otimes_k k_s)_{1/l}^{\text{Gal}(k_s/k)} \oplus \bigoplus_{r=\dim(V)}^{i-1} K_0^{\hat{i}, \text{et}}(\mathbb{A}_r^{\text{dg}} \otimes_k k_s)_{1/l}^{\text{Gal}(k_s/k)}. \end{array}$$

This implies the following equivalence:

$$(6.11) \quad T_{\text{nc}}^l(\text{perf}_{\text{dg}}(X_L)) \Leftrightarrow T_{\text{nc}}^l(\mathbb{C}_L^{\text{dg}}) + T_{\text{nc}}^l(\mathbb{A}_{\dim(V)}^{\text{dg}}) + \cdots + T_{\text{nc}}^l(\mathbb{A}_{i-1}^{\text{dg}}).$$

Note that all the above holds *mutatis mutandis* with  $X_L$  replaced by  $Y_L$ . Hence, the semi-orthogonal decomposition (6.4) leads to the equivalence:

$$(6.12) \quad T_{\text{nc}}^l(\text{perf}_{\text{dg}}(Y_L)) \Leftrightarrow T_{\text{nc}}^l(\mathbb{B}_{j-1}^{\text{dg}}) + \cdots + T_{\text{nc}}^l(\mathbb{B}_{\dim(L^\perp)}^{\text{dg}}) + T_{\text{nc}}^l(\mathbb{C}_L^{\text{dg}'}) .$$

The assumption (c) of Theorem 2.2 and the semi-orthogonal decompositions (6.1)-(6.2) imply that the conjectures  $T_{\text{nc}}^l(\mathbb{A}_r^{\text{dg}})$  and  $T_{\text{nc}}^l(\mathbb{B}_r^{\text{dg}})$ , with  $0 \leq r \leq i-1$ , hold. Consequently, the right-hand side of (6.15), resp. (6.16), reduces to the conjecture  $T_{\text{nc}}^l(\mathbb{C}_L^{\text{dg}})$ , resp.  $T_{\text{nc}}^l(\mathbb{C}_L^{\text{dg}'})$ . Moreover, since the functor  $\text{perf}(X_L) \rightarrow \mathbb{C}_L \rightarrow \text{perf}(Y_L)$  is of Fourier-Mukai type, the dg categories  $\mathbb{C}_L^{\text{dg}}$  and  $\mathbb{C}_L^{\text{dg}'}$  are Morita equivalent. This implies that the conjectures  $T_{\text{nc}}^l(\mathbb{C}_L^{\text{dg}})$  and  $T_{\text{nc}}^l(\mathbb{C}_L^{\text{dg}'})$  are equivalent. Consequently, the proof of Theorem 2.2 follows now from the equivalences  $T^l(X_L) \Leftrightarrow T_{\text{nc}}^l(\text{perf}_{\text{dg}}(X_L))$  and  $T^l(Y_L) \Leftrightarrow T_{\text{nc}}^l(\text{perf}_{\text{dg}}(Y_L))$  established in Theorem 2.1.

We now prove the equivalence  $\overline{T}^l(X_L) \Leftrightarrow \overline{T}^l(Y_L)$ . Recall from [27, Thm. 7.1] that the functor  $- \otimes_k k_s: \text{dgc}at(k) \rightarrow \text{dgc}at(k_s)$  preserves smooth proper dg categories and gives rise to a  $\mathbb{Q}_l$ -linear functor  $- \otimes_k k_s: \text{NChow}(k)_{\mathbb{Q}_l} \rightarrow \text{NChow}(k_s)_{\mathbb{Q}_l}$ . Consequently, the above direct sum decomposition (6.5) in the  $\mathbb{Q}_l$ -linearized category  $\text{NChow}(k)_{\mathbb{Q}_l}$  gives rise to the following direct sum decomposition

$$U(\text{perf}_{\text{dg}}(X_L) \otimes_k k_s)_{\mathbb{Q}_l} \simeq U(\mathbb{C}_L^{\text{dg}} \otimes_k k_s)_{\mathbb{Q}_l} \oplus U(\mathbb{A}_{\dim(V)}^{\text{dg}} \otimes_k k_s)_{\mathbb{Q}_l} \oplus \cdots \oplus U(\mathbb{A}_{i-1}^{\text{dg}} \otimes_k k_s)_{\mathbb{Q}_l}$$

in the  $\mathbb{Q}_l$ -linearized category  $\text{NChow}(k_s)_{\mathbb{Q}_l}$ . Similarly to (the proof of) Proposition 6.7, the functors  $K_n^{\hat{i}, \text{et}}(-)_{1/l}: \text{dgc}at(k_s) \rightarrow \text{Vect}(\mathbb{Q}_l)$  are additive invariants. This yields, in particular, the following  $\mathbb{Q}_l$ -linear functor:

$$(6.13) \quad K_0^{\hat{i}, \text{et}}(-)_{1/l}: \text{NChow}(k_s)_{\mathbb{Q}_l} \longrightarrow \text{Vect}(\mathbb{Q}_l).$$

Making use of the functor (6.13), the canonical  $\mathbb{Q}_l$ -linear homomorphism (4.4) (with  $\mathcal{A} = \text{perf}_{\text{dg}}(X_L)$ ) may be described as the induced homomorphism

$$(6.14) \quad \begin{array}{c} \text{Hom}_{\text{NChow}(k_s)_{\mathbb{Q}_l}}(U(k_s)_{\mathbb{Q}_l}, U(\text{perf}_{\text{dg}}(X_L) \otimes_k k_s)_{\mathbb{Q}_l}) \\ \downarrow \\ \text{Hom}_{\text{Vect}(\mathbb{Q}_l)}(K_0^{\hat{i}, \text{et}}(k_s)_{1/l}, K_0^{\hat{i}, \text{et}}(\text{perf}_{\text{dg}}(X_L) \otimes_k k_s)_{1/l}); \end{array}$$

note that (6.14) takes values in  $\bigcup_{k'/k} K_0^{\hat{i}, \text{et}}(\text{perf}_{\text{dg}}(X_L) \otimes_k k_s)_{1/l}^{\text{Gal}(k_s/k')}$ . Therefore, thanks to the above direct sum decomposition of  $U(\text{perf}_{\text{dg}}(X_L) \otimes_k k_s)_{\mathbb{Q}_l}$ , the induced homomorphism (6.14) identifies with the (diagonal)  $\mathbb{Q}_l$ -linear homomorphism:

$$\begin{array}{c} K_0(\mathbb{C}_L^{\text{dg}} \otimes_k k_s)_{\mathbb{Q}_l} \oplus \bigoplus_{r=\dim(V)}^{i-1} K_0(\mathbb{A}_r^{\text{dg}} \otimes_k k_s)_{\mathbb{Q}_l} \\ \downarrow \\ \bigcup_{k'/k} K_0^{\hat{i}, \text{et}}(\mathbb{C}_L^{\text{dg}} \otimes_k k_s)_{1/l}^{\text{Gal}(k_s/k')} \oplus \bigoplus_{r=\dim(V)}^{i-1} \bigcup_{k'/k} K_0^{\hat{i}, \text{et}}(\mathbb{A}_r^{\text{dg}} \otimes_k k_s)_{1/l}^{\text{Gal}(k_s/k')}. \end{array}$$

This implies the following equivalence:

$$(6.15) \quad \overline{T}_{\text{nc}}^l(\text{perf}_{\text{dg}}(X_L)) \Leftrightarrow \overline{T}_{\text{nc}}^l(\mathbb{C}_L^{\text{dg}}) + \overline{T}_{\text{nc}}^l(\mathbb{A}_{\dim(V)}^{\text{dg}}) + \cdots + \overline{T}_{\text{nc}}^l(\mathbb{A}_{i-1}^{\text{dg}}).$$

Note that all the above holds *mutatis mutandis* with  $X_L$  replaced by  $Y_L$ . Hence, the semi-orthogonal decomposition (6.4) leads to the equivalence:

$$(6.16) \quad \overline{T}_{\text{nc}}^l(\text{perf}_{\text{dg}}(Y_L)) \Leftrightarrow \overline{T}_{\text{nc}}^l(\mathbb{B}_{j-1}^{\text{dg}}) + \cdots + \overline{T}_{\text{nc}}^l(\mathbb{B}_{\dim(L^\perp)}^{\text{dg}}) + \overline{T}_{\text{nc}}^l(\mathbb{C}_L^{\text{dg}'}) .$$

The remainder of the proof is now similar to the proof of  $T^l(X_L) \Leftrightarrow T^l(Y_L)$ .

We now prove the equivalence  $\text{SS}^l(X_L) \Leftrightarrow \text{SS}^l(Y_L)$ . Note that similarly to (the proof of) Proposition 6.7, the following functors are additive invariants

$$K_n^{\hat{i}, \text{et}}(- \otimes_k k_s)_{1/l}: \text{dgc}at(k) \longrightarrow \text{Rep}_{\text{Gal}(k_s/k)}(\mathbb{Q}_l) \quad n \in \mathbb{Z},$$

where  $\text{Rep}_{\text{Gal}(k_s/k)}(\mathbb{Q}_l)$  stands for the category of (continuous)  $l$ -adic representations of  $\text{Gal}(k_s/k)$ . Hence, they yield, in particular, the following  $\mathbb{Q}_l$ -linear functors:

$$(6.17) \quad K_0^{\hat{i}, \text{et}}(- \otimes_k k_s)_{1/l}, K_1^{\hat{i}, \text{et}}(- \otimes_k k_s)_{1/l}: \text{NChow}(k)_{\mathbb{Q}_l} \longrightarrow \text{Rep}_{\text{Gal}(k_s/k)}(\mathbb{Q}_l).$$

Therefore, by applying the  $\mathbb{Q}_l$ -linear functors (6.17) to the above direct sum decomposition (6.5), we obtain the equivalence:

$$\mathrm{SS}_{\mathrm{nc}}^l(\mathrm{perf}_{\mathrm{dg}}(X_L)) \Leftrightarrow \mathrm{SS}_{\mathrm{nc}}^l(\mathbb{C}_L^{\mathrm{dg}}) + \mathrm{SS}_{\mathrm{nc}}^l(\mathbb{A}_{\dim(V)}^{\mathrm{dg}}) + \cdots + \mathrm{SS}_{\mathrm{nc}}^l(\mathbb{A}_{i-1}^{\mathrm{dg}}).$$

All the above holds *mutatis mutandis* with  $X_L$  replaced by  $Y_L$ . Consequently, the above semi-orthogonal decomposition (6.4) leads to the equivalence:

$$\mathrm{SS}_{\mathrm{nc}}^l(\mathrm{perf}_{\mathrm{dg}}(Y_L)) \Leftrightarrow \mathrm{SS}_{\mathrm{nc}}^l(\mathbb{B}_{j-1}^{\mathrm{dg}}) + \cdots + \mathrm{SS}_{\mathrm{nc}}^l(\mathbb{B}_{\dim(L^\perp)}^{\mathrm{dg}}) + \mathrm{SS}_{\mathrm{nc}}^l(\mathbb{C}_L^{\mathrm{dg}'}).$$

The remainder of the proof is now similar to the proof of  $\mathrm{T}^l(X_L) \Leftrightarrow \mathrm{T}^l(Y_L)$ .

The proof of the equivalence  $\mathrm{T}^p(X_L) \Leftrightarrow \mathrm{T}^p(Y_L)$  is similar to the proof of  $\mathrm{T}^l(X_L) \Leftrightarrow \mathrm{T}^l(Y_L)$ : simply replace  $\mathbb{Q}_l$  by  $\mathbb{Q}_p$  and the  $\mathbb{Q}_l$ -linear functor (6.9) by the  $\mathbb{Q}_p$ -linear functor (4.10).

Finally, we prove the equivalence  $\mathrm{B}(X_L) \Leftrightarrow \mathrm{B}(Y_L)$ . As above, the semi-orthogonal decomposition (6.3) gives rise to the following direct sum decomposition

$$(6.18) \quad U(\mathrm{perf}_{\mathrm{dg}}(X_L))_{\mathbb{Q}} \simeq U(\mathbb{C}_L^{\mathrm{dg}})_{\mathbb{Q}} \oplus U(\mathbb{A}_{\dim(V)}^{\mathrm{dg}})_{\mathbb{Q}} \oplus \cdots \oplus U(\mathbb{A}_{i-1}^{\mathrm{dg}})_{\mathbb{Q}}$$

in the category  $\mathrm{NChow}(k)_{\mathbb{Q}}$ . As proved in [34, §6], given any smooth proper dg category  $\mathcal{A}$ , we have a natural isomorphism:

$$(6.19) \quad \mathrm{Hom}_{\mathrm{NNum}(k)_{\mathbb{Q}}}(U(k)_{\mathbb{Q}}, U(\mathcal{A})_{\mathbb{Q}}) \simeq K_0(\mathcal{A})_{\mathbb{Q}} / \sim_{\mathrm{num}}.$$

Therefore, by applying  $\mathrm{Hom}_{\mathrm{NChow}(k)_{\mathbb{Q}}}(U(k)_{\mathbb{Q}}, -)$  and  $\mathrm{Hom}_{\mathrm{NNum}(k)_{\mathbb{Q}}}(U(k)_{\mathbb{Q}}, -)$  to the direct sum decomposition (6.18), we obtain the equivalence of:

$$\mathrm{B}_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(X_L)) \Leftrightarrow \mathrm{B}_{\mathrm{nc}}(\mathbb{C}_L^{\mathrm{dg}}) + \mathrm{B}_{\mathrm{nc}}(\mathbb{A}_{\dim(V)}^{\mathrm{dg}}) + \cdots + \mathrm{B}_{\mathrm{nc}}(\mathbb{A}_{i-1}^{\mathrm{dg}}).$$

All the above holds *mutatis mutandis* with  $X_L$  replaced by  $Y_L$ . Consequently, the above semi-orthogonal decomposition (6.4) leads to the equivalence:

$$\mathrm{B}_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(Y_L)) \Leftrightarrow \mathrm{B}_{\mathrm{nc}}(\mathbb{B}_{j-1}^{\mathrm{dg}}) + \cdots + \mathrm{B}_{\mathrm{nc}}(\mathbb{B}_{\dim(L^\perp)}^{\mathrm{dg}}) + \mathrm{B}_{\mathrm{nc}}(\mathbb{C}_L^{\mathrm{dg}'}).$$

The remainder of the proof is now similar to the proof of  $\mathrm{T}^l(X_L) \Leftrightarrow \mathrm{T}^l(Y_L)$ .

## 7. PROOF OF PROPOSITION 2.5

As proved in [7, Thms. 1.3 and 1.7], the dg category  $\mathrm{perf}_{\mathrm{dg}}(\mathrm{Gr}(r, U_1))$  is Morita equivalent to a finite dimensional  $k$ -algebra of finite global dimension  $A$ . Let us write  $J(A)$  for the Jacobson radical of  $A$ ,  $V_1, \dots, V_s$  for the simple (right)  $A/J(A)$ -modules and  $D_1 := \mathrm{End}_{A/J(A)}(V_1), \dots, D_s := \mathrm{End}_{A/J(A)}(V_s)$  for the associated division  $k$ -algebras. Thanks to the Artin-Wedderburn theorem, the quotient  $A/J(A)$  is Morita equivalent to the product  $D_1 \times \cdots \times D_s$ . Moreover, the center of  $D_i$  is a finite field extension  $l_i$  of  $k$  and  $D_i$  is a central simple  $l_i$ -algebra. As proved in [39, Thm. 3.15], we have the following direct sum decomposition

$$U(\mathbb{A}_0^{\mathrm{dg}})_{\mathbb{Q}} = U(\mathrm{perf}_{\mathrm{dg}}(\mathrm{Gr}(r, U_1)))_{\mathbb{Q}} \simeq U(A/J(A))_{\mathbb{Q}} \simeq U(l_1)_{\mathbb{Q}} \oplus \cdots \oplus U(l_s)_{\mathbb{Q}}$$

in the category  $\mathrm{NChow}(k)_{\mathbb{Q}}$ . Similarly to the proof of Theorem 2.2, we hence obtain the following equivalences:

$$\mathrm{C}(\mathbb{A}_0^{\mathrm{dg}}) \Leftrightarrow \mathrm{C}_{\mathrm{nc}}(l_1) + \cdots + \mathrm{C}_{\mathrm{nc}}(l_s) \quad \text{with } \mathrm{C} \in \{\mathrm{T}^l, \overline{\mathrm{T}}^l, \mathrm{SS}^l, \mathrm{T}^p, \mathrm{B}\}.$$

Note that since the  $k$ -schemes  $\mathrm{Spec}(l_i)$  are 0-dimensional, the conjectures  $\mathrm{C}(\mathrm{Spec}(l_i))$ , with  $\mathrm{C} \in \{\mathrm{T}^l, \overline{\mathrm{T}}^l, \mathrm{SS}^l, \mathrm{T}^p, \mathrm{B}\}$ , hold. Consequently, the proof follows now from the equivalences  $\mathrm{C}(\mathrm{Spec}(l_i)) \Leftrightarrow \mathrm{C}_{\mathrm{nc}}(l_i)$  established in Theorem 2.1.

## 8. PROOF OF THEOREM 2.15

**Item (i).** Following [23, §3.5] (see also [2, §1.6]), let us write  $\mathcal{Z}$  for the center of  $\mathcal{C}l_0(q)|_L$  and  $\text{Spec}(\mathcal{Z}) =: \tilde{\mathbb{P}}(L) \rightarrow \mathbb{P}(L)$  for the *discriminant cover* of  $\mathbb{P}(L)$ . As explained in *loc. cit.*,  $\tilde{\mathbb{P}}(L) \rightarrow \mathbb{P}(L)$  is a 2-fold cover which is ramified over the divisor  $D := \mathbb{P}(L) \cap \Delta_1$ . Since by assumption  $\dim(L) = 2$ , we have  $\dim(D) = 0$ . Consequently, since  $D$  is smooth,  $\tilde{\mathbb{P}}(L)$  is also smooth. Let us write  $\mathcal{F}$  for the sheaf of noncommutative algebras  $\mathcal{C}l_0(q)|_L$  considered as a sheaf of noncommutative algebras over  $\tilde{\mathbb{P}}(L)$ . As proved in *loc. cit.*, since by assumption we have  $\mathbb{P}(L) \cap \Delta_2 = \emptyset$ ,  $\mathcal{F}$  is a sheaf of Azumaya algebras over  $\tilde{\mathbb{P}}(L)$  of rank  $2^{(d/2)-1}$ . Moreover, the category  $\text{perf}(\mathbb{P}(L); \mathcal{C}l_0(q)|_L)$  is equivalent (via a Fourier-Mukai type functor) to  $\text{perf}(\tilde{\mathbb{P}}(L); \mathcal{F})$ . This leads to a Morita equivalence between the dg categories  $\text{perf}_{\text{dg}}(\mathbb{P}(L); \mathcal{C}l_0(q)|_L)$  and  $\text{perf}_{\text{dg}}(\tilde{\mathbb{P}}(L); \mathcal{F})$ . Making use of [39, Thm. 2.1], we hence obtain the following isomorphisms

$$U(\text{perf}_{\text{dg}}(\mathbb{P}(L); \mathcal{C}l_0(q)|_L))_{\mathbb{Q}} \simeq U(\text{perf}_{\text{dg}}(\tilde{\mathbb{P}}(L); \mathcal{F}))_{\mathbb{Q}} \simeq U(\text{perf}_{\text{dg}}(\tilde{\mathbb{P}}(L)))_{\mathbb{Q}}$$

in the category  $\text{NChow}(k)_{\mathbb{Q}}$ . Similarly to the proof of Theorem 2.2, this leads to the following equivalences:

$$\text{C}_{\text{nc}}(\text{perf}_{\text{dg}}(\mathbb{P}(L); \mathcal{C}l_0(q)|_L)) \Leftrightarrow \text{C}_{\text{nc}}(\text{perf}_{\text{dg}}(\tilde{\mathbb{P}}(L))) \quad \text{with } C \in \{\mathbb{T}^l, \overline{\mathbb{T}}^l, \text{SS}^l, \text{T}^p, \text{B}\}.$$

Since  $\tilde{\mathbb{P}}(L)$  is a curve, the conjectures  $\text{C}(\tilde{\mathbb{P}}(L))$ , with  $C \in \{\mathbb{T}^l, \overline{\mathbb{T}}^l, \text{SS}^l, \text{T}^p, \text{B}\}$ , hold. Consequently, thanks to Corollary 2.6, the proof follows now from the equivalence  $\text{C}(\tilde{\mathbb{P}}(L)) \Leftrightarrow \text{C}_{\text{nc}}(\text{perf}_{\text{dg}}(\tilde{\mathbb{P}}(L)))$  established in Theorem 2.1.

**Item (ii).** Following [23, §3.6] (see also [2, §1.7]), let us write  $\widehat{\mathbb{P}}(L)$  for the *discriminant stack* associated to the pull-back  $q|_L$  along  $\mathbb{P}(L) \subset \mathbb{P}(S^2(W^\vee))$  of the flat quadric fibration  $q: \mathcal{H} \rightarrow \mathbb{P}(S^2(W^\vee))$ . As explained in *loc. cit.*, since by assumption  $1/2 \in k$ ,  $\widehat{\mathbb{P}}(L)$  is a smooth Deligne-Mumford stack. Moreover, using the fact that  $\widehat{\mathbb{P}}(L)$  is a square root stack and that the critical locus of the flat quadric fibration  $q|_L$  is the divisor  $D$ , we conclude from [14, Thm. 1.6] that  $\text{perf}(\widehat{\mathbb{P}}(L)) = \langle \text{perf}(D), \text{perf}(\mathbb{P}(L)) \rangle$ . Hence, similarly to the proof of Theorem 2.2, we obtain the equivalences:

$$(8.1) \quad \text{C}_{\text{nc}}(\text{perf}_{\text{dg}}(\widehat{\mathbb{P}}(L))) \Leftrightarrow \text{C}(D) + \text{C}(\mathbb{P}(L)) \quad \text{with } C \in \{\mathbb{T}^l, \overline{\mathbb{T}}^l, \text{SS}^l, \text{T}^p, \text{B}\}.$$

Let us write  $\mathcal{F}$  for the sheaf of noncommutative algebras  $\mathcal{C}l_0(q)|_L$  considered as a sheaf of noncommutative algebras over  $\widehat{\mathbb{P}}(L)$ . As proved in [23, §3.6] (see also [2, §1.7]), since by assumption we have  $\mathbb{P}(L) \cap \Delta_2 = \emptyset$ ,  $\mathcal{F}$  is a sheaf of Azumaya algebras over  $\widehat{\mathbb{P}}(L)$ . Moreover, the category  $\text{perf}(\mathbb{P}(L); \mathcal{C}l_0(q)|_L)$  is equivalent (via a Fourier-Mukai type functor) to  $\text{perf}(\widehat{\mathbb{P}}(L); \mathcal{F})$ . This leads to a Morita equivalence between the dg categories  $\text{perf}_{\text{dg}}(\mathbb{P}(L); \mathcal{C}l_0(q)|_L)$  and  $\text{perf}_{\text{dg}}(\widehat{\mathbb{P}}(L); \mathcal{F})$ . Making use of Corollary 2.14, we hence obtain the following equivalences:

$$(8.2) \quad \text{C}(X_L) \Leftrightarrow \text{C}_{\text{nc}}(\text{perf}_{\text{dg}}(\widehat{\mathbb{P}}(L); \mathcal{F})) \quad \text{with } C \in \{\mathbb{T}^l, \overline{\mathbb{T}}^l, \text{SS}^l, \text{T}^p, \text{B}\}.$$

Since by assumption  $\dim(L) = 2$ , we have  $\dim(\mathbb{P}(L)) = 1$ . Using the fact that the Brauer group of every smooth curve over a finite field  $k$  is trivial, we hence conclude that in (8.2) we can replace the dg category  $\text{perf}_{\text{dg}}(\widehat{\mathbb{P}}(L); \mathcal{F})$  by the dg category  $\text{perf}_{\text{dg}}(\widehat{\mathbb{P}}(L))$ . Consequently, since  $\dim(D) = 0$ , the proof follows now from

the combination of the above equivalences (8.1)-(8.2) with the fact that the Tate conjecture and its variants hold in dimensions  $\leq 1$ .

*Remark 8.3* (Intersection of even dimensional quadrics). Let  $X_L$  be as in Corollary 2.14. Assume that  $d$  is even, that  $\dim(L) \geq 2$ , that  $\mathbb{P}(L) \cap \Delta_2 = \emptyset$ , and that the divisor  $\mathbb{P}(L) \cap \Delta_1$  is smooth. Under these assumptions, a proof similar to the one of Theorem 2.15 gives rise to the following equivalences

$$(8.4) \quad C(X_L) \Leftrightarrow C(\tilde{\mathbb{P}}(L)) \quad \text{with} \quad C \in \{\mathbb{T}^l, \bar{\mathbb{T}}^l, \mathbb{S}\mathbb{S}^l, \mathbb{T}^p, \mathbb{B}\},$$

where  $\tilde{\mathbb{P}}(L)$  is the discriminant 2-fold cover of  $\mathbb{P}(L)$ .

## 9. PROOF OF THEOREM 2.17

Note first that since  $p \nmid n$ , the integer  $n$  is invertible in  $k$ . Making use of [38, Thm. 1.1 and Rk. 1.3], we hence conclude that the noncommutative Chow motive  $U(\text{perf}_{\text{dg}}(\mathcal{X}))_{\mathbb{Q}}$  is a direct summand of  $\bigoplus_{\sigma \subseteq G} U(\text{perf}_{\text{dg}}(X^\sigma \times \text{Spec}(k[\sigma])))_{\mathbb{Q}}$ . Therefore, arguments similar to those used in the proof of Theorem 2.2 yield the searched implications (2.18). Assume now moreover that  $k$  contains the  $n^{\text{th}}$  roots of unity. In this case, [38, Cor. 1.5(i)] implies that  $U(\text{perf}_{\text{dg}}(\mathcal{X}))_{\mathbb{Q}}$  is a direct summand of  $\bigoplus_{\sigma \subseteq G} U(\text{perf}_{\text{dg}}(X^\sigma))_{\mathbb{Q}}^{\oplus r_\sigma}$ , where  $r_\sigma$  are certain positive integers. Hence, as above, arguments similar to those used in the proof of Theorem 2.2 yield the implications  $\sum_{\sigma \subseteq G} C(X^\sigma) \Rightarrow C(\mathcal{X})$ , with  $C \in \{\mathbb{T}^l, \bar{\mathbb{T}}^l, \mathbb{S}\mathbb{S}^l, \mathbb{T}^p, \mathbb{B}\}$ .

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