Birational Geometry of 3-fold Mori Fibre Spaces

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Abstract

In this paper we study standard 3-fold conic bundles $X/\mathbb{P}^2$ and stable cubic del Pezzo fibrations $X/\mathbb{P}^1$. These are the key examples of 3-fold Mori fibre spaces (Mfs). We begin systematically to chart the *geography* of these varieties, namely the classification of their deformation families and discrete invariants. Given a 3-fold Mfs X/S, we aim to understand the set of all Mfs Y/T with X birational to Y. For example, we say that X/S is *birationally rigid* if this is a 1-element set. We state conjectures on the rigidity of Mfs, with the goal of approaching optimal criteria. Our main results are summarised in Tables 1 and 2 and Figure 1. We want this to become your standard guidebook to Mori fibre spaces and their birational geometry.

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1 Introduction

1.1 Mori fibre spaces

A 3-fold Mori fibre space—we often use the acronym Mfs—is a 3-fold X together with an extremal contraction $f\colon X \to S$ of *fibering type*. This means that X has $\mathbb{Q}$-factorial terminal singularities, $-K_X$ is ample on fibres, the relative Picard rank $\rho = \operatorname{rk} N^1(X) - \operatorname{rk} N^1(S)$ equals 1, and $\dim S < \dim X$. This structure is the higher-dimensional generalisation of minimally ruled surfaces and $\mathbb{P}^2$.

This paper is devoted to *strict* 3-fold Mori fibre spaces, that is, we always assume $\dim S \geq 1$. There are two cases: *conic bundles* when $\dim S = 2$ and *del Pezzo fibrations*—we write dP_k fibrations when fibres are del Pezzo surfaces of degree k—when $\dim S = 1$. We also always assume that the base variety S is rational.

Definition 1. Let $f\colon X \to S$, $g\colon Y \to T$ be Mori fibre spaces. A birational map

$$\varphi\colon X \dashrightarrow Y$$

is *square* if it maps a general fibre of f isomorphically to a general fibre of g. We say that $X \to S$ and $Y \to T$ are *square birational*, or *square equivalent*, when there is a square birational map between them.

1.2 Geography

Our first goal in this paper is to begin a systematic study of the *geography* of strict Mori fibre spaces. We focus on the case of *standard* conic bundles $X \to \mathbb{P}^2$ and *stable* dP_3 fibrations $X \to \mathbb{P}^1$.

Recall that a conic bundle $X \to S$ is standard if X is nonsingular; this in turn implies that S is also nonsingular, and the discriminant locus $\Delta \subset S$ is a nodal curve. Every conic bundle is square birational to a standard conic bundle (it may be necessary to blow up S). With qualifications, the assumption is not really restrictive. In Section 3 of this paper we begin a systematic study of standard conic bundles $X \to \mathbb{P}^2$. Necessarily, X embeds in a projectivised rank 3 vector bundle on $\mathbb{P}^2$, and we learn how to write down the bundle and equations of X explicitly. As an illustration, we construct standard conic bundles for all ramification data with a discriminant curve of degree 7. (There are 4 deformation families.) Our methods apply to higher degrees and to surfaces other than $\mathbb{P}^2$; what we do here is a starting point for future work.

The definition of stable del Pezzo fibration is technical; in this paper we almost always work with the slightly stronger condition that X be nonsingular. Again, a dP fibration is always square equivalent to a stable one. In Section 4 of this paper we begin a systematic study of the geography of stable dP_3 fibrations $X \to \mathbb{P}^1$. Necessarily, X is a member of a linear system $|3M + nL|$ on a rational scroll $\mathbb{F}(0, a, b, c)$. We determine the invariants n, a, b, c such that a general element of the linear system is a stable dP_3 fibration. We study the degree 3 case because it is the most interesting, but our methods apply to degree 1 and 2. Grinenko has studied del Pezzo fibrations of degree 1 and 2 in a series of recent papers [Gri01a, Gri01b, Gri00b, Gri00a, Gria], but he always works under the assumption that the total space X is Gorenstein. In the case of dP fibrations of degree 1 and 2, this assumption is too restrictive.

1.3 Sarkisov links

Both in the case of conic bundles and del Pezzo fibrations X/S, we study in detail nonsquare Sarkisov links originating from X/S. Recall that Sarkisov links are the "elementary" building blocks of the birational geometry of Mfs: any birational map between Mfs can be factored as a chain of Sarkisov links. We usually, but not always, assume that X is nonsingular and sufficiently general in moduli. To understand Sarkisov links, we are led to compute explicitly the Mori cone of X and the cone of mobile divisors on X. In our examples X has rank 2, so these cones are 2-dimensional, and we just need to determine the two edges in terms of explicit loci on X. The mobile cone of X is partitioned into chambers (in our case these are just wedges) corresponding to moves of the 2-ray game [Cor00] starting with X. We study these moves explicitly to see if the 2-ray game leads to a Sarkisov link from $X \to S$ to a new Mori fibre space. When X is embedded in a scroll $\mathbb{F}$ over S the moves are often, but not always, induced by moves of $\mathbb{F}$.

1.4 Rigid Mori fibre spaces and known criteria

Definition 2. [CR00, Foreword] The *pliability* of a Mori fibre space $X \to S$ is the set

$$\mathcal{P}(X/S) = \{\text{Mfs } Y/T \mid Y \text{ is birational to } \quad X\}/\sim$$

where $\sim$ denotes *square birational equivalence*. We say that $X \to S$ is *birationally rigid* when $\mathcal{P}(X/S)$ consists of one element.

A few general criteria for birational rigidity have been known for quite some time, see [IP96], [Cor00]. These criteria and their proofs are based on exploiting properties of the 1-cycle K_X^2. The importance of the following condition was first recognised in a series of brilliant papers by A. Pukhlikov. His point of view is explained for example in [Puk00].

Definition 3. [Puk98a, Puk98b, Puk97] We say that a variety X satisfies the K^2 *condition* if K_X^2 is not in the interior of the Mori cone $\overline{\mathrm{NE}}\,X$ of effective 1-dimensional cycles on X.

Theorem 4. *[Sar82, Sar80, Sar79], [Puk98a, Puk98b, Puk97] Let X be a nonsingular 3-fold and $X \to S$ a conic bundle or a dP_3 fibration (satisfying an additional "genericity" condition on the singularities—see Theorem 22 below). If X satisfies the K^2 condition, then $X \to S$ is birationally rigid.*

More recent results explore examples that lie on the boundary between rigid and nonrigid [CM, Mel], [Grib], [Sob02]. These papers study Mfs X/S which are either *bi-rigid* in the sense that they have exactly two models as Mori fibre spaces (more precisely, $\mathcal{P}(X/S)$ is a set with two elements), or that are rigid in a subtle way, that is, there exist Mfs Y/T and nonsquare maps $X \dashrightarrow Y$, but Y/T is always square birational to X/T. In this paper we construct many examples of varieties that are likely to behave similarly.

1.5 Conjectures

The known general criteria for rigidity of Mori fibre spaces are not optimal. Our second goal in this paper is to state some conjectures, based on the intuition built on looking at the examples we construct. Some of these conjectures we expect to be able to prove in the near future, others, we feel, are definitely harder, and some are probably wrong. The K^2 condition is not natural. Our conjectures are stated in term of a geometrically more natural condition, whose importance was also recognised in work by Grinenko [Gric, Conjecture 1.5] [Gri00a, Conjecture 1.6] [Gria, Conjecture 2.5].

Definition 5. A variety X *satisfies condition* $(*)$ if the anticanonical class $-K$ is not in the interior of the cone of mobile divisors of X.

Note that condition $(*)$ is stronger than the K^2 condition. Indeed, if $-K$ is in the interior of the mobile cone, we can write for some $n > 0$

$$-nK = M_1 + M_2 = H_1 + H_2$$

where $M_1 \sim H_1$ have no component in common, and, similarly, $M_2 \sim H_2$ have no component in common. Then writing

$$n^2K^2 = \sum M_i H_j$$

shows that K^2 is in the interior of the Mori cone.

In Sections 3.5 and 4.2 we state some conjectures on the rigidity of standard conic bundles and stable dP_3 fibrations in terms of condition $(*)$. For many of the examples that we construct which seem to lie on the boundary between rigid and nonrigid, it would be natural to conjecture that they are bi-rigid, or that they are still rigid, though they do not satisfy condition $(*)$. We have resisted the temptation to make such conjectures, but we invite our readers to make their own and possibly prove some theorems.

It is by now well understood how to use Pukhlikov's K^2 condition to prove rigidity of Mfs, but it seems difficult to go further. On the other hand, we have as yet no experience or success using condition $(*)$ to prove rigidity of Mfs. We hope that this paper will help to introduce a new point of view on the birational geometry of Mori fibre spaces, where the geometric properties of $-K$ play more prominent a role through condition $(*)$.

1.6 Tables and Figures

Tables 1 and 2, and Figure 1 summarise and collect key information which is obtained in many calculations throughout the paper. Table 1 shows the list of standard conic bundles over $\mathbb{P}^2$ with discriminant of degree 7 and contains information about their nonsquare Sarkisov links to alternative models as Mori fibre spaces. Figure 1 depicts the geography of dP_3 fibrations and contains information about the general member of each family, showing whether it satisfies condition $(*)$ and whether we know an alternative model as a Mori fibre space. This information is supplemented in Table 2, which lists all families of dP_3 fibration for which we know an element that has a nonsquare Sarkisov link to some Mfs Y/T and contains detailed information about the link.

1.7 The Appendix

The Appendix sets out our notation for rational scrolls and information about their birational maps. A scroll $\mathbb{F}$ is a quotient of affine space by an action of the group $G = \mathbb{C}^\times \times \mathbb{C}^\times$. Though one is not normally aware of this, the quotient depends on the choice of a G-linearisation. Different G-linearisations produce different quotients. In this manner we can understand the 2-ray game originating from the scroll. In the Appendix, we introduce notation for this which is used throughout the paper when constructing Sarkisov links of Mfs $X \subset \mathbb{F}$; indeed, it is often the case that moves of $\mathbb{F}$ induce moves of X.

1.8 Various cones

Let X be a projective variety. We denote $N^1(X) = N^1(X, \mathbb{R})$ the vector space of Cartier divisors on X with real coefficients, modulo *numerical* equivalence, and $N_1(X) = N_1(X, \mathbb{R})$ the dual space of 1-dimensional cycles on X with real coefficients, modulo numerical equivalence. Throughout this paper, it is crucial to be aware of various cones associated to X:

The Mori cone $\overline{\mathrm{NE}}_1(X)$, or simply $\overline{\mathrm{NE}}(X)$, is the closure of the cone in $N_1(X)$ generated by the effective 1-cycles.

The ample cone $\overline{\mathrm{NA}}^1(X)$ is the cone in $N^1(X)$ generated by ample divisors; Kleiman's criterion states that it is the dual cone to the Mori cone.

The mobile cone $\overline{\mathrm{NM}}^1(X)$ is the cone in $N^1(X)$ generated by mobile divisors. We say that a divisor D on X is *mobile* if a positive multiple nD moves in a linear system with no fixed divisor.

The quasieffective cone $\overline{\mathrm{NE}}^1(X)$ is the closure of the cone in $N^1(X)$ generated by effective divisors. Divisor in this cone are called *quasieffective*.

1.9 Disclaimer

This paper was prepared against a tight (for us) deadline. We did not test some of the conjectures as much as we would have liked, and it is likely that some of the definitions, calculations, etc. contain mistakes. The responsibility for these is of course ours, but we still want to apologise for the inconvenience we may be causing our readers.

2 The Sarkisov category

The *Sarkisov category* is the category of Mori fibre spaces and birational maps between them. We begin with a simple result stating that Mfs that have a model as a Fano 3-fold belong to finitely many deformation families.

Proposition 6. *There is a finite collection of algebraic families of Mori fibre spaces with the following property: if $X \to S$ is birational to a Fano 3-fold Y, then X/S is square birational to a member of one of the families in the collection.*

Proof. We briefly explain the idea of the proof. By the Sarkisov program [Cor95], nonrigid Mfs are parametrised by nonsquare links of the Sarkisov program. These are in turn parametrised by weak terminal Fano 3-folds of rank 2, and these are bounded by [KMMT00]. $\square$

We hope that we will soon be able to prove the following.

Conjecture 7. *If $X \to S$ is a Mori fibre space, the pliability $\mathcal{P}(X/S)$ is in a natural way an algebraic variety.*

The key difficulty is this. Let $\mathcal{X} \to T$ be a family of Fano 3-folds with a birational selfmap $\sigma\colon \mathcal{X} \dashrightarrow \mathcal{X}$ that maps fibres birationally into fibres and hence induces a birational selfmap of the base T. If this induced birational selfmap of T has infinite order and $t_1 \in T$ is a general point, then the fibre X_{t_1} is birational to X_{t_2}, and then to X_{t_3} and so on. It seems possible that all the values $t_i \in T$, but not the whole of T, may contribute to the pliability.

Recent experience [CPR00] [CM, Mel] suggests the following rather optimistic conjectures.

