Fano 3-folds, K3 surfaces and graded rings

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Abstract. Explicit birational geometry of 3-folds represents a second phase of Mori theory, going beyond the foundational work of the 1980s. This paper is a tutorial and colloquial introduction to the explicit classification of Fano 3-folds (also known by the older name $\mathbb{Q}$-Fano 3-folds), a subject that we hope is nearing completion. With the intention of remaining accessible to beginners in algebraic geometry, we include examples of elementary calculations of graded rings over curves and K3 surfaces. For us, K3 surfaces have at worst Du Val singularities and are polarised by an ample Weil divisor (you might prefer to call these $\mathbb{Q}$-K3 surfaces); they occur as the general elephant of a Fano 3-fold, but are also interesting in their own right. A second section of the paper runs briefly through the classical theory of nonsingular Fano 3-folds and Mukai's extension to indecomposable Gorenstein Fano 3-folds. Ideas sketched out by Takagi at the Singapore conference reduce the study of $\mathbb{Q}$-Fano 3-folds with $g \geq 2$ (and a suitable assumption on the general elephant) to indecomposable Gorenstein Fano 3-folds together with unprojection data.

Much of the information about the anticanonical ring of a Fano 3-fold or K3 surface is contained in its Hilbert series. The Hilbert function is determined by orbifold Riemann–Roch (the Lefschetz formula of Atiyah, Singer and Segal, see Reid [**YPG**]); using this, we can treat the Hilbert series as a simple collation of the genus and a basket of cyclic quotient singularities. Many hundreds of families of K3s and Fano 3-folds are known, among them a large number with $g \leq 0$, and Takagi's methods do not apply to these. However, in many cases, the Hilbert series already gives firm indications of how to construct the variety by biregular or birational methods. A final section of the paper introduces the K3 database in Magma, that manipulates these huge lists without effort.

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1. Introduction

1.1. The graded ring $R(X, A)$ of a polarised variety. Let X be an irreducible projective variety over $\mathbb{C}$, and A an ample divisor on X (see below for more explanation). For $n \geq 0$, we write

$$H^0(X, nA) = H^0(X, \mathcal{O}_X(nA)) = \mathcal{L}(nA) = \{f \in \mathbb{C}(X) \mid \operatorname{div} f + nA \geq 0\}$$

for the *Riemann–Roch space* (RR space) of nA. That is, $H^0(X, nA)$ is the finite dimensional vector space of rational (or meromorphic) functions $f \in \mathbb{C}(X)$ with divisor of poles $\leq nA$. Our basic construction is the graded ring

$$R(X, A) = \bigoplus_{n \geq 0} H^0(X, nA), \tag{1.1.1}$$

where the product is simply multiplication of rational functions

$$H^0(X, nA) \times H^0(X, mA) \to H^0(X, (n+m)A) \quad \text{by} \quad (f, g) \mapsto fg.$$

This just says that if f, g are rational functions with poles $\leq nA, mA$ then fg has poles $\leq (n+m)A$.

Our special interest is the case when the ring $R(X, A)$ can be described by explicit generators and relations, for example, as a polynomial ring or a polynomial ring divided by a principal ideal (geometrically, a hypersurface). When this is possible, it corresponds to embedding X in projective space and determining the defining equations of the image variety. It frequently happens in higher dimensions that the generators x_i of $R(X, A)$ have different weights, so we have to work with weighted projective spaces (*w.p.s.*) and weighted homogeneous ideals. Dolgachev [**D**] and Fletcher [**Fl**] are useful as general references on w.p.s. and their complete intersections (*c.i.*).

Remarks 1.2. Although we intend to allow X to have mild singularities, please think of it in the first instance as nonsingular. The point of the RR spaces $H^0(X, nA) = \mathcal{L}(nA)$ is this: because X is projective, the only globally defined regular functions (holomorphic functions) on it are the constants. By allowing poles along a divisor nA, we get a finite dimensional space of rational (or meromorphic) functions $H^0(X, nA)$; as we discuss below, the RR theorem predicts $\dim H^0(X, nA)$ in good cases. Choosing a basis $x_0, x_1, \ldots, x_k \in H^0(X, nA)$ defines a rational map

$$\varphi_{nA} \colon X \dashrightarrow \mathbb{P}^k \quad \text{by} \quad P \mapsto (x_0(P) : \cdots : x_k(P));$$

there are straightforward criteria that determine when φ_{nA} is a morphism $X \to \mathbb{P}^k$ or an embedding $X \hookrightarrow \mathbb{P}^k$. The divisor nA is *very ample* if it defines an embedding $\varphi_{nA} \colon X \hookrightarrow \mathbb{P}^k$, and A is *ample* if nA is very ample for some n.

