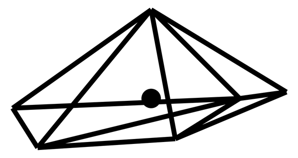
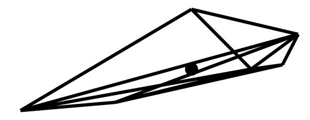


Graded Ring Database

`http://www.maths.warwick.ac.uk/grdb/`



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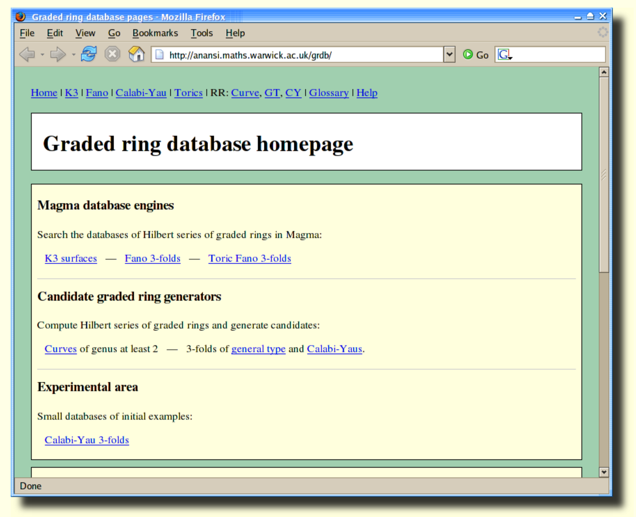
The database

The graded ring database allows one to search lists of graded rings of some polarised varieties including orbifold curves, K3 surfaces and both Fano and Calabi–Yau 3-folds. It is possible to search the database using various criteria such as genus, codimension, weights of minimal generators, singularities and types of projection and unprojection—the screenshot to the right illustrates this for K3 surfaces.

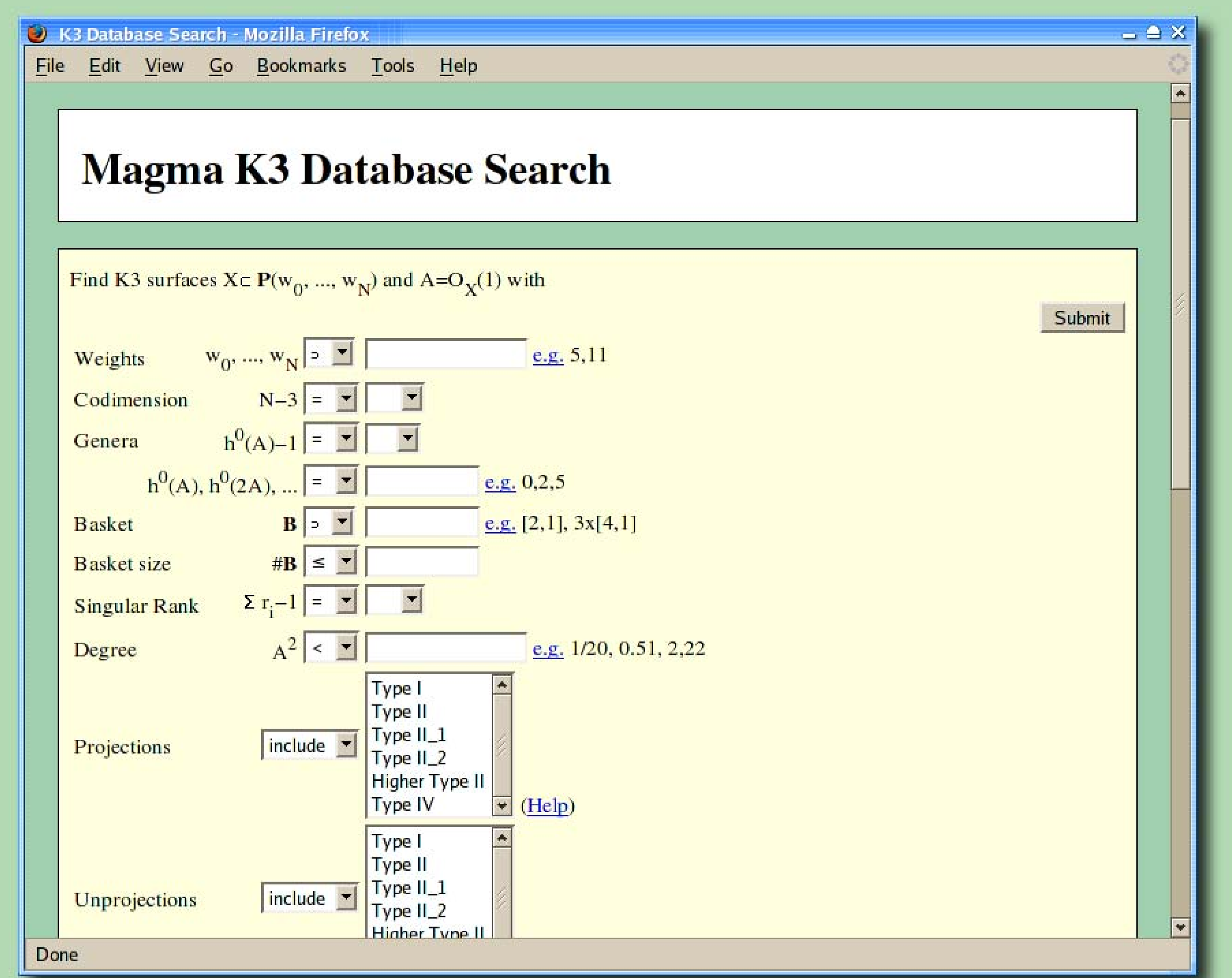
The lists in the database are based on power series generated by Riemann–Roch (RR) formulas, an approach sometimes called the *graded ring method*.