Conjecture 8. *For the total space X of a 3-fold Mfs $X \to S$ to be rational is a topological property. In other words, if X/S and Y/T are 3-fold Mori fibre spaces, X is rational and Y is diffeomorphic to X, then Y is also rational.*

Conjecture 9. *For a 3-fold Mori fibre space to be rigid is a topological property.*

In particular, being rigid or rational is constant along algebraic families where all fibres are diffeomorphic.

Conjecture 10. *The pliability $\mathcal{P}(X/S)$ of a 3-fold Mori fibre space is in a natural way a topological invariant.*

3 Conic bundles

We do not remind you of the known sufficient conditions for rigidity of conic bundles or Iskovskikh's conjectural characterisation of conic bundles with rational total space [Isk96, Isk91a, Isk91b, Isk87] [Cor00]. We do illustrate a method to write down equations of a conic bundle with assigned discriminant. This method is based on Catanese's work on the Babbage conjecture [Cat81] and is explicitly computable, unlike the traditional abstract approach via Brauer groups. We focus on conic bundles over $\mathbb{P}^2$ with discriminant a nodal plane curve of degree 7. Motivated by these examples, we state some conjectures on the rigidity of conic bundles.

3.1 Conic bundles and Brauer groups

For details on this section see [AM72], [Sar82]. Let $X \to S$ be a *standard* conic bundle over a rational surface S. By definition, this means that X is nonsingular and the relative Picard rank is 1, a consequence of which is that S is itself nonsingular and the discriminant curve $\Delta \subset S$ is nodal. Over Δ, the fibre is generically the sum of two lines, which specifies a 2-to-1 *admissible cover* $N \to \Delta$. (When C is nonsingular, admissible just means étale; in general, an admissible 2-to-1 cover is required to ramify over both branches of each node of C.) Together, we refer to the double cover $N \to \Delta$ and the embedding $\Delta \subset S$ as the *ramification data* of the conic bundle.

It is well known that there is a standard conic bundle $X \to S$ with any preassigned ramification data $\Delta \subset S$ and 2-to-1 admissible cover $N \to \Delta$. The total space X is not unique, but any two choices are square birational over the base.

The traditional proof uses the exact sequence [AM72] (in which S is a rational surface with field of rational functions K)

$$0 \to \operatorname{Br} S \to \operatorname{Br} K \to \bigoplus_{\text{curves } C \subset S} H^1(K(C), \mathbb{Q}/\mathbb{Z}) \to \bigoplus_{\text{points } P \in S} \mu^{-1} \to \mu^{-1} \to 0$$

and the fact, due to Platonov and reproduced in [Sar82], that every element of order 2 in the Brauer group $\operatorname{Br} K$ can be represented by a quaternion algebra. See the references for details and explanations.

In the following subsections, we illustrate an effective proof of this statement in two steps: first we learn how to specify effectively a 2-to-1 cover $N \to \Delta$, then we write down explicit equations for a conic bundle X.

3.2 Catanese's theory

Let C be a nonsingular curve of degree d in $\mathbb{P}^2$. (With a bit of care, the theory works unchanged for nodal curves.) We fix coordinates u_0, u_1, u_2 on $\mathbb{P}^2$ and denote $S = k[u_0, u_1, u_2]$ the homogeneous coordinate ring. We are interested in admissible 2-to-1 covers of C. We summarise the part of Catanese's paper [Cat81] which is relevant to us; see also [Dix02]. The theory works identically for covers which in addition ramify over a specified hyperplane section of C, and we treat these as well.

An admissible 2-to-1 cover corresponds to a line bundle $\mathcal{L}$ on C with a choice of isomorphism: either $\mathcal{L}^2 = \mathcal{O}_C$, or $\mathcal{L}^2 = \mathcal{O}_C(-1)$. Denote by $i\colon C \hookrightarrow \mathbb{P}^2$ the inclusion. By [Cat81], Theorems 2.16 and 2.19, $i_*\mathcal{L}$ is a Cohen-Macaulay $\mathcal{O}_{\mathbb{P}^2}$-module, so it has a 2-step symmetric locally free resolution:

$$0 \to \oplus \mathcal{O}_{\mathbb{P}^2}(-l_i) \xrightarrow{A} \oplus \mathcal{O}_{\mathbb{P}^2}(-r_i) \to i_*\mathcal{L} \to 0.$$

In other words, A is a symmetric $n \times n$ matrix whose entries are homogeneous forms on $\mathbb{P}^2$.

Note that $\det A$ is a homogeneous equation of C. On the other hand, writing C as a symmetric determinantal specifies the line bundle $\mathcal{L}$ and the double cover.

Denote by d_i the degree of the i-th diagonal entry a_{ii} of A; $d = \sum d_i$ is a partition of d. The degree of a_{ij} is $(d_i + d_j)/2$, so that all the d_is have the same parity. We have the following numerology:

$$\begin{aligned} r_i &= (d + e - d_i)/2 \\ l_j &= (d + e + d_j)/2 \end{aligned} \quad \text{where} \quad e = \begin{cases} 0 & \text{if} \quad \mathcal{L}^2 = \mathcal{O}_C \\ 1 & \text{if} \quad \mathcal{L}^2 = \mathcal{O}_C(1). \end{cases}$$

Let

$$M = \bigoplus_{n \geq 0} M_n \quad \text{where} \quad M_n = H^0\big(\mathbb{P}^2, i_*\mathcal{L}(n)\big)$$

be the Serre module of the sheaf $i_*\mathcal{L}$. The direct summand $\mathcal{O}_{\mathbb{P}^2}(-r_i)$ in the presentation corresponds to a generator m_i of degree r_i of M. The matrix A encodes the structure of M as a graded module over the homogeneous coordinate ring $S = k[u_0, u_1, u_2]$ of $\mathbb{P}^2$, as well as the identity $m_i m_j = b_{ij} \in S$ in which $(b_{ij}) = B = \operatorname{ad} A$ is the adjugate matrix of A.

We work out the well-known example of theta characteristics of plane quartics in detail; this corresponds to the case $d = 4$, $e = 1$ of the theory. Here $\mathcal{L}$ is a line bundle with $\mathcal{L}^2 = \mathcal{O}(-1)$, and $\mathcal{L}(1)$ is a theta characteristic since $K_C = \mathcal{O}(1)$. A plane quartic has genus $g = 3$ and $\deg \mathcal{L} = -2$, so Riemann–Roch gives

$$h^0(C, \mathcal{L}(1)) - h^1(C, \mathcal{L}(1)) = 0.$$

We have two cases: $h^0(\mathcal{L}(1)) = 0$ (even theta characteristic), and $h^0(\mathcal{L}(1)) = 1$ (odd theta characteristic). We work out the odd case.

By assumption, we have exactly one generator $m \in M_1 = H^0(C, \mathcal{L}(1))$ of degree 1. Next, $\mathcal{L}(2)$ has degree 6 and Riemann–Roch calculates

$$h^0(C, \mathcal{L}(2)) = 6 + 1 - 3 = 4,$$

so M_2 is based by 4 elements: they are u_0m, u_1m, u_2m and a new generator n. Noting $h^0(C, \mathcal{L}(3)) = 8$, we see that there must be a relation in degree 2 between the 9 elements u_iu_jm, u_kn; this relation must take the form $nf_1 = mf_2$ where f_1 is a linear form in u_0, u_1, u_2 and f_2 a quadratic form. The matrix A has the form

$$A = \begin{pmatrix} f_1 & f_2 \\ f_2 & f_3 \end{pmatrix}.$$

The whole structure is specified by writing the curve $C = (f_1f_3 - f_2^2 = 0)$ as a symmetric determinantal. This presentation reveals a preferred bitangent $f_1 = 0$, which "is" the odd section m.

In the even case, you will find 4 generators $m_1, \ldots, m_4$ of degree 2, and 4 linear relations among the 12 elements u_im_j: A is a 4×4 symmetric matrix of linear forms.

3.3 Plane curves of degree 7

We study the case $d = 7$, $e = 0$, that is, line bundles $\mathcal{L}$ with $\mathcal{L}^2 = \mathcal{O}_C$ on plane curves C of degree $\deg C = 7$. Here $K_C = \mathcal{O}(4)$, so $\mathcal{L}(2)$ is a theta characteristic. A plane curve of degree 7 has genus $g = 15$ and canonical degree $\deg K_C = 28$. Riemann–Roch and Serre duality give at once

$$\begin{aligned} h^0\big(C, \mathcal{L}(n)\big) - h^1\big(C, \mathcal{L}(n)\big) &= 7(n-2) \\ h^1\big(C, \mathcal{L}(n)\big) &= h^0\big(C, K_C \otimes \mathcal{L}(n)^*\big) \\ &= h^0\big(C, \mathcal{L}(4-n)\big). \end{aligned}$$

We use these equations without comment throughout. It is clear that $p_1 = h^0(C, \mathcal{L}(1))$ and $p_2 = h^0(C, \mathcal{L}(2))$ determine all the $p_n = h^0(C, \mathcal{L}(n))$: indeed, the Hilbert series of $i_*\mathcal{L}$ as an S-module is

$$P(t) = \sum_{n \geq 0} h^0(C, \mathcal{L}(n))t^n = \\ p_1t + p_2t^2 + (7 + p_1)t^3 + 14t^4 + \cdots + 7it^{i-2} + \cdots.$$

There are four cases, corresponding to the four possible values of $p_2 = h^0(C, \mathcal{L}(2))$. We show later that the first two cases occur on a generic curve, while the last two cases can only occur on curves with special moduli. In the four cases, the degrees of the diagonal entries of the matrix A are the summands of the four different partitions of the number $d = 7$ into odd summands.

1. $h^0(C, \mathcal{L}(2)) = 0$ (generic even theta),
2. $h^0(C, \mathcal{L}(2)) = 1$ (generic odd theta),
3. $h^0(C, \mathcal{L}(2)) = 2$ (special even theta),
4. $h^0(C, \mathcal{L}(2)) = 3$ (special odd theta).

3.3.1 $h^0(C, \mathcal{L}(2)) = 0$

We have $h^0(C, \mathcal{L}(3)) = 7$ generators of degree 3. Multiplying these by u_0, u_1, u_2, we get 21 elements in the 14-dimensional space M_4, so there are 7 equations between them. In this case A is a 7 by 7 symmetric matrix of linear forms.

3.3.2 $h^0(C, \mathcal{L}(2)) = 1$

Again $h^0(C, \mathcal{L}(3)) = 7$, but here we assume a generator m of degree 2, giving $u_i m$ in M_3, so we need 4 more generators $n_1, \dots, n_4$ in M_3. In this case A is a 5×5 symmetric matrix of homogeneous forms of degrees

$$\begin{pmatrix} 3 & 2 & 2 & 2 & 2 \\ & 1 & 1 & 1 & 1 \\ & & 1 & 1 & 1 \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix}$$

in which the degrees of the diagonal elements express $d = 7 = 3 + 1 + 1 + 1 + 1$ as a partition into odd summands.

3.3.3 $h^0(C, \mathcal{L}(2)) = 2$

This case corresponds to the partition $7 = 3 + 3 + 1$. The assumption is that there are two generators m_1, m_2 in M_2, so, as usual, $h^2(C, \mathcal{L}(3)) = 7$ then shows that we need one more generator n in degree 3. We conclude that A is a 3×3 symmetric matrix of homogeneous forms of degrees

$$\begin{pmatrix} 3 & 3 & 2 \\ & 3 & 2 \\ & & 1 \end{pmatrix}. \tag{1}$$

We show below that not all plane septics have a theta-characteristic of this type. More precisely, the locus of those who do is of codimension 1 in moduli.

Example 11 (Counting moduli)**.** Let $C = (\det A = 0) \subset \mathbb{P}^2$ for A a symmetric 3×3 matrix of forms of degrees given by matrix format (1) above. The matrix A depends on $3 \times 10 + 2 \times 6 + 3 = 45$ parameters, and A is uniquely determined by the choice of the generators of the Serre module. In this case, that choice amounts to the choice of m_1, m_2 in M_2 (which has the 4 degrees of freedom of the basis of a 2-dimensional vector space) and the choice of the additional generator n in M_3 (7 degrees of freedom) for a total of $4+7=11$ degrees of freedom. The space of septics of the given form has dimension $45-11=34$, whereas the space of all septics has dimension 35.

3.3.4 $h^0(C, \mathcal{L}(2)) = 3$

This case corresponds to the partition $7 = 5+1+1$. Here $p_1 = 1$ and the element m in M_1 generates a basis $u_i m$ of M_2. This time $h^0(C, \mathcal{L}(3)) = 8$, so we need two additional generators n_1, n_2 in degree 3. The matrix A is a 3×3 matrix of homogeneous forms of degrees

$$\begin{pmatrix} 5 & 3 & 3 \\ & 1 & 1 \\ & & 1 \end{pmatrix}.$$

Once again, the locus of septics who own a theta characteristic of this type is of codimension 1 in moduli.

3.4 Conic bundles over $\mathbb{P}^2$ with discriminant of degree 7

We construct standard conic bundles over $\mathbb{P}^2$ with ramification data a 2-to-1 admissible cover $N \to \Delta \subset \mathbb{P}^2$ of a plane curve of degree 7. We make four deformation families of such 3-folds, following the analysis in the previous section. We show that general members X of the first three families have an alternative model as a Mori fibre space. (This may be true for all their members, but we didn't check that). Conjecture 17 below implies that a general member of the fourth family is birationally rigid. This is a striking prediction, because this example is far from the numerical range where the known rigidity criteria apply.