1.3. The RR theorem for a curve. Please skip this stuff if you already know it. If not, we strongly advise you to make an effort to *commit it to memory*, since it is one of the central points of algebraic geometry. A nonsingular projective curve C (or compact Riemann surface) has a *genus* g that can be defined in several alternative ways:

(1) C is homeomorphic to the traditional picture of a surface with g holes.
(2) C has Euler number $e(C) = 2 - 2g$.
(3) The tangent bundle to C has degree $2 - 2g$, or equivalently, the canonical divisor class K_C has degree $2g - 2$. Here K_C is defined as the class of the divisor of a rational (or meromorphic) differential.

(4) $\dim H^0(C, \mathcal{O}_C(K_C)) = \dim \mathcal{L}(K_C) = g$. In words, g equals the dimension of the vector space of global regular (or holomorphic) differentials on C.
(5) etc.; for example, you can define g as the integer appearing in the RR theorem for curves (1.4.1).

THEOREM 1.4 (RR for curves). *Let $D = \sum n_i P_i$ be a divisor on C. Then*

$$\dim H^0(D) - \dim H^0(K_C - D) = 1 - g + \deg D, \tag{1.4.1}$$

where $\deg D = \sum n_i$. *(Recall that $H^0(D) = \mathcal{L}(D)$ and $H^0(K_C - D) = \mathcal{L}(K_C - D)$, and in this language the formula is*

$$\dim \mathcal{L}(D) - \dim \mathcal{L}(K_C - D) = 1 - g + \deg D.)$$

When $\deg D > 2g - 2$, we get $H^0(K_C - D) = 0$, so (1.4.1) simplifies to a formula for $\dim H^0(D)$ in terms of topological invariants of C and D.

EXAMPLE 1.5. As a baby case of our graded ring methods, we compute the canonical ring $R(C, K_C)$ of a curve of genus 3, and conclude that C is a plane quartic curve under suitable extra assumptions.

Take C to be a nonsingular projective curve of genus 3, with polarising divisor K_C. The starting point is that RR gives $\dim H^0(nK_C)$:

$$\dim H^0(nK_C) \;=\; \begin{cases} 1 & \text{if } n = 0, \\ g & \text{if } n = 1, \\ (2n-1)(g-1) & \text{if } n \geq 2. \end{cases} \tag{1.5.1}$$

The last number comes from (1.4.1), with the r.h.s. equal to $1 - g + n(2g - 2)$.

Now $\dim H^0(C, K_C) = 3$. As we said above, choosing a basis x_1, x_2, x_3 of $H^0(C, K_C)$ defines a rational map $\varphi_{K_C} \colon C \dashrightarrow \mathbb{P}^2$. A traditional and easy argument based on RR applied to the divisors $K_C - P$ and $K_C - P - Q$ proves that φ_{K_C} is a morphism, either defining an isomorphism $C \cong C_4 \subset \mathbb{P}^2$ of C with a plane quartic curve C_4, or a generically 2-to-1 cover $C \to Q_2 \subset \mathbb{P}^2$ over a conic.

We argue somewhat more algebraically on the ring

$$R(C, K_C) = \bigoplus_{n \geq 0} H^0(C, nK_C)$$

using (1.5.1); in degree two, $\dim H^0(C, 2K_C) = 6$. But we already know 6 elements of $H^0(C, 2K_C)$, namely the 6 quadratic monomials

$$S^2 x_i = \{x_1^2, x_1x_2, x_2^2, x_1x_3, x_2x_3, x_3^2\}.$$

We assume that these monomials are linearly independent, so form a basis of $H^0(C, 2K_C)$. Likewise, in degree 3, $\dim H^0(C, 3K_C) = 10$, and there are $10 = \binom{5}{2}$ cubic monomials $S^3 x_i$, so we assume that they are linearly independent, and again form a basis. In degree 4, however, we necessarily find a relation, since $\dim H^0(C, 4K_C) = 14$, but there are $15 = \binom{6}{2}$ quartic monomials $S^4 x_i$.