As well as searching pre-computed lists, one can generate candidate graded rings from RR input data in the cases of curves of general type and 3-folds.

The scripts behind the pages make calculations of graded rings using the Magma computer algebra system. Selected results can be accessed using SQL.



The pages at `magma.maths.usyd.edu.au` contain more information about Magma.



The graded ring method

A *polarised variety* (X, A) is a variety X together with a choice of ample divisor A. Such a pair (X, A) has an associated graded ring, the homogeneous coordinate ring of X in its embedding by A,

$$R(X, A) = \bigoplus_{n \geq 0} H^0(X, nA) \quad \text{so that} \quad X = \operatorname{Proj} R(X, a) \subset \mathbb{P}^N$$

where $\mathbb{P}^N$ is some weighted projective space (wps).

The graded ring method attempts to reconstruct $R(X, A)$, or some related ring, from its Hilbert series $P_X(t)$. For example, an integral domain with Hilbert series $P(t) = 1+t+2t^2+3t^3+4t^4+\cdots$ must have generators in degrees 1,2 and 3 suggesting the projective model $X_6 \subset \mathbb{P}^2(1, 2, 3)$, the standard Weierstrass model of an elliptic curve.

The graded ring database uses Magma to generate these Hilbert series and find the corresponding weights using a number of special tricks, unprojection being the most powerful.

The main ingredient of any such calculation is a RR formula for $h^0(X, nA)$. These exist for curves, varieties with isolated quotient singularities and some other classes of variety. The best-developed database is the K3 database that contains 24,099 K3 surfaces; see Brown, *A database of polarised K3 surfaces*, to appear in Experimental Mathematics.

A guiding problem is the classification of Mori–Fano 3-folds, which are known to be contained in finitely many families. The databases include a list of about 40,000 candidates which include all semistable examples. See Altınok–Brown–Reid, *Fano 3-folds, K3 surfaces and graded rings*, Contemp. Math. 314, 2002, pp.25–61.

Subcanonical orbifold curves

The orbifold canonical class of a curve C is of the form $K_C + \sum((r-1)/r)p$ for points $p \in C$ and stabilisers $\mathbb{Z}/r\mathbb{Z}$. For a positive integer k dividing $2g-2$ and each $r-1$, there is a RR formula for (C, A) with $A = D + \sum(b/r)p$, where $K_C = kD$ and $r - 1 = kb$:

$$P_C(t) = \frac{1 + q_1 t + q_2 t^2 + \cdots + t^{k+2}}{(1-t)^2} + \sum \frac{t^{\lceil r/b \rceil} + \cdots + t^r}{(1-t)(1-t^r)}$$

where the first numerator is palindromic, or *Gorenstein symmetric*, with coefficients encoding invariants of (C, D) and the second numerator encodes the carry modulo r, in that it has $+t^e$ each time the multiple eb passes a multiple of r.

The example of an elliptic curve E polarised by various multiples of a point $p \in E$ is closely related:

$$E_6 \subset \mathbb{P}(1,2,3) \text{ is polarised by } p, \quad E_4 \subset \mathbb{P}(1,1,2) \text{ is polarised by } 2p, \text{ and so on.}$$

The embedding $(E, (n+1)p)$ is a Type I unprojection of (E, np), and these fit together into a sequence of unprojections

$$\cdots \dashrightarrow \left(E_{2,2} \subset \mathbb{P}^3\right) \dashrightarrow \left(E_3 \subset \mathbb{P}^2\right) \dashrightarrow \left(E_4 \subset \mathbb{P}(1,1,2)\right) \dashrightarrow \left(E_6 \subset \mathbb{P}(1,2,3)\right).$$

The Type I unprojection theorem of Papadakis–Reid applies inductively to show that each embedding exists exactly as the Hilbert series predicts.

The same idea can be employed using fractional divisors on curves to generate infinite unprojection families of orbifold curves. For example, a curve C of genus 1 polarised by divisors of degree $n/(4n+1)$ lives in a sequence

$$\cdots \dashrightarrow \left(C_{15,18,19,22,26} \subset \mathbb{P}(2,5,9,13,17)\right) \dashrightarrow \left(C_{15,18} \subset \mathbb{P}(2,5,9,13)\right) \dashrightarrow \left(C_{20} \subset \mathbb{P}(2,5,9)\right).$$

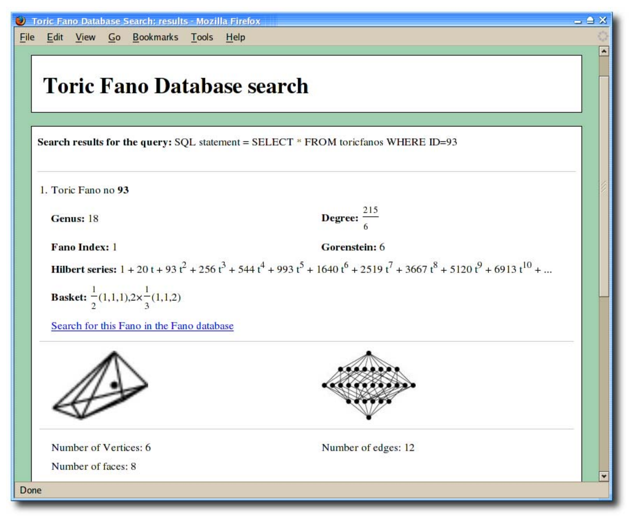
The codimension 3 embedding is defined by the five maximal Pfaffians of a skew 5×5 matrix.

Toric Fano 3-folds

The screenshot to the right shows some of the data of a toric Fano 3-fold. The picture is the dual polygon of its toric fan.

The list of 634 toric $\mathbb{Q}$-factorial terminal Fano 3-folds, classified by Al Kasprzyk, is included in the database. It is just one of several such lists of toric varieties.

For details, see the paper:



Kasprzyk, *Toric Fano 3-folds with terminal singularities*, to appear in Tohoku Math. J.

Calabi–Yau 3-folds

A Calabi–Yau 3-fold X has $K_X = 0$. The famous database of 500 million such 3-folds by Kreuzer and Skarke is at `hep.itp.tuwien.ac.at/~kreuzer/CY/` .

The Buckley–Szendrői RR formula computes the Hilbert series for Calabi–Yau 3-folds with canonical singularities. For example, setting basket and initial plurigenera as

$$\mathcal{B} = \left\{ \frac{1}{3}(1,1,1), \quad \frac{1}{5}(1,2,2) \right\} \quad \text{and} \quad h^0(X, A) = 2, \quad h^0(X, 2A) = 5$$

and running the graded ring game (with some twists) results in the suggested 3-fold

$$X \subset \mathbb{P}^7(1,1,2,2,3,3,4,5) \quad \text{embedded in codimension 4.}$$

To prove that this embedding exists, we project from each of the singular points to see images

$$\mathbb{P}^2 \subset Y_{6,8} \subset \mathbb{P}^5(1,1,2,2,3,5) \quad \text{and} \quad \mathbb{P}(1,2,2) \subset Z_{5,6,6,7,8} \subset \mathbb{P}^6(1,1,2,2,3,3,4).$$

The first is a complete intersection in a toric variety so is already in the Kreuzer–Skarke lists. It can be unprojected using Papadakis' recent Type II_1 formulas. The second is a codimension 3 Pfaffian, and the methods of Tom & Jerry Type I unprojection apply—Kerber has done these calculations for Fano 3-folds and the results are in the database.

Graded rings on Calabi–Yaus is in progress: the RR formula is already available, and in due course lists of 3-folds, such as the well-known 7555 quasismooth hypersurfaces in wps, will appear.