3.4.1 $h^0(\Delta, \mathcal{L}(2)) = 0$

Let $N \to \Delta$ be as in 3.3.1. A conic bundle with this ramification data was known classically, and it is birational to the Fano 3-fold $Y = Y_{2,2,2} \subset \mathbb{P}^6$, the codimension 3 complete intersection of three quadrics in $\mathbb{P}^6$. Starting from Y, the link $Y \dashrightarrow X$ begins with the projection from a line $\ell \subset Y$. The details of this construction are well known, but see also the following example.

3.4.2 $h^0(C, \mathcal{L}(2)) = 1$

Let $N \to \Delta$ be as in 3.3.2. We want explicit equations for a conic bundle with ramification data $N \to \Delta$. Consider the variety

$$Z = \{{}^t\mathbf{x}A\mathbf{x} = 0\} \subset \mathbb{F} = \mathbb{F}(1,0,0,0,0).$$

This is a bundle of 3-dimensional quadrics immersed in a scroll over $\mathbb{P}^2$ with ramification data $N \to \Delta$. Indeed, a singular 3-dimensional quadric is, generically, a cone over $\mathbb{P}^1 \times \mathbb{P}^1$, and the two rulings give back the cover $N \to \Delta$. It is natural to try to construct our conic bundle from Z by a kind of "dimensional reduction".

Claim 1 We can choose coordinates in the scroll $\mathbb{F}$ such that the matrix A has the form

$$A = \begin{pmatrix} B & \mathbf{b} \\ {}^t\mathbf{b} & 0 \end{pmatrix}$$

where $\mathbf{b}$ is a 1×4 column vector of linear forms. To see that this is possible, we look for a coordinate change of the form:

$$\begin{pmatrix} x_0 \\ \xi \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ \beta & M \end{pmatrix} \begin{pmatrix} x_0 \\ \xi \end{pmatrix}$$

where ${}^t\xi = (x_1, x_2, x_3, x_4)$, $M = (m_{ij}) \in GL(4, \mathbb{C})$, and β is a 4×1 column vector of linear forms in u_0, u_1, u_2. The coordinate change brings A in the wanted form if and only if the column vector with entries $(0, m_{14}, m_{24}, m_{34}, m_{44})$ is a nonzero null-vector of A. It is easy to see that such a vector exists; indeed the 4×4 symmetric submatrix (a_{ij}) of A is a matrix of linear forms and corresponds to a net of 3-dimensional quadrics. A base point of the net of quadrics gives the null-vector we want.

Equations of X Define

$$X = \{{}^t\mathbf{x}B\mathbf{x} = \mathbf{b} \cdot \mathbf{x} = 0\} \subset \mathbb{F}(1, 0, 0, 0)$$

where $\mathbb{F}(1, 0, 0, 0)$ is the subscroll $\{x_4 = 0\} \subset \mathbb{F}(1, 0, 0, 0, 0)$. One can easily check that X is a conic bundle with the required ramification data $N \to \Delta$. Notice that, in terms of projectivised vector bundles on $\mathbb{P}^2$, the linear form expresses X as a conic in a projectivised nonsplit rank 3 vector bundle on $\mathbb{P}^2$.

Claim 2 If $\mathbf{b}$ is sufficiently general, we may further change coordinates so that

$$\mathbf{b} = \begin{pmatrix} 0 \\ u_0 \\ u_1 \\ u_2 \end{pmatrix}.$$

Model as a Fano 3-fold We construct a Sarkisov link from X to a codimension 3 Fano 3-fold $Y = Y_{2,3,3,3,3,3} \subset \mathbb{P}(1^6, 2)$. The ideal of Y is generated by the 4×4 Pfaffians of a 5×5 antisymmetric matrix of homogeneous forms of degrees

$$\begin{pmatrix} 2 & 2 & 2 & 2 \\ & 1 & 1 & 1 \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}$$

on $\mathbb{P}(1^6, 2)$. Assuming that X is generic, we exhibit a birational map, in fact a link of the Sarkisov program, from X to a Fano 3-fold of the type just described. This alternative model of X as a Mori fibre space demonstrates, in particular, that X is not birationally rigid.

If we choose coordinates as in Claim 3.4.2, then $X \subset \mathbb{F}(1,0,0,0)$ is given by equations

$$\begin{cases} {}^t\mathbf{x}B\mathbf{x} = 0 \\ u_0x_1 + u_1x_2 + u_2x_3 = 0 \end{cases}$$

where B is a 4×4 symmetric matrix of forms of degrees

$$\begin{pmatrix} 3 & 2 & 2 & 2 \\ & 1 & 1 & 1 \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}$$

in the coordinates u_0, u_1, u_2.

The morphism $p\colon \mathbb{F} \to \mathbb{P}^5 = \mathbb{P}H^0(\mathbb{F}, M)^*$ given by

$$(x_1, x_2, x_3, x_4, x_5, x_6) = (x_1, x_2, x_3, u_0x_0, u_1x_0, u_2x_0)$$

identifies $\mathbb{F}$ with the blow up of $\mathbb{P}^5$ along the 2-plane $\Pi = \{x_4 = x_5 = x_6 = 0\}$. Under this morphism X maps to $p(X) = \overline{Y}_{2,3}$, a generic complete intersection of a quadric and a cubic in $\mathbb{P}^5$ containing the plane Π. To find the equations of $\overline{Y}$, note that we can find unique homogeneous quadrics q_0, q_1, q_2 in six variables, such that

$${}^t\mathbf{x}B\mathbf{x} = \sum_{i=0}^{2} u_iq_i(x_1, x_2, x_3, u_0x_0, u_1x_0, u_2x_0).$$

(If B is generic, the quadrics q_0, q_1, q_2 are also generic.) It is easy to see that $\overline{Y}$ is given by the following equations:

$$\begin{pmatrix} x_1 & x_2 & x_3 \\ q_0 & q_1 & q_2 \end{pmatrix} \begin{pmatrix} x_4 \\ x_5 \\ x_6 \end{pmatrix} = 0.$$

Starting from $\overline{Y}_{2,3}$, we construct Y as an *unprojection*, that is, the contraction of the Weil divisor $\Pi \subset \overline{Y}$; see [Rei], [ABR02] for details.

3.4.3 $h^0(\Delta, \mathcal{L}(2)) = 2$

Let $N \to \Delta$ be as in 3.3.3. In this case, we can immediately write down equations for a conic bundle with this ramification data. Indeed, A is a 3×3 symmetric matrix of homogeneous forms of degrees

$$\begin{pmatrix} 3 & 3 & 2 \\ & 3 & 2 \\ & & 1 \end{pmatrix}.$$

We can take

$$X = \{{}^t\mathbf{x}A\mathbf{x} = 0\} \in |2M + L| \subset \mathbb{F}(1,1,0).$$

We now exhibit an alternative model of X as a dP_3 fibration. We begin with a preliminary discussion of the geometry of the scroll $\mathbb{F} = \mathbb{F}(1,1,0)$, which we think of as a *geometric* quotient

$$\mathbb{F} = \mathbb{A}/\!\!/G$$

of $\mathbb{A} = \mathbb{C}^6$, with coordinates $u_0, u_1, u_2, x_0, x_1, x_2$, by the group $G = \mathbb{C}^\times \times \mathbb{C}^\times$, with coordinates λ, μ, acting in the usual way (see the Appendix). Indeed, a G-linearisation of the trivial line bundle on $\mathbb{A}$ is the same as a character $\chi\colon G \to \mathbb{C}^\times$. The group G acts on global sections $f\colon \mathbb{A} \to \mathbb{C}$ by $gf(x) = \chi(g)f(g^{-1}x)$. We say that a linearisation is *useful* if the (open) subset

$$\mathbb{A}^{ss}_\chi = \{x \in \mathbb{A} \mid \exists\, f \in \mathcal{O}^\chi_\mathbb{A},\ f(x) \neq 0\}$$

of semistable points is nonempty, where

$$\mathcal{O}^\chi_\mathbb{A} = \{f\colon \mathbb{A} \to \mathbb{C} \mid f(gx) = \chi(g)f(x)\}$$

is the set of G-invariant sections. We have that

$$\mathbb{A}^{ss}_\chi = (\mathbb{C}^3 \setminus \{0\}) \times (\mathbb{C}^3 \setminus \{0\}) \quad \Longleftrightarrow \quad \chi \in \mathbb{R}_+[L] + \mathbb{R}_+[M].$$

There are other useful linearisations. In fact, the cone of useful linearisations is the cone $\mathbb{R}_+[L]+\mathbb{R}_+[M-L]$. This cone is naturally partitioned in two chambers, and

$$\mathbb{A}^{ss}_\chi = (\mathbb{C}^4 \setminus \{0\}) \times (\mathbb{C}^2 \setminus \{0\}) \quad \Longleftrightarrow \quad \chi \in \mathbb{R}_+[M] + \mathbb{R}_+[M-L].$$

gives the second chamber. When choosing a linearisation in the second chamber, the geometric quotient is a different scroll over $\mathbb{P}^1$:

$$\mathbb{A}/\!\!/G = \mathbb{F}' = \mathbb{F}(0,1,1,1).$$

Crossing the wall separating the two chambers corresponds to a birational map $\mathbb{F} \dashrightarrow \mathbb{F}'$. The equation of X, viewed as the equation of a hypersurface $Y \subset \mathbb{F}'$, is a section of the line bundle $\mathcal{O}_{\mathbb{F}'}(3M-L)$. The most convincing way to see this is to make the substitutions

$$\begin{aligned} u_0, u_1, u_2, x_0 &\mapsto y_1, y_2, y_3, y_0 \\ x_1, x_2 &\mapsto t_0, t_1 \end{aligned}$$

and to think of $t_0, t_1, y_0, y_1, y_2, y_3$ as natural coordinates on the scroll $\mathbb{F}'$; the action of λ, μ on these coordinates is

$$\begin{aligned} \lambda\colon (t_0, t_1, y_0, y_1, y_2, y_3) &\mapsto (\lambda^{-1}t_0, \lambda^{-1}t_1, y_0, \lambda y_1, \lambda y_2, \lambda y_3) \\ \mu\colon (t_0, t_1, y_0, \dots, y_3) &\mapsto (\mu t_0, \mu t_1, \mu y_0, y_1, y_2, y_3). \end{aligned}$$

To recover the standard presentation of the scroll $\mathbb{F}' = \mathbb{F}(0,1,1,1)$, we change coordinates in G: the 1-parameter subgroups $\lambda' \to (\lambda'^{-1}, 1)$ and $\mu' \to (\mu', \mu')$ act as

$$\begin{aligned}\lambda' \colon (t_0, t_1, y_0, y_1, y_2, y_3) \mapsto& (\lambda' t_0, \lambda' t_1, y_0, \lambda'^{-1} y_1, \lambda'^{-1} y_2, \lambda'^{-1} y_3)\\ \mu' \colon (t_0, t_1, y_0, y_1, y_2, y_3) \mapsto& (t_0, t_1, \mu' y_0, \mu' y_1, \mu' y_2, \mu' y_3).\end{aligned}$$

These calculations show that, with the stated substitutions, $H^0\big(\mathbb{F}', 3M - L\big)$ is canonically identified with

$$H^0\big(\mathbb{F}, 3M - (-L + M)\big) = H^0\big(\mathbb{F}, 2M + L\big).$$

It is easy to see that the map $X \dashrightarrow Y$ is a flop which, by what we just said, is a Sarkisov link (of type IV according to [Cor95]) from the conic bundle $X/\mathbb{P}^2$ to a dP_3 fibration $Y/\mathbb{P}^1$.

3.4.4 $h^0(\Delta, \mathcal{L}(2)) = 3$

Let $N \to \Delta$ be as in 3.3.4. In this case, A is a 3×3 symmetric matrix of homogeneous forms of degrees

$$\begin{pmatrix} 5 & 3 & 3 \\ & 1 & 1 \\ & & 1 \end{pmatrix}$$

and we can take

$$X = \{ {}^t\mathbf{x} A \mathbf{x} = 0 \} \in |2M + L| \subset \mathbb{F}(2,0,0).$$

The divisor $E = \{x_2 = 0\} \cap X$ is the exceptional divisor of a (2,1)-contraction $g\colon X \to Y$ with $K = 0$, that is, E contracts to a curve of strictly canonical singularities on Y. This is a bad link and no alternative model as a Mfs is produced. We suspect that X is birationally rigid. Observe that this follows from Conjecture 17.

Table 1 summarises the examples discussed so far.

3.5 Conjectures on rigid conic bundles

We briefly sketch a few more examples of conic bundles, mainly nonstandard or over surfaces other than $\mathbb{P}^2$, then state some conjectures.

Example 12. We show that a general codimension 3 Fano 3-fold $Y_{3,3,4,4,4} \subset \mathbb{P}(1^5, 2, 3)$ is linked to a *nonstandard* conic bundle X. The equations of Y are the Pfaffians of a 5×5 antisymmetric matrix of homogeneous forms which, in suitable coordinates, can be written as follows:

$$\begin{pmatrix} z & a_1 & a_2 & a_3 \\ & b_1 & b_2 & b_3 \\ & & x_5 & x_4 \\ & & & x_3 \end{pmatrix}.$$

$h^0(\Delta, \mathcal{L}(2))$	model over $\mathbb{P}^2$	link	other model
0	$X_{2M+L,M+L,M+L}$ $\subset \mathbb{P}^2 \times \mathbb{P}^4$	flop and (2,1)-contraction	$Y_{2,2,2} \subset \mathbb{P}^6$
1	$X_{2M+L,M+L}$ $\subset \mathbb{F}(1, 0^3)$	flop and (2,0)-contraction	$Y_{2,3,3,3,3} \subset \mathbb{P}(1^6, 2)$
2	$X_{2M+L} \subset \mathbb{F}(1,1,0)$	flop	$Y_{3M-L} \subset \mathbb{F}(1^3, 0)$
3	$X_{2M+L} \subset \mathbb{F}(2,0,0)$	bad K trivial (2,1)-contraction	X rigid ?

Table 1: Conic bundles over $\mathbb{P}^2$ with discriminant Δ of degree 7

(Here as usual z is a coordinate of weight 3, y is a coordinate of degree 2, and $x_1, \dots, x_5$ are coordinates of weight 1. The as and the bs are homogeneous forms of degree 2 in the xs and y.) Projecting Y to $\mathbb{P}(1^5, 2)$ we obtain a complete intersection $\overline{Y}_{3,3}$ given by equations:

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} = 0,$$

containing the weighted plane $\Pi = \{x_3 = x_4 = x_5 = 0\}$. We now construct a model as a conic bundle. First we construct a suitable ambient space. Consider the quotient

$$\mathbb{F} = \big((\mathbb{C}^3 \setminus \{0\}) \times (\mathbb{C}^4 \setminus \{0\})\big)/(\mathbb{C}^\times \times \mathbb{C}^\times)$$

by the action

$$\begin{aligned} \lambda &\colon (u_0, u_1, u_2, x_0, x_1, x_2, y) \mapsto (\lambda u_0, \lambda u_1, \lambda u_2, \lambda^{-1} x_0, x_1, x_2, y) \\ \mu &\colon (u_0, u_1, u_2, x_0, x_1, x_2, y) \mapsto (u_0, u_1, u_2, \mu x_0, \mu x_1, \mu x_2, \mu^2 y). \end{aligned}$$

We can think of $\mathbb{F}$ as a scroll over $\mathbb{P}^2$ with fibre the weighted projective space $\mathbb{P}(1^3, 2)$. We define a morphism $f\colon \mathbb{F} \to \mathbb{P}(1^5, 2)$ by setting:

$$(u_0, u_1, u_2, x_0, x_1, x_2, y) \mapsto (x_1, x_2, u_0 x_0, u_1 x_0, u_2 x_0, y).$$

The proper preimage $X = f^{-1}\overline{Y}$ is a general complete intersection of the form $X = X_{2M+L,2M+L} \subset \mathbb{F}$. The ramification data $N \to \Delta$ is a 2-to-1 covering of a special plane octic with a triple point $P \in \Delta$, so X is not a standard conic bundle. It is easy to see that X has an index 2 orbifold point over P. Because of the singular point, the threshold invariant $\tau(X/\mathbb{P}^2)$ [Cor00, §4.1] is 5/2, and not the expected 8/2. Blowing up the point $P \in \mathbb{P}^2$ expresses X as a conic bundle over $\mathbb{F}^1$ with discriminant of relative degree 5, and this in turn has a model as a dP_4 fibration over $\mathbb{P}^1$.

Example 13. A general codimension 3 Fano 3-fold $Y_{4,4,4,4,4} \subset \mathbb{P}(1^3, 2^4)$ is linked to a nonstandard conic bundle $X/\mathbb{P}^2$. The discriminant Δ is a plane curve of degree 9 with four ordinary triple points, so X is not a standard conic bundle. It is easy to see that X has index 2 orbifold singularities over the singular points of Δ. Please do your own calculations.

Example 14. Takagi and Reid [Tak02], [Rei] construct a codimension 4 Fano 3-fold $Y \subset \mathbb{P}(1^6, 2^2)$ with $h^0(-K) = g + 2 = 6$ and $-K^3 = 7$ with 2 index 2 orbifold points. There are two deformation families with these invariants, corresponding to the "Tom" and "Jerry" formats of unprojection. If Y is a general Fano in the Tom family, then Y is linked to a general conic bundle $X \to \mathbb{P}^2$ with discriminant curve of degree 6. The cover $N \to \Delta$ corresponds to the partition $6 = 2 + 2 + 2$, but we did not yet carry out all the necessary calculations to understand the model X explicitly.

Example 15. There is no reason why we should only work with conic bundles over $\mathbb{P}^2$, since Catanese's theory works essentially unchanged for curves on a weighted projective plane. For example, there are two Fano 3-folds in codimension 4 and 5, $Y \subset \mathbb{P}(1^5, 2^3)$ and $Y \subset \mathbb{P}(1^5, 2^4)$, with $-K^3 = 11/2$ and 6, see [Tak02], which can be linked to conic bundles over $\mathbb{P}(1, 1, 2)$ with discriminants in $\mathcal{O}(8)$ and $\mathcal{O}(6)$.

We have written down explicit equations of several families of conic bundles with discriminant of low degree. We believe that the *explicit geometry* of these varieties will play an increasingly more prominent role in the study of their birational geometry. We state the following rather optimistic conjecture.

Definition 16. We say that a standard conic bundle $\pi\colon X \to S$ satisfies condition $(*)$ if $-K_X \notin \operatorname{Int} \overline{\mathrm{NM}}^1 X$.

Conjecture 17. *A standard conic bundle over $\mathbb{P}^2$ is birationally rigid if it satisfies condition $(*)$.*

We feel like making the following conjecture for which we have little evidence.

Conjecture 18. *A standard conic bundle over $\mathbb{P}^2$ is birationally rigid if the discriminant has degree ≥ 9.*

4 dP_3 fibrations

Convention 19. In this section a dP_3 *fibration* is a 3-fold X together with a morphism $f\colon X \to \mathbb{P}^1$ satisfying the following conditions:

1. The nonsingular fibres of f are cubic surfaces.
2. X has Gorenstein terminal singularities (these are precisely the isolated hypersurface singularities with a DuVal section). In particular $X \subset \mathbb{F} = \mathbb{P}(E)$ is naturally embedded in a rational scroll over $\mathbb{P}^1$ (a natural choice is $E = f_*\mathcal{O}(-K_X)$).
3. X has Picard rank $\rho = 2$, that is, the morphism $f\colon X \to \mathbb{P}^1$ is extremal.

4. The local rings of X are unique factorisation domains, that is, Weil divisors on X are Cartier.

In addition, we often assume that X is nonsingular, or at least that $f\colon X \to \mathbb{P}^1$ is *semistable* in the sense of [Kol97].

Remark 20. Corti and Kollár [Cor96] [Kol97] show that if $f\colon X \to \mathbb{P}^1$ is a Mori fibre space dP_3 fibration, then $X/\mathbb{P}^1$ is square birational to a semistable $g\colon Y \to \mathbb{P}^1$; in particular Y is Gorenstein and the conditions above are satisfied. It is therefore not restrictive to limit our attention to semistable fibrations.

In this section, we aim to do two things. First, we want to determine the geography of dP_3 fibrations, that is, determine all integers n, a, b, c such that a general member $X \in |3M + nL|$ on $\mathbb{F}(0, a, b, c)$ is a dP_3 fibration in our sense. Second, we want to state some conjectures on the birational geometry of dP_3 fibrations. M. Grinenko has been studying del Pezzo fibrations systematically in a series of recent papers [Gri01a, Gri01b, Gri00b]. Because he is primarily concerned with fibres of degree 1 and 2, there is little overlap between his work and what we do here.

4.1 The K^2 condition

Definition 21. We say that X satisfies the K^2 *condition* if

$$K_X^2 \notin \operatorname{Int} \overline{\mathrm{NE}}\, X.$$

Theorem 22. *[Puk97] Let $f\colon X \to \mathbb{P}^1$ be a dP_3 fibration satisfying the following technical conditions:*

- *the total space X is nonsingular, and f has Lefschetz singularities, that is, it has only ordinary critical points, with distinct critical values.*
- *If X_b is a singular fibre, then there are exactly six lines of X_b passing through the unique singular point.*

If in addition X satisfies the K^2 condition, then X is birationally rigid.

Remark 23. It follows from the above theorem that, under its assumptions, Iskovskikh's conjecture [Isk95] holds.

If a dP_3 fibration $X \to S$, with X nonsingular, is nonrigid, then $X \to S$ belongs to one of finitely many algebraic families. (This can be proved using Proposition 6.) On the other hand, as we see in Section 4.3, infinitely many families do not satisfy the K^2 condition. This shows that Pukhlikov's theorem is not optimal.

4.2 Conjectures

Definition 24. We say that X satisfies the *condition* $(*)$ if $-K \notin \operatorname{Int} \overline{\mathrm{NM}}^1 X$.

Remark 25. In Section 4.4 we study (among other things) condition $(*)$ for the general members of families of dP_3. Though we don't prove it completely, we believe that condition $(*)$ is satisfied by a general member of all but a handful of families of dP_3 fibrations listed in Table 2.

Grinenko has recently made the following striking conjecture.

Conjecture 26. *[Gric, Conjecture 1.5] [Gri00a, Conjecture 1.6] [Gria, Conjecture 2.5] A dP_3 fibration with nonsingular total space is birationally rigid if it satisfies condition $(*)$.*

In the remaining part of this subsection we make a few comments on the meaning of the conjecture.

Conjecture 27. *Same as 26, only assuming that $X/\mathbb{P}^1$ is semistable in the sense of [Kol97].*

Conjecture 28. *Let $X \to \mathbb{P}^1$ be a dP_3 fibration, with X nonsingular. Let $X' \to \mathbb{P}^1$ be a 3-fold Mori fibre space, square birational to $X/\mathbb{P}^1$. If $X/\mathbb{P}^1$ satisfies condition $(*)$, then so does $X'/\mathbb{P}^1$.*

Conjecture 29. *Same as 28, only assuming that $X/\mathbb{P}^1$ is semistable.*

More experimentation is needed before we can have any confidence in these conjectures. Here we only briefly touch on these matters in the Example in Section 4.4.4.

Proposition 30. *Conjecture 26 follows from Conjecture 28, and Conjecture 27 from Conjecture 29.*

This indicates that further progress is likely to come from a systematic study of square birational maps involving a semistable dP_3 fibration and a Mori fibre space.

Proof. We sketch the proof. If $X \to \mathbb{P}^1$ is not rigid, there is a Mfs $Y \to T$ and a nonsquare birational map $X \dashrightarrow Y$. Applying the Sarkisov program as in [Cor95] gives a Mfs $f' \colon X' \to \mathbb{P}^1$ square birational to $X \to \mathbb{P}^1$, and a linear system

$$\mathcal{H}' \subset |-nK_{X'} + f'^{*}A|$$

where A is a divisor on $\mathbb{P}^1$ of strictly negative degree. It follows that X' does not satisfy condition $(*)$. ☐

4.3 Geography for dP_3 fibrations

4.3.1 Notation and basic numerology

Notation A dP_3 fibration $X \to \mathbb{P}^1$, as defined formally in Convention 19, always admits model as a hypersurface in a 4-fold scroll $\mathbb{F}$ that is a $\mathbb{P}^3$ bundle over $\mathbb{P}^1$. We fix the notation in use throughout this section.

1. $X \in |3M + nL| \subset \mathbb{F} = \mathbb{F}(0, a, b, c)$, where a, b, c, n are integers with $0 \leq a \leq b \leq c$. We write $d = a + b + c$.
2. We write u, v, x, y, z, t for the homogeneous coordinates of $\mathbb{F}$, where u, v are the homogeneous coordinates on the base $\mathbb{P}^1$ and x, y, z, t are the fibre coordinates.
3. We denote L and M the natural basis of $\mathrm{Pic}(\mathbb{F})$. These line bundles have sections $u \in H^0(\mathbb{F}, L)$ and $x \in H^0(\mathbb{F}, M)$, and we sometimes identify L, M with the actual divisors $u = 0$, $x = 0$.
4. We write $\Gamma = \{y = z = t = 0\}$ and $B = \{z = t = 0\}$. Note that Γ generates an extremal ray of $\overline{\mathrm{NE}}(\mathbb{F})$.
5. We denote by $F \in \mathcal{O}_{\mathbb{F}}(3M + nL)$ the equation of X. The polynomial F is a sum $\sum \alpha m$ of *terms* αm where the sum ranges over the *fibre monomials* m that are cubics in x, y, z, t with coefficients $\alpha = \alpha(u, v)$ to fix up homogeneity. We write both $\alpha m \in F$ and $m \in F$ to mean that the term (with implicit coefficient α if not mentioned) appears in F with nonzero coefficient.

Basic numerology

1. $M^3 L = 1$, $M^4 = d$.
2. $-K_X = -K_{|X}$, where $-K = M + (2 - d - n)L$ is a divisor on $\mathbb{F}$.
3. $X \cdot \Gamma = n$. Apart from the trivial case $X \in |3M|$ on $\mathbb{P}^1 \times \mathbb{P}^3$ (a constant family of cubic surfaces) there are two cases:

$$\begin{cases} n \geq 0 & \text{and } 3M + nL \text{ is nef and big,} \\ n < 0. \end{cases}$$

   When $3M + nL$ is nef and big, it is base point free and a general $X \in |3M + nL|$ is nonsingular. It then follows from the Lefschetz hyperplane theorem that $\rho(X) = 2$ and X is a dP_3 fibration in our sense. Almost everything we say below refers to the much more interesting case when $n < 0$.

4.3.2 What the picture says

We plot the families of dP_3 fibrations on a graph of n against d as in Figure 1 and we refer to this picture as *geography*. We regard the figure as a pictorial statement of a theorem that we now spell out.

The figure summarises information obtained in calculations carried out in the remaining part of the paper; in particular the figure displays the following.

1. Pairs (n, d) for which there is a family of dP_3 fibrations $X \in |3M + nL|$ in $\mathbb{F}(0, a, b, c)$ with $d = a + b + c$.

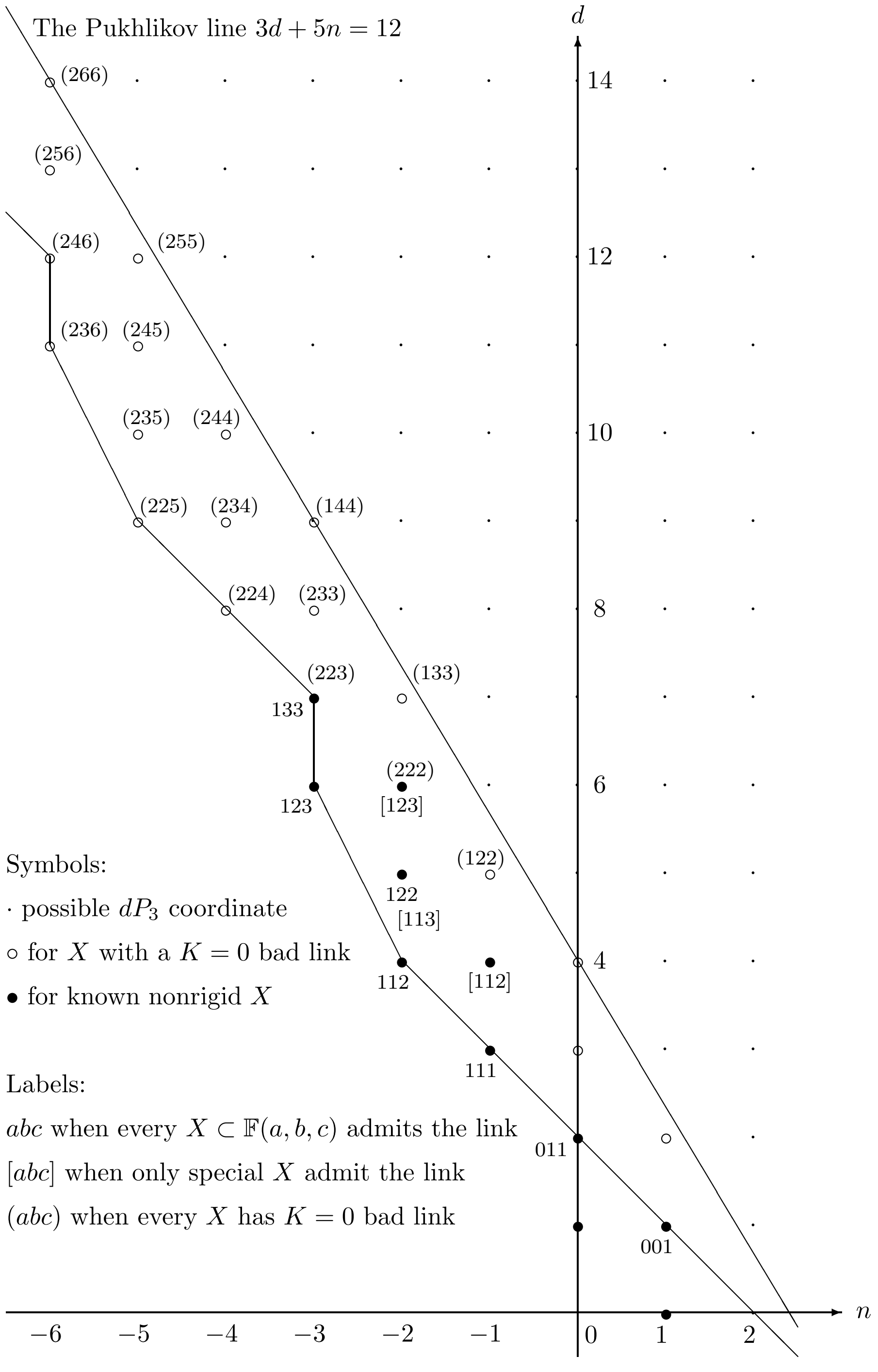

Figure 1: Geography of $|3M + nL| \subset \mathbb{F}(a, b, c)$ with $d = a + b + c$

2. Triples (a,b,c) such that a general or a special member of the corresponding family is known to be nonrigid.

3. Triples (a,b,c) such that a general member of the corresponding family does not satisfy condition $(*)$. We do not prove it completely, but we believe that the figure displays all such triples.

We plan to use the picture as a primary testing ground for Grinenko's Conjecture and as a starting point possibly to prove it. Here we explain what the picture and its various elements mean. Precise details are worked out in the following subsections.

Families of dP_3 fibrations We mark points (n,d) of Figure 1 by a dot $\cdot$ if and only if there are values $0 \leq a \leq b \leq c$ with $d = a+b+c$, such that a general $X \in |3M + dL|$ in $\mathbb{F}(0,a,b,c)$ is a dP_3 fibration in our sense. A dot can be a bullet $\bullet$ or a circle $\circ$; we explain what these mean below. The geography consists of the first quadrant minus the origin, and the region to the right of a curve with a periodic behaviour which we define by the picture itself. A point of the picture may house several deformation families: for example, (n,d) is $(-2,5)$ for a family of dP_3 fibrations in $\mathbb{F}(1,1,3)$ and also for a different family that lies in $\mathbb{F}(1,2,2)$. We explain how this region is drawn in Section 4.3.3 below.

Nonrigid families A bullet $\bullet$ marks a point (n,d) corresponding to some family of dP_3 fibrations for which we know that a member X is nonrigid. When X is general in its family, we specify the family by writing *abc* under the bullet, indicating the invariants of $\mathbb{F}(0,a,b,c)$. In some cases, we only know that a *special* X in the family is nonrigid. We indicate this by writing [*abc*] instead. In all cases when we know that X is nonrigid, the 2-ray game from $X/\mathbb{P}^1$ is a nonsquare link of the Sarkisov program: we refer to this as *the* link and we say that X *admits the link*. We describe the link explicitly in Section 4.4 and summarise our calculations in Table 2.

Condition $(*)$ We do not prove it completely, but we believe that condition $(*)$ does not hold for a general member $X \in |3M + nL|$ in $\mathbb{F}(a,b,c)$ if and only if (n,d) has $\bullet$ with *abc*. After the argument of Section 1.5, we only study points to the left of the line $3d+5n-12=0$; the arguments of Section 4.4.2 show immediately that about two thirds of those with $n < 3$ satisfy condition $(*)$, and we believe the remaining cases to be only slightly more difficult. Grinenko's conjecture is then equivalent to the statement that Table 2 is a complete list of families of dP_3 fibrations with nonrigid general element.

K **trivial contractions** Unless there is already a bullet at this point, we mark (n,d) with a circle $\circ$ when we know that there is a corresponding family of dP_3 fibrations such that the 2-ray game starting with a general member X terminates with a K-trivial contraction and a variety with strictly canonical singularities. We specify the family by writing (*abc*) above the circle. Note that, in each example of $[abc]$, a general X in the family has a K-trivial bad link and we do not repeat (abc) there. These examples are not our main interest here, but they are cases where the question of rigidity is particularly intriguing.

The Pukhlikov line The figure also shows the line $3d + 5n = 12$. As we show in Lemma 36 below, the general member of a family satisfies the K^2 condition if the point (n, d) lies to the right of this line. (For n negative, this is if and only if, which leaves out a couple of cases with $n > 0$ where we are not completely sure.) The graph shows that the line leaves out a thin strip of the geography, containing infinitely many families.

4.3.3 How the geography is obtained

In this subsection we make the statements from which the geography of dP_3 fibrations is derived.

Proposition 31. *Figure 1 marks all pairs (n, d) for which there are integers $0 \le a \le b \le c$ such that:*

1. *The relative surface $B = \{z = t = 0\}$ is not contained in the base locus of the linear system $|3M + nL|$. (When $a = b$, we require that a general member of the linear system does not contain a surface of the form $\{s = t = 0\}$ where s is some section of $\mathcal{O}_{\mathbb{F}}(M - bL)$.)*
2. *The curve $\Gamma = \{y = z = t = 0\}$ is contained in the base locus of $|3M+nL|$ with multiplicity at most one, that is, when $n < 0$, a general member X in the linear system is nonsingular generically along Γ.*

The conditions imply that a general member $X \in |3M + nL|$ is nonsingular outside Γ and possibly has isolated singularities along Γ. In fact, X is a dP_3 fibration.

Proposition 32. *Let (n, d) and $X \in |3M + nL|$ in $\mathbb{F}(0, a, b, c)$ be as in Proposition 31 above. Then:*

1. *For any X, $\operatorname{Pic}(X) = H^2(X) = \mathbb{Z}^2$, that is, X has Picard rank 2.*
2. *If all fibres of $f\colon X \to \mathbb{P}^1$ are reduced and irreducible, we have an exact sequence*
$$0 \to \mathbb{Z}[L] \to A_2(X) \to \operatorname{Pic}(X_\eta) \to 0$$
where $A_2(X)$ is the group of 2-dimensional cycles, that is, Weil divisors on X modulo rational equivalence, and X_η is the generic fibre.
3. *If X is general, then $\operatorname{Pic}(X_\eta) = \mathbb{Z}[-K]$.*
4. *If X is general, then X has isolated singularities of cA type.*

Corollary 33. *With the same assumptions, if X is general, then $X/\mathbb{P}^1$ is a dP_3 fibration.*

Proof of Proposition 31 The proposition is a direct consequence of the following more precise lemma. Indeed, it is elementary to check that the inequalities in the lemma specify the region marked in Figure 1.

Lemma 34. *Consider the linear system $|3M + nL|$ on $\mathbb{F}(0, a, b, c)$ as above.*

1. *$B \not\subset \operatorname{Bs}|3M + nL|$ if and only if $n \geq -3a$.*

2. *If $n < 0$ and $a = b$, then all X contain a surface $t = \ell(z, y) = 0$, where ℓ is a linear form if and only if $n = -3a$.*

3. *If $n < 0$, so that $\Gamma \subset \operatorname{Bs}|3M+nL|$, then a general member $X \in |3M+nL|$ is nonsingular generically along Γ if and only if $n \geq c$. In this case also $n + d \geq 2a$.*

Proof. If the first inequality fails, then every monomial in F is divisible by z or t so X contains B.

If $-n = 3a > 0$ and $a = b$, we can refine this analysis slightly. We can write $F = f_3(y, z) + tf_2(x, y, z, t)$, where f_2, f_3 are homogeneous forms of the indicated degrees. If $\ell(y, z)$ is a linear factor of f_3, then X necessarily contains the surface $t = \ell(z, y) = 0$.

Suppose that $n < 0$, so that $a > -(1/3)n > 0$ by the first inequality. The fibre monomial x^3 cannot now occur in any term of F, so the inclusion $\Gamma \subset X$ is clear.

The total space X is nonsingular generically along Γ if and only if, for general values of u, v, the polynomial F contains at least one of the terms x^2y, x^2z, x^2t. If $x^2t \in F$, then $n \geq -c$, and that is the weakest inequality of the three. Finally

$$d + n = (a + b) + (c + n) \geq 2a$$

with equality only if $n = -c$ and $a = b$. $\square$

Remark 35. Some of our statements in Section 4.3.2 follow from Lemma 34. Let σ be the closed subcone of $\overline{\mathrm{NM}}^1(X)$ generated by the mobile rays L, $D_z = (z = 0) \cap X$. Then writing $-K_X = (2 - a - c - n)L + D_z$ shows that

(a) $2 - a - c - n > 0$ if and only if $-K_X \in \operatorname{Int}\sigma$,

(b) $2 - a - c - n \geq 0$ if and only if $-K_X \in \sigma$.

(c) If condition $(*)$ holds for $(n; a, b, c)$, then $2 - a < c + n$.

It is easy to solve these inequalities for a, c, n with the results of Lemma 34. In case (a), the solutions are exactly those 4-tuples $(n; a, b, c)$ for which (n, d) is marked by $\bullet_{abc}$. In case (b), the solutions are exactly those $(n; a, b, c)$ for which (n, d) is marked by $\circ^{(abc)}$. Condition $(*)$ holds for each of the solutions that have $a = 2$; it fails only at the special member of some solutions when $a = 1$, which are the [abc] cases. Part (c), which is virtually no condition at all when $n \geq 0$.

Proof of Proposition 32 We briefly sketch the proof, which is standard. We take on the first assertion first. If $n \geq 0$, then X is nef and big. Denote $U = \mathbb{F} \setminus X$ the open complement. The Lefschetz hyperplane theorem implies that $H_c^{8-i}(U) = H_i(U) = (0)$ for $i > 4$ hence $H_c^2(U) = H_c^3(U) = (0)$ and from the standard exact sequence

$$H_c^2(U) \to H^2(\mathbb{F}) \to H^2(X) \to H_c^3(U)$$

we deduce that $H^2(X) = H^2(\mathbb{F}) = \mathbb{Z}^2$. Since $H^1(X, \mathcal{O}_X) = (0)$, it follows that $\operatorname{Pic}(X) = \mathbb{Z}^2$, based by L and M. When $n < 0$ X is not nef and we can proceed in various ways. For example, we can observe that the proper transform $X' \subset \mathbb{F}'$ is nef and big on the variety $\mathbb{F}'$ obtained from $\mathbb{F}$ by flipping the curve Γ (see the Appendix for information on $\mathbb{F}'$). By Lefschetz then $H^2(X') = \mathbb{Z}^2$. The birational map $X' \dashrightarrow X$ is either an isomorphism in codimension 1, or else X' contains the whole of the flipped set $\mathbb{P}(a, b, c)$. In either case, we conclude that $H^2(X) = \mathbb{Z}^2$ and the statement follows.

The exact sequence in 2. is standard.

Consider now the generic fibre S of $X \to \mathbb{P}^1$ which is a nonsingular cubic surface embedded in $\mathbb{P}^3$ over the function field $\mathbb{C}(s)$ of $\mathbb{P}^1$ with coordinates x, y, z, t. See [KSC03] for a tutorial in the elementary methods of such cubics. We claim that S has Picard rank 1. An example of such S is the surface

$$S_0 = py^3 + qz^3 + rt^3 + x^2 t$$

where $p, q, r \in \mathbb{C}(s)$ are chosen generally. It is easy to calculate the 27 lines on such a cubic surface, and then to figure out the action of the Galois group $\operatorname{Gal}(K/\mathbb{C}(s))$ for any extension $K/\mathbb{C}(s)$ in which the lines are geometric: for general p, q, r, the Galois group is $\mathbb{Z}/3 \times \mathbb{Z}/3 \times \mathbb{Z}/2$ and the 27 lines split into orbits of size 3,6,9,9. One checks by explicit computation that the smaller orbits are made of unions of entire hyperplane sections over K so there are no Galois orbits of disjoint lines. Segre's theorem, [KSC03] Theorem 2.16, then implies that the Picard rank of S_0 is 1. Since a special surface S_0 in the family has rank 1, so does the general surface S. This shows statement 3.

The final claim follows from an elementary monomial argument. Indeed $x^2 t \in F$ with coefficient a homogeneous polynomial $\varphi(u, v)$. The singularities of X along Γ are at the zeros of φ and if X is general we may assume that these all have multiplicity 1. If $u|\varphi$, for example, the singularity at $u = 0$ is of the form $ut + \cdots = 0$. This implies that the general surface section has a singularity of type A_k there. □

4.3.4 The K_X^2 condition and Pukhlikov's Theorem

Lemma 36. *If $X \to \mathbb{P}^1$ is a dP_3 fibration, then*

$$K_X^2 \in \operatorname{Int} \overline{\mathrm{NE}}(X) \quad \textit{implies that} \quad 3d + 5n < 12.$$

If in addition $n < 0$, then

(a) *the 1-cycles* M^2L *and* $\Gamma\subset X$ *bound the Mori cone* $\overline{\mathrm{NE}}(X)$

(b) $K_X^2\in\operatorname{Int}\overline{\mathrm{NE}}(X)$ *if and only if* $3d+5n<12$.

Proof. Suppose first that $n<0$. The general fibre of $X\to\mathbb{P}^1$ is a cubic surface, so M^2L, a line in the fibre, lies on one extreme ray of $\overline{\mathrm{NE}}(X)$. Since $a>0$, there is a projective morphism from the scroll $\mathbb{F}$ that contracts Γ, so the curve $\Gamma\subset X$ generates the other ray of $\overline{\mathrm{NE}}(X)$, proving (a).

Part (b) follows from (a) by calculating the 1-cycle K_X^2 in the basis Γ, M^2L. Denoting $i\colon X\hookrightarrow\mathbb{F}$ for the inclusion and noting that

$$\Gamma=(M-aL)(M-bL)(M-cL)=M^3-dM^2L,$$

we calculate $i_*K_X^2=(-K)^2X$ on $\mathbb{F}$ as

$$\begin{aligned}\big(M+(2-d-n)L\big)^2(3M+nL) &= 3M^3+\big(6(2-d-n)+n\big)M^2L\\ &= 3\Gamma+(12-3d-5n)M^2L.\end{aligned}$$

Now let n be any integer. Suppose that $K_X^2\in\operatorname{Int}\overline{\mathrm{NE}}(X)$. We calculate in $\mathrm{N}_1(\mathbb{F})$ omitting i_*. Let σ be the closed subcone of effective divisors spanned by K_X^2 and M^2L. If $\Gamma\notin\sigma$, then K_X^2 is a strictly convex combination of Γ and M^2L so the inequality holds. If $\Gamma\in\sigma$, then some multiple $\Sigma=k\Gamma$ of the class of Γ (on $\mathbb{F}$) contains an effective curve that lies on X. But $\Gamma\subset\mathbb{F}$ can be contracted, so Σ must be supported on Γ. Therefore $\Gamma\subset X$ and Γ generates an extremal ray of $\overline{\mathrm{NE}}(X)$. The inequality follows as before. $\square$

4.4 Nonrigid dP_3 fibrations

We summarise all the examples of nonrigid dP_3 fibrations that we know in Table 2. We do not dwell on the well-known cases:

- $(n,d)=(0,1)$ links to the cubic 3-fold $Y=Y_3\subset\mathbb{P}^4$, the dP_3 fibration being the pencil of $\mathbb{P}^3$s through a plane intersecting Y in a cubic curve
- $(n,d)=(1,0)$ links to $Y=\mathbb{P}^3$, the dP_3 fibration being the pencil of any pair of transverse cubic surfaces in Y
- $(n,d)=(1,1)$ links to $Y_{3,3}\subset\mathbb{P}(1^5,2)$, the dP_3 fibration being a pencil of divisors having maximal vanishing at the singular point; see [CM], [BZ].

We discuss the table below and then make three detailed studies of examples.

4.4.1 Overview of Table 2

Each entry of Table 2 represents a family of dP_3 fibrations

$$X\in|3M+nL|\subset\mathbb{F}(0,a,b,c)$$

for which the 2-ray game on some member X results in a Sarkisov link to another model of X as a Mfs. The general member X of families 1–7 is nonsingular. In

No.	n	a, b, c	μ	Link of $-\mu K_X - L$	other model
1	1	$0, 0, 1$	3	9-flop then $(2, 0)$ to $\frac{1}{2}(1, 1, 1)$ singularity	$Y'_{3,3} \subset \mathbb{P}^5(1^5, 2)$ general in its family
2	0	$0, 1, 1$	1	3-flop	dP_3 fibration, same numerology as X
3	-1	$1, 1, 1$	1	flop	conic bundle over $\mathbb{P}^2$ with $\deg \Delta = 7$
4	-2	$1, 1, 2$	1	flop then $(2, 1)$ to linear $\mathbb{P}^1 \cong \ell \subset Y'$	$Y'_4 \subset \mathbb{P}^4(1^4, 2)$
5	-2	$1, 2, 2$	1	Francia antiflip then flop	dP_2 fibration with $\frac{1}{2}(1, 1, 1)$ on 1 fibre
6	-3	$1, 2, 3$	1	Francia antiflip then $(2, 0)$ to $P \in Y'$	$Y'_6 \subset \mathbb{P}^4(1^3, 2, 3)$ P a $c\mathrm{D}_4$ singularity
7	-3	$1, 3, 3$	1	toric antiflip $(1, 1, -1, -3)$	dP_1 fibration with $\frac{1}{3}(1, 1, 2)$ on 1 fibre
$8a$	-1	$1, 1, 2$	5	$(1, 1, -1, -1, -3)$, 7-flop, $(2, 0)$	$Y' \subset \mathbb{P}(1^4, 2, 3, 4)$ general, $P = \frac{1}{4}(1, 1, 3)$
$8b$	-1	$1, 1, 2$	3	$(1, 1, -1, -1, -4)$, 3-flop, $(2, 0)$	$Y' \subset \mathbb{P}^5(1^4, 2^2)$?
9	-2	$1, 1, 3$	3	$(1, 1, -1, -1, -4)$, 3-flop, $(2, 0)$	$Y' \subset \mathbb{P}^5(1^2, 2^2, 3, 5)$?
10	-2	$1, 2, 3$	3	$(1, 1, -1, -2, -7)$, $(2, 0)$	$Y' \subset \mathbb{P}^5(1^2, 2^3, 3)$?

Table 2: Nonrigid dP_3 fibrations $X \in |3M + nL| \subset \mathbb{F}(0, a, b, c)$

families 3–7, every X (not just the general member) admits the link as described. Families 8–10 necessarily have a singularity on Γ, which is described (in new coordinates) for general X as follows.

No.	$8a$	$8b$	9	10
Equation	$xy = zt$	$xy = z^3 + t^3$	$xy = z^3 + t^3$	$xy = z^3 + t^6$

In families 8–10, only special members X admit the link as described.

Our method for calculating the other model is to calculate the graded ring of $-\mu K_X - L$. In these four cases, though, this does not present the other model well, since, we believe, the contraction is to a non-Gorenstein singularity $P \in Y'$. We have not yet made the required calculations, but it seems likely that the other model in families 8b, 9, 10 is better presented as a complete intersection in weighted projective spaces $\mathbb{P}^5(1^4, 2^2)$, $\mathbb{P}^5(1^2, 2^2, 3, 5)$ and $\mathbb{P}^5(1^2, 2^3, 3)$ respectively, the index of $P \in Y'$ being 2, 5, 2 respectively.

We draw attention to family 2. This links pair of dP_3 fibrations $X \dashrightarrow X'$ that lie in the same family. We guess that, in general, they are not isomorphic and that they form a bi-rigid pair. Indeed, if they were isomorphic, this link would be an 'untwisting' link (in the sense of [CPR00]), and we would be inclined to guess that X is rigid. We know little about this case.

The numbering of cases in the first column is arbitrary. Apart from this, the information for each entry is separated into three columns, and we describe the contents of each of these in turn:

1. This lists the integers n and a, b, c that determine the family in question, as well as the $\mu > 0$ for which $-\mu K_X - L$ determines an edge of the mobile cone $\overline{\mathrm{NM}}^1(X)$. Note that, as Conjecture 26 predicts, condition $(*)$ is not satisfied by these examples

2. This describes the link. We express the antiflips as $\mathbb{C}^\times$ quotients by listing the characters of a $\mathbb{C}^\times$ action; we comment further below. The word 'flop' means the flop of a single rational curve. We say 'n-flop' when, for general X, an analytic neighbourhood of the flopping curve consists of the disjoint union of n rational curve neighbourhoods. The notation $(2, p)$ indicates a divisorial contraction to a point (when $p = 0$) or a line (when $p = 1$).

3. This final column gives a sketch of the other model.

We say more about the antiflips. The Francia antiflip replaces a $\mathbb{P}^1$ in the nonsingular locus having normal bundle $\mathcal{O}(-1)\oplus\mathcal{O}(-2)$ by a $\mathbb{P}^1$ passing through an index 2 terminal quotient singularity. This is worked out in Section 4.4.2. The notation $(1, 1, -1, -3)$ denotes a toric 3-fold antiflip similar to the Francia flip, but with an index 3 singularity. An example of the hypersurface antiflips $(1, 1, -1, -1, -m)$ is described in Section 4.4.3: typically, these replace a single rational curve passing through a terminal Gorenstein point with a bouquet of $m - 1$ rational curves meeting in an index m singular point. See [Bro99] for details and lists of these flips.

Other models as strict Mfs Four cases link to another strict Mfs. Family 2 is discussed above, and family 3 is linked to the conic bundle of Section 3.4.2.

The other two, families 5 and 7, share two novelties. First, the use of weighted scrolls for describing dP_1 and dP_2 fibrations contrasts with Grinenko's use of finite morphisms to nonsingular scrolls. As calculated in Section 4.4.2, the other model of family 5 is the general element

$$X' \in |4M - L| \quad \text{in the weighted scroll} \quad \begin{pmatrix} 0 & 0 & 1 & 2 & 1 & 1 \\ 1 & 1 & 0 & -1 & -1 & -1 \end{pmatrix}.$$

Similarly, one sees the other model of family 7 as the general element

$$X' \in |6M - 3L| \quad \text{in the weighted scroll} \quad \begin{pmatrix} 0 & 0 & 2 & 3 & 1 & 1 \\ 1 & 1 & -1 & -2 & -1 & -1 \end{pmatrix}.$$

Second, these examples present nonrigid dP_1 and dP_2 fibrations. Grinenko [Gria] has complete classifications of such nonrigid dP_k in the Gorenstein case, and the nonrigid examples are rare. The specimens here have singularities of index 2 and 3, so they are not subject to Grinenko's classifications, but they do invite one to extend Grinenko's results to the higher index case.

4.4.2 Family 5: general members are nonrigid

We work out the link arising from the 2-ray game for family 5 using the wall-crossing methods of Section 3.4.3 and the Appendix. The calculations are similar for families 2–7. Consider any dP_3 fibration

$$X\colon (F=0) \in |3M-2L| \subset \mathbb{F} = \mathbb{F}(0,1,2,2).$$

The polynomial F is a combination of the monomials of the Newton polygon

$\deg_{u,v}$	fibre monomial
0	xy^2, x^2z, x^2t
1	y^3, xyz, xyt
2	$y^2z, y^2t, xz^2, xzt, xt^2$
3	yz^2, yzt, yt^2
4	z^3, z^2t, zt^2, t^3

where $\deg_{u,v}$ denotes the degree in u, v of the coefficient. We know that F must involve both $xy^2 + \varphi_1(u,v)y^3$ and $x^2z + x^2t$ nontrivially. We continue to use the notation $\Gamma\colon (y=z=t=0) \subset X$.

The chamber defining $\mathbb{F}$ is $\mathbb{R}_+[L] + \mathbb{R}_+[M]$. Crossing the wall into the next chamber $\mathbb{R}_+[M] + \mathbb{R}_+[M-L]$ corresponds to a birational map $\mathbb{F} \dashrightarrow \mathbb{F}_1$ that factors through the contraction of $\Gamma \subset \mathbb{F}$ and the extraction of a surface $E_1\colon (u = v = 0) \subset \mathbb{F}_1$. In fact, $E_1 \cong \mathbb{P}(1,2,2)$ with coordinates y, z, t.

The birational image $X_1 \subset \mathbb{F}_1$ of X is still $F = 0$ so $X_1 \cap E_1$ is a $\mathbb{P}(1,2)$: when $u = v = 0$ we can set $x = 1$, as usual in projective geometry, so the intersection is $y^2 + z + t = 0$ in $\mathbb{P}(1,2,2)$. We see that $-K\Gamma = (M-L)\Gamma = -1$, so the map $X \dashrightarrow X_1$ is an antiflip of Γ and the link proceeds with X_1 if and only if X_1 has terminal singularities. We check this condition in coordinates.

We calculate one patch of X_1 in detail, and leave the others to the reader. The unstable locus of the quotient defining $\mathbb{F}_1$ is

$$(u=v=x=0) \cup (y=z=t=0)$$

so the open set $xz \neq 0$ is a well-defined affine patch U on $\mathbb{F}_1$. There is a residual $\mathbb{Z}/2$ action (the stabiliser of z-axis by the μ-action) defining the chart $\mathbb{A}^4 \to U$ that acts on coordinates u, v, y, t of $\mathbb{A}^4$ by the character $(1,1,1,0)$. The equation of X_1 in U includes the monomial t, and the equation of the antiflipped curve $\Gamma_1 \subset X_1$ is $u = v = t = 0$. An analytic neighbourhood of X_1 near the origin in U is isomorphic to $t = 0$ in the quotient $\mathbb{A}^4/(\mathbb{Z}/2)(1,1,1,0)$, and this is terminal. The flip $X_1 \to X$ is the Francia flip.

We cross to the next chamber giving $\mathbb{F}_1 \dashrightarrow \mathbb{F}'$ which factors as the contraction of $\mathbb{P}^2 \subset \mathbb{F}_1$ (with coordinates u, v, x) and the extraction of a $\mathbb{P}^1 \subset \mathbb{F}'$ (with coordinates z, t). The birational image of X is $X' \subset \mathbb{F}'$. Since F involves $xy^2 + uy^3$, the intersection of X with the exceptional $\mathbb{P}^2$ is a line Σ_1. Clearly $-K\Sigma_1 = 0$ so the map $X_1 \dashrightarrow X'$ is a flop and, in particular, X' has terminal singularities.

The variety $\mathbb{F}'$ has a morphism $\mathbb{F}' \to \mathbb{P}^1$ given by the ratio z, t; fibres are $\mathbb{P}(1,1,2,1)$ with coordinates u, v, x, y. To see this, make row operations on the character matrix (basis changes in $\mathbb{C}^\times \times \mathbb{C}^\times$ that do not alter the quotients):

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & -1 & -2 & -2 \end{pmatrix} \sim \begin{pmatrix} -1 & -1 & -1 & 0 & 1 & 1 \\ 1 & 1 & 2 & 1 & 0 & 0 \end{pmatrix}.$$

The right-hand matrix is better adapted to the unstable locus $(u = v = x = y = 0) \cup (z = t = 0)$ for $\mathbb{F}'$. The appropriate basis of $\mathrm{Pic}(\mathbb{F}')$ is M', L', the line bundles corresponding to the characters $\binom{0}{1}$ and $\binom{1}{0}$ in the basis of the right-hand matrix.

The induced map $X' \to \mathbb{P}^1$ describes X' as a dP_2 fibration: since $x^2t \in F$, it is clear that

$$X' \subset |4M' - L'| \subset \mathbb{F}'.$$

One can check that X' is a general in its family if X was to start with.

In this example, every step of the 2-ray game on X was inherited (by computing birational images) from the steps of the 2-ray game on $\mathbb{F}$, itself an easy toric calculation. Most examples we know have this feature.

Definition 37. Let $X \subset \mathbb{F}$ be a dP_3 fibration in a scroll $\mathbb{F} \to \mathbb{P}^1$ (of any dimension). We say that *the link follows the scroll* if the birational images of X in the 2-ray game of $\mathbb{F}$, together with the birational maps induced between them, make up the 2-ray game of X.

This property is one of the characteristics of *Mori dream spaces*, as introduced by Keel–Hu [HK00]. Our less precise definition is a convenient shorthand for the purposes of this paper only.

4.4.3 Family 8: special members are nonrigid

Consider dP_3 fibrations in the family

$$X\colon (F = 0) \in |3M - L| \subset \mathbb{F} = \mathbb{F}(0,1,1,2).$$

The polynomial F is a combination of the monomials of the Newton polygon

$\deg_{u,v}$	fibre monomial
0	x^2y, x^2z
1	xy^2, xyz, xz^2, x^2t
2	$y^3, y^2z, yz^2, z^3, xyt, xzt$
3	y^2t, yzt, z^2t, xt^2
4	yt^2, zt^2
5	t^3

.

We define

$$\mathrm{val}(F) = \min\{\deg_{u,v} m \mid m \in F \text{ a monomial with nonzero coefficient}\}$$

to separate cases of F according to their leading term in u, v.

Case 1: val$(F) = 0$**.** This is the general case: the coefficient of x^2y or x^2z in F is not zero and so the birational link of such X follows the scroll. Since $-K_X = M - L$, one calculates that the final divisorial contraction is trivial against $-K_X$, so this is a bad link on X.

Case 2: val$(F) = 1$**.** Every term of F is divisible by u or v so we write

$$F = uf - vg \quad \text{with} \quad f, g \in |3M - 2L|.$$

Since u and v do not vanish simultaneously on X, the rational section

$$\xi = f/v = g/u \quad \text{of} \quad \mathcal{O}_X(3M - 3L) \tag{2}$$

is regular: $\xi \in H^0(X, 3M - 3L)$. We regard ξ as a new variable and, using equations (2), recompute the link starting from

$$X\colon (v\xi = f, u\xi = g) \in |3M - 2L| \cap |3M - 2L| \subset \mathbb{F}^5$$

where the weighted scroll $\mathbb{F}^5$ is defined by the $\mathbb{C}^\times \times \mathbb{C}^\times$ action

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 3 & 1 \\ 1 & 1 & 0 & -1 & -1 & -3 & -2 \end{pmatrix}$$

on $\mathbb{A}^7$ with coordinates u, v, x, y, z, ξ, t. We have re-embedded X isomorphically in a larger scroll: projection from ξ is the isomorphism to the original embedding. Necessarily $F \ni x^2t$, since otherwise X would be singular along Γ (the negative section $y = z = \xi = t = 0$), and then either $f \ni x^2t$, or $g \ni x^2t$. Possibly after renaming u, v, we may assume that $f \ni x^2t$. This case now divides into two subcases.

Case 2a: val$(F) = 1$**,** val$(g) = 0$**.** We can write the equations of X in the form

$$\begin{pmatrix} f_2 & f_1 + \xi & f_3 \\ g_1 + \xi & g_2 & g_3 \end{pmatrix} \begin{pmatrix} u \\ v \\ x \end{pmatrix} = 0. \tag{3}$$

Following the link of $\mathbb{F}^5$, we see $X \dashrightarrow X_1$. We solve (3) by

$$\eta = \frac{f_2g_2 - (f_1 + \xi)(g_1 + \xi)}{x} = \frac{f_2g_3 - f_3(g_1 + \xi)}{v} = \frac{(f_1 + \xi)g_3 - f_3g_2}{u} \tag{4}$$

and conclude that $\eta \in H^0(X_1, 5M - 6L)$ since the semistable locus of the action defining X_1 does not include $u = v = x = 0$. Once more, we make a new scroll $\mathbb{F}^6$, by including η among the coordinates, and re-embed $X \subset \mathbb{F}^6$:

$$X\colon (\mathrm{Pfaff}_{4\times 4} M = 0) \subset \mathbb{F}^6$$

where $\mathbb{F}^6$ is defined by the $\mathbb{C}^\times \times \mathbb{C}^\times$ action

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 3 & 5 & 1 \\ 1 & 1 & 0 & -1 & -1 & -3 & -6 & -2 \end{pmatrix}$$

on $\mathbb{A}^8$ with coordinates $u, v, x, y, z, \xi, \eta, t$. Manipulating (4), the equations of $X \subset \mathbb{F}^6$ are the 4×4 Pfaffians of a 5×5 skew symmetric matrix

$$M = \begin{pmatrix} \eta & f_2 & f_1 + \xi & f_3 \\ & g_1 + \xi & g_2 & g_3 \\ & & x & -v \\ & & & u \end{pmatrix}.$$

The 2-ray game of X follows the link of the scroll $\mathbb{F}^6$. We sketch the steps:

1. Antiflip $X \dashrightarrow X_1$. This factors as $X \to Z_1 \leftarrow X_1$, by the contraction of Γ to Z_1, both maps being the restrictions of maps of the scroll. The exceptional locus $u = v = 0$ of $Z_1 \leftarrow X_1$ is defined by the equations of X_1 in $\mathbb{P}^4(1, 1, 3, 5, 1)$ with coordinates y, z, ξ, η, t, since that $\mathbb{P}^4$ is exceptional in the scroll. We set $x = 1$ to see that $t = 0$, since $f_3 \ni xt$, and also that η is eliminated. So antiflipped locus is

   $$\big\{\text{three } \mathbb{P}(1,3)\text{s meeting in a point}\big\} = \big(g_3(y, z) = 0\big) \subset \mathbb{P}(1, 1, 3).$$

   One can check in coordinates, as in Section 4.4.2, that this point is an index 3 terminal singularity on X_1. In Table 2, this antiflip is denoted by $(1, 1, -1, -1, -3)$.

2. Flop $X_1 \dashrightarrow X'$. Since $-K_{X'} = M - L$, curves that are contracted are trivial against $-K_{X'}$. See [BZ] for a detailed calculation, that also counts the number of contracted curves.

3. Divisorial contraction to a point $X' \to Y'$. The divisor $F = (t = 0)$ on the scroll is contracted to a point, so $F \cap X'$ is too. The linear system $|6M - 5L|$ (and its multiples) define the morphism, so since

   $$-6K_X = 6(M - L) = (6M - 5L) + L$$

   is negative on only the flipped curves, it must be relatively ample. Therefore $X' \to Y'$ is extremal, and Y' has terminal singularities.

The result Y' is No. 6 of Altınok's list of codimension 3 Fano 3-folds [Alt98]. This Sarkisov link is already known to us, calculated from the Y' end of the link in [BZ] following Example 9.16 of [Rei].

Case 2b: $\mathrm{val}(F) = 1$**,** $\mathrm{val}(g) = 1$**.** Now g is further specialised:

$$g = uf_1 + vg_1 \quad \text{with} \quad f_1, g_1 \in |3M - 3L|.$$

As usual, since u, v do not vanish simultaneously on X and $u\xi - g$ is identically zero we write

$$u\xi = uf_1 + vg_1 \quad \text{and} \quad \eta = (\xi - f_1)/v = g_1/u \tag{5}$$

and conclude that $\eta \in H^0(X, 3M-4L)$. Again we include η as a new variable and compute the link again starting from

$$X \in |3M-2L| \cap |3M-3L| \subset \mathbb{F}^5$$

$$\text{determined by the action} \quad \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 3 & 1 \\ 1 & 1 & 0 & -1 & -1 & -4 & -2 \end{pmatrix}.$$

Unlike case 2a, here we have eliminated ξ from the coordinates using the equation $\xi = v\eta + f_1$ of (5). Substituting for ξ calculates the equations of $X \subset \mathbb{F}^5$:

$$v^2\eta + vf_1 = f, \quad u\eta = g_1.$$

The 2-ray game of X follows the link of the scroll $\mathbb{F}^5$. The final contraction in the link is given by the linear system $|3M-4L|$ and its multiples on X'.

4.4.4 Unstable dP_3 fibrations and condition $(*)$

Consider the family of dP_3 fibrations

$$X\colon (F=0) \in |3M-4L| \subset \mathbb{F}(0,2,2,4).$$

A general X has a 2-ray game that follows the scroll, but this gives a K_X-trivial bad link. Nevertheless, we find a special X which is nonrigid.

The polynomial F is a combination of the monomials of the Newton polygon

degree	fibre monomial
0	xy^2, xyz, xz^2, x^2t
2	$y^3, y^2z, yz^2, z^3, xyt, xzt$
4	y^2t, yzt, z^2t, xt^2
6	yt^2, zt^2
8	t^3

and we impose conditions on the coefficient polynomials $\alpha(u,v)$: we require u^i to divide the coefficient of fibre monomials according to the table

i	fibre monomial
1	xyt, xzt
2	y^2t, yzt, z^2t
3	xt^2
4	yt^2, zt^2
6	t^3

with coefficients otherwise general. The reader can check that a general such X is nonsingular away from a $c\mathrm{D}_4$ singularity at the point $(0,1;0,0,0,1)$. Note that X satisfies condition $(*)$ because, even though it is special in the family, the defining equation of X involves x^2t with nontrivial coefficient, hence it is still true that the 2-ray game from X follows the scroll (and ends in a bad link).

Following Corti–Kollár [Cor96], [Kol97], X is unstable with respect to the weight system $w = (3, 2, 2, 0)$. Indeed, F is divisible by u^6 after the substitution

$$u^3x', u^2y', u^2z', t' \quad \text{for} \quad x, y, z, t.$$

Cancelling the u^6 factor gives a dP_3 fibration

$$X_{\text{st}} \in |3M - L| \subset \mathbb{F}(0, 1, 1, 1)$$

that is square birational to $X \to \mathbb{P}^1$. Note that the fibre at $u = 0$ has an Eckardt point:

$$X_{\text{st}} \cap (u = 0) = \big(tf_2(x, y, z) = g_3(y, z)\big).$$

Even though it is not general in its family, X_{st} fails condition $(*)$, and X_{st} does have a Sarkisov link, following the scroll, to a conic bundle over $\mathbb{P}^2$.

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A Birational transformations of scrolls

A.1 Definition of scrolls

We set our notation for rational scrolls and toric links between them. Our treatment follows closely [Rei97, Chapter 2].

Throughout the Appendix, we consider actions of the group $G = \mathbb{C}^\times \times \mathbb{C}^\times$ on affine space. The elements of G are ordered pairs (λ, μ) where $\lambda, \mu \in \mathbb{C}^\times$. We denote by $\mathbb{X} = \mathrm{Hom}(G, \mathbb{C}^\times)$ the lattice of characters of G, with basis the coordinate functions (projections on the two factors) χ_1, χ_2 such that $\chi_1(\lambda, \mu) = \lambda$ and $\chi_2(\lambda, \mu) = \mu$. The dual lattice $\mathbb{X}^* = \mathrm{Hom}(\mathbb{X}, \mathbb{Z})$ is based by the 1-parameter subgroups e_1, e_2 such that $e_1(\lambda) = (\lambda, 1)$ and $e_2(\mu) = (1, \mu)$. Sometimes we abuse notation and write λ for the element $e_1(\lambda) \in G$ (and, similarly, μ for $e_2(\mu)$). Occasionally we abuse even further and identify λ with the coordinate function $\chi_1 \colon G \to \mathbb{C}^\times$ (and, similarly, we identify μ with χ_2).

We now define rational scrolls. Fix a base $\mathbb{P} = \mathbb{P}^k$, with homogeneous coordinates $u_0, \dots, u_k$; in this paper, we only work with $k = 1$ or $k = 2$. Consider now $\mathbb{C}^{n+1}$, with coordinates $x_0, \dots, x_n$. Fix integers $a_0, \dots, a_n$, usually nonnegative and in increasing order. Consider the action of G on the affine space $\mathbb{A} = \mathbb{C}^{k+1} \times \mathbb{C}^{n+1}$, where the two factors of G act by

$$\begin{aligned}\lambda \colon (u_0, \dots, u_k, x_0, \dots, x_n) &\mapsto (\lambda u_0, \dots, \lambda u_k, \lambda^{-a_0} x_0, \dots, \lambda^{-a_n} x_n)\\ \mu \colon (u_0, \dots, u_k, x_0, \dots, x_n) &\mapsto (u_0, \dots, u_k, \mu x_0, \dots, \mu x_n).\end{aligned}$$

We summarise this action by writing down the matrix:

$$\begin{pmatrix} 1 & \dots & 1 & -a_0 & \dots & -a_n \\ 0 & \dots & 0 & 1 & \dots & 1 \end{pmatrix}.$$

By definition, the scroll $\mathbb{F} = \mathbb{F}(a_0, \dots, a_n)$ is the following quotient:

$$\mathbb{F}(a_0, a_1, \dots, a_n) = \big(\mathbb{C}^{k+1} \setminus \{0\}\big) \times \big(\mathbb{C}^{n+1} \setminus \{0\}\big)/G.$$

It is clear that $\mathbb{F}$ is a $\mathbb{P}^n$-bundle over $\mathbb{P}^k$.

A.2 Line bundles on scrolls

There is a 1-to-1 correspondence between line bundles on the scroll $\mathbb{F}$ and characters $\chi\colon G \to \mathbb{C}^\times$ of the group G. To a character χ we associate a *G-linearisation* of the trivial line bundle over $\mathbb{A}$, by acting with χ in the direction of fibres. Taking the quotient of the G-linearisation by G, we form a bundle on $\mathbb{F}$. Let us denote by L_χ the resulting bundle. It is easy to chase through the definition and see that the sections of L_χ are the global eigenfunctions with eigenvalue χ:

$$H^0(\mathbb{F}, L_\chi) = \{f\colon (\mathbb{C}^{k+1} \setminus \{0\}) \times (\mathbb{C}^{n+1} \setminus \{0\}) \to \mathbb{C} \mid f(gx) = \chi(g) f(x)\}.$$

Sometimes, we abuse notation and confuse the line bundle and the corresponding character.

We denote by L and M the line bundles corresponding to χ_1 and χ_2. By what we just said, the sections of L are the functions $f\colon (\mathbb{C}^{k+1}\setminus\{0\})\times(\mathbb{C}^{n+1}\setminus\{0\}) \to \mathbb{C}$ such that

$$\begin{aligned} f(\lambda u_0, \dots, \lambda u_k, \lambda^{-a_0} x_0, \dots, \lambda^{-a_n} x_n) &= \lambda f(u_0, \dots, u_k, x_0, \dots, x_n) \\ f(u_0, \dots, u_k, \mu x_0, \dots, \mu x_n) &= f(u_0, \dots, u_k, x_0, \dots, x_n). \end{aligned}$$

It follows that $H^0(\mathbb{F}, L)$ is based by the coordinate functions $u_0, \dots, u_n$ and L is the pull-back $\pi^*\mathcal{O}(1)$ by the natural morphism $\pi\colon \mathbb{F} \to \mathbb{P}^k$. Similarly, the group $H^0(\mathbb{F}, M)$ of global sections of M is based by the monomials

$$u_0^{w_0} \cdots u_k^{w_k} x_i \quad \text{where} \quad w_0 + \cdots + w_k = a_k.$$

A.3 $\mathbb{F}$ as a geometric quotient

Using the language of G-linearisations, we can view $\mathbb{F}$ as a *geometric* quotient $\mathbb{A}^{k+n+2}/\!\!/G$ in the sense of Geometric Invariant Theory. If G acts on $\mathbb{A} = \mathbb{A}^{k+n+2}$ as before and $\chi\colon G \to \mathbb{C}^\times$ is a character, then G acts on functions $f \in \mathcal{O}_\mathbb{A}$ by $gf(x) = \chi(g) f(g^{-1}x)$, and we denote $\mathcal{O}_\mathbb{A}^\chi$ the invariants. The set of semistable points is by definition

$$\mathbb{A}^{\mathrm{ss}}_\chi = \{x \in \mathbb{A} \mid \exists f \in \mathcal{O}^\chi_\mathbb{A},\ f(x) \neq 0\}.$$

The group G acts with finite stabilisers (in fact, freely) on the set of semistable points, and the geometric quotient is by definition

$$\mathbb{A}/\!\!/G = \mathbb{A}^{\mathrm{ss}}_\chi/G.$$

A.4 Linearisations and geometric quotients

Different linearisations lead to different quotients. We state what is going on and leave the elementary proofs to the reader. We say that a linearisation is *useful* if the set of semistable points is nonempty. To fix ideas, let us assume that $0 = a_0 \leq a_1 \leq \cdots \leq a_n$. The cone of useful linearisations is the cone

$$(\mathbb{R}_+[L] + \mathbb{R}_+[M - a_{n-1}L]) \cap \mathbb{X}$$

This cone is partitioned into chambers corresponding to different geometric quotients. For example if $\chi \in \mathbb{R}_+[L] + \mathbb{R}_+[M]$, then $\mathbb{A}^{\mathrm{ss}}_\chi = (\mathbb{C}^{k+1} \setminus \{0\}) \times (\mathbb{C}^{n+1} \setminus \{0\})$ and we get our $\mathbb{F}$ back. However there are the other chambers

$$\sigma_i = (\mathbb{R}_+[M - a_{i-1}L] + \mathbb{R}_+[M - a_iL]) \cap \mathbb{X},$$

whenever $a_{i-1} < a_i$, and if $\chi \in \sigma_i$

$$\mathbb{A}^{\mathrm{ss}}_\chi = (\mathbb{C}^{k+1+i} \setminus \{0\}) \times (\mathbb{C}^{n+1-i} \setminus \{0\}).$$

The corresponding quotient

$$\mathbb{F}_i = \mathbb{A}^{\mathrm{ss}}_\chi / G$$

for $\chi \in \sigma_i$ is birational to $\mathbb{F}$. The sequence of birational maps $\mathbb{F} = \mathbb{F}_0 \dashrightarrow \mathbb{F}_1 \dashrightarrow \cdots$ is a 2-ray game in the sense of [Cor00, Section 2.2]. For example if $a_1 > 0$, the move $\mathbb{F} \dashrightarrow \mathbb{F}_1$ is the antiflip of the section $\Gamma = \{x_1 = x_2 = \cdots = x_n = 0\}$ which generates one of the two extremal rays of $\overline{\mathrm{NE}}(\mathbb{F})$.

A.5 Generalisations

The above can be generalised slightly to the action of G on $\mathbb{A}^{n+1}$ given by the matrix

$$\begin{pmatrix} \alpha_0 & \dots & \alpha_n \\ \beta_0 & \dots & \beta_n \end{pmatrix}.$$

We use this notation in the text as a shorthand for the action

$$x_0, \dots, x_n \mapsto \lambda^{\alpha_0} x_0, \dots, \lambda^{\alpha_n} x_n$$
$$x_0, \dots, x_n \mapsto \mu^{\beta_0} x_0, \dots, \mu^{\beta_n} x_n.$$

Here we assume:

1. the rows are linearly independent (so we get a faithful action of G) and all columns are nonzero,
2. all $\beta_i \geq 0$, and $\beta_n > 0$,
3. the ratios α_i / β_i are in decreasing order.

As before we denote L, M the G-linearisations corresponding to the characters χ_1, χ_2. The cone of useful linearisations

$$(\mathbb{R}_+[\alpha_0 M - \beta_0 L] + \mathbb{R}_+[\alpha_{n-1} M - \beta_{n-1} L]) \cap \mathbb{X}$$

is partitioned into chambers

$$\sigma_i = (\mathbb{R}_+[\alpha_{i-1} M - \beta_{i-1} L] + \mathbb{R}_+[\alpha_i M - \beta_i L]).$$

(when $\alpha_{i-1}/\beta_{i-1} > \alpha_i/\beta_i$). Choosing a character $\chi \in \sigma_i$ gives a semistable locus $\mathbb{A}^{\mathrm{ss}}_\chi = (\mathbb{C}^{i+1} \setminus \{0\}) \times (\mathbb{C}^{n-i} \setminus \{0\})$ and a geometric quotient $\mathbb{F}_i = \mathbb{A}^{\mathrm{ss}}_\chi / G$.

A.6 Alternative viewpoints

There are at least two other points of view on birational maps of scrolls.

We can identify the cone of useful G-linearisations with the mobile cone $\overline{\mathrm{NM}}^1(\mathbb{F})$ of the scroll. From this point of view, the main focus is the bigraded ring

$$\bigoplus_{e,n} H^0(\mathbb{F}, eM + nL).$$

We can also view scrolls, and their generalisations, as special cases of rank 2 toric varieties.