The conclusion is the prediction that the graded ring $R(C, K_C)$ has the simplest form $R(C, K_C) = \mathbb{C}[x_1, x_2, x_3]/(f_4)$; and the corresponding map $\varphi_{K_C} \colon C \to \mathbb{P}^2$ is an embedding with image C_4 given by $f_4(x_1, x_2, x_3) = 0$.

REMARK 1.6. The assumption that the quadratic monomials $S^2 x_i$ are linearly independent is of course the hyperelliptic dichotomy. In the nonhyperelliptic case,

the image $\varphi_{K_C}(C)$ cannot be contained in a curve of degree ≤ 3 union a finite set, so that once the relation f_4 has been detected, one sees that the ring homomorphism

$$\mathbb{C}[x_1, x_2, x_3]/(f_4) \to R(C, K_C)$$

must be injective. Then it is surjective, because in degree n the quotient ring has dimension $\binom{n+4}{2} - \binom{n}{2} = \dim H^0(C, nK_C)$.

In the hyperelliptic case, the canonical ring is

$$\mathbb{C}[x_1, x_2, x_3, y]/(Q_2, F_4),$$

where $Q_2(x) = 0$ is the equation of the image conic, y the new generator in degree 2 required to compensate for this relation, and $F_4 : y^2 = f_4(x)$. Then $C \to Q_2$ is the double cover ramified in the 8 points $Q_2 = f_4 = 0$.

We can even consider a degenerating family with quadratic relation $\lambda y = Q_2(x)$ that consists of a nonhyperelliptic curve if $\lambda \neq 0$ and a hyperelliptic curve if $\lambda = 0$.

Example 1.7. Consider now a nonhyperelliptic curve C of genus 6. Its canonical embedding $\varphi\colon C_{10} \hookrightarrow \mathbb{P}^5$ has image of codimension 4. As opposed to the hypersurface case, we do not have any surefire way of predicting the equations of a variety of codimension ≥ 4. This is a major preoccupation of the rest of the paper. In this case, we happen to be lucky, but it still involves a case division into Brill–Noether special and Brill–Noether general:

Case 1. C is trigonal or isomorphic to a plane quintic. Trigonal means that C has a linear system g^1_3; equivalently, C can be represented as a triple cover $C \to \mathbb{P}^1$. It is well known that then $\varphi_{K_C}(C)$ is contained in a quartic surface scroll (typically, $\mathbb{P}^1 \times \mathbb{P}^1$ embedded by $\mathcal{O}(1,2)$); its equations are the 6 quadrics defining the scroll and 3 cubics defining C inside the scroll. The case of a plane quintic is similar, with $\varphi_{K_C}(C)$ contained in the Veronese image of $\mathbb{P}^2$, and defined by the 6 quadratic equations of the Veronese surface and 3 cubics defining C in it.

Case 2. C is not trigonal and not isomorphic to a plane quintic. Then the canonical image $\varphi_{K_C}(C)$ is a c.i. in $\mathrm{Grass}(2,5)$:

$$C \;=\; \mathrm{Grass}(2,5) \cap Q_2 \cap H_1 \cap \cdots \cap H_4 \subset \mathbb{P}^9.$$

Remarks 1.8. Here $\mathrm{Grass}(2,5)$ is the Grassmann variety of 2-dimensional vector subspaces of $\mathbb{C}^5$ in its Plücker embedding in $\mathbb{P}^9 = \mathbb{P}(\bigwedge^2 \mathbb{C}^5)$. Equivalently, it is the determinantal variety defined by the 4×4 diagonal Pfaffians $\mathrm{Pf}_{ij.kl}$ of the 5×5 skew matrix

$$\begin{pmatrix} x_{12} & x_{13} & x_{14} & x_{15} \\ & x_{23} & x_{24} & x_{25} \\ & & x_{34} & x_{35} \\ & & & x_{45} \end{pmatrix},$$

where $\pm\,\mathrm{Pf}_{ij.kl} = x_{ij}x_{kl} - x_{ik}x_{jl} + x_{il}x_{jk}$ for $\{i,j,k,l\} \subset \{1,2,3,4,5\}$. (We only write the upper triangular elements. For the whole 5×5 matrix, just write zeros in the 5 diagonal entries and the skew elements $-x_{ji}$.)

Although Case 2 is a 19th century result, the direct statement and proof is due to Mukai ([**Mu1**], Section 5), and is a first substantial case of his general program. He proves that there exists a unique rank 2 stable vector bundle E on C with $\det E = K_C$ and $H^0(C, E) = \mathbb{C}^5$; moreover, E is generated by its H^0, and $\bigwedge^2 H^0(C, E) \twoheadrightarrow H^0(C, \bigwedge^2 E)$. By the universal mapping property of the

Grassmann variety, E and its sections define a morphism $\psi_E\colon C \to \mathrm{Grass}(2,5)$, making φ_{K_C} a linear section of the Plücker embedding:

$$\begin{array}{ccc} C & \hookrightarrow & \mathbb{P}^5 \\ \downarrow & & \cap \\ \mathrm{Grass}(2,5) & \hookrightarrow & \mathbb{P}^9 \end{array}$$

Since $g = 6 \geq 2 \cdot 3$, standard Brill–Noether arguments give the existence of a decomposition $K_C = \xi + \eta$ with divisors ξ and η such that $H^0(C,\xi) = 2$ and $H^0(C,\eta) = 3$. RR (1.4.1) gives $\deg \xi = 4$, and the case assumptions imply that $|\xi|$ is a free g^1_4 and $|\eta|$ a free g^2_6. For every such decomposition $K_C = \xi + \eta$, the bundle E appears as the unique extension $\xi \hookrightarrow E \twoheadrightarrow \eta$ for which the boundary map $H^0(\eta) \to H^1(\xi)$ is zero.

Example 1.9. As in Example 1.5, let C be a nonsingular quartic curve; however, instead of K_C, we choose a point $P \in C$ and consider the *fractional divisor* $A = K_C + \frac{1}{2}P$. The RR space of nA is given by the same formula:

$$H^0(C,nA) = \mathcal{L}(nA) = \big\{ f \in \mathbb{C}(C) \;\big|\; \operatorname{div} f + nA \geq 0 \big\}.$$

The novel point is that a rational function cannot have a pole of fractional order. Thus although the definition allows f a pole of order $n/2$ at P, in practice this restricts it to have pole of order $\lfloor \frac{n}{2} \rfloor$.

The first time this has any effect is when $n = 2$, when

$$H^0(C,2A) = H^0(C,2K_C + P)$$

has dimension 1 bigger than $H^0(C,2K_C)$. Thus the ring $R(C,A)$ needs a new generator y in degree 2. In explicit terms, if $P = (1,0,0) \in C \subset \mathbb{P}^2$ then the equation of C is $f_4 = x_2A + x_3B = 0$ where A, B are cubics in x_1, x_2, x_3, and $y = B/x_2 = -A/x_3$ is a rational section of $\mathcal{O}_C(2)$ with pole at P. It follows that

$$R(C,A) = \mathbb{C}[x_1,x_2,x_3,y]/(yx_2 - B, yx_3 + A).$$

Remarks 1.10. The ring $\mathbb{C}[x_1,x_2,x_3,y]/(yx_2 - B, yx_3 + A)$ constructed in Example 1.9 corresponds to embedding $C \hookrightarrow \mathbb{P}(1,1,1,2)$ as a $(3,3)$ c.i. Here the w.p.s. $\mathbb{P}(1,1,1,2)$ is the cone over the Veronese surface, and C passes through the cone point $(0,0,0,1)$ as a nonsingular branch, with the equations $yx_2 = B, yx_3 = -A$ determining x_2, x_3 as implicit functions.

The fractional divisor $K_C + \frac{r-1}{r}P$ is the *orbifold canonical class* of C, where P is viewed as an orbifold point of order r. The ring $R(C,A)$ is again Gorenstein (see Watanabe [**W**]); we treat the case $K_C + \frac{2}{3}P$ in Example 3.7. More generally, we could take any *basket* of orbifold points P_i of order r_i on C and consider the orbifold canonical class $A = K_C + \sum \frac{r_i - 1}{r_i} P_i$. The affine cone $\operatorname{Spec} R(C,A)$ corresponds to a $\mathbb{C}^*$ fibration over C with isotropy μ_{r_i} over P_i, the $\mathbb{C}^*$ analog of a Seifert fibre space. The fact that $R(C,A)$ is a Gorenstein ring means that there is a category of orbifold sheaves on C (more precisely, modules over its affine cone $\mathcal{C}C = \operatorname{Spec} R(C,A)$) having a nice form of Serre duality.

A curve with an orbifold point is a substantial first case of the cyclic quotient singularities needed in higher dimension. For example, our construction of $C_{3,3} \subset \mathbb{P}(1,1,1,2)$ extends in the obvious way to a K3 surface $S_{3,3} \subset \mathbb{P}(1^4,2)$, Fano 3-fold $V_{3,3} \subset \mathbb{P}(1^5,2)$, etc.

2. Classic Fano 3-folds

The subject starts in the 1930s, when Fano studies projective 3-folds having canonical curve sections. His definition is projective: $V = V^3_{2g-2} \subset \mathbb{P}^{g+1}$ should have canonical curves as its codimension 2 linear sections:

$$V \cap H_1 \cap H_2 \;=\; C_{2g-2} \subset \mathbb{P}^{g-1},$$

but this is more-or-less equivalent to assuming that $-K_V$ is very ample (see below). Iskovskikh modernised Fano's treatment in the 1970s. His starting point is a nonsingular 3-fold V with $-K_V$ ample. He analyses the rather few exceptions to $-K_V$ very ample, and corrects and reworks Fano's classification. From the 1980s onwards Mukai discovers a new interpretation of Fano and Iskovskikh's results in terms of linear sections of projective homogeneous varieties, and generalises them to indecomposable Gorenstein Fano 3-folds (see below for explanation).

Definition 2.1. A *nonsingular Fano* 3*-fold* is a nonsingular irreducible projective variety of dimension 3 with ample $-K_V$. Likewise, a *Gorenstein Fano* 3*-fold* is a normal irreducible projective 3-fold V with at worst canonical Gorenstein singularities and $-K_V$ ample. The *genus* of a Fano 3-fold is the integer g defined by $\dim H^0(V, -K_V) = g+2$. In the nonsingular case, we deduce that $2g-2 = (-K_V)^3$.

We say that V is *prime* if $\operatorname{Cl} V = \mathbb{Z} \cdot (-K_V)$ (here $\operatorname{Cl} V$ is the Weil divisor class group); and V is *indecomposable* if there does not exist any decomposition $-K_V = A + B$ where A, B are Weil divisors such that $|A|, |B|$ are nontrivial linear systems.

The point of these definitions is to exclude easy cases such as $\mathbb{P}^3$ (for which $-K = \mathcal{O}(4)$) and quadric or cubic hypersurfaces $Q_2, F_3 \subset \mathbb{P}^4$ (for which $-K = \mathcal{O}(3)$ or $\mathcal{O}(2)$ respectively), that can be handled by simpler methods. The prime condition is Fano and Iskovskikh's assumption that V is factorial and has rank $\rho(V) = \operatorname{rank} \operatorname{Pic} V = 1$ and index 1; indecomposable is Mukai's generalisation. Nonsingular Fano 3-folds with $\rho \geq 2$ were treated in detail by Mori and Mukai [**MM1**]–[**MM2**].

Example 2.2. A quartic 3-fold containing a plane $\Pi \subset V_4 \subset \mathbb{P}^4$ is indecomposable. If Π is the x_3, x_4, x_5 coordinate plane defined by $x_1 = x_2 = 0$ then V is defined by an equation $x_1 A + x_2 B = 0$, and this is singular at the points $\Pi \cap (A = B = 0)$ (in general 9 ordinary double points).

2.3. General theory. An *elephant* of V is a surface $S \in |-K_V|$ with at worst Du Val singularities (rational double points). If V is a Gorenstein Fano 3-fold, an elephant is known to exist by a theorem of Shokurov and Reid [**R1**]. It is a K3 surface: in fact $K_S = (K_V + S)_{|S} = 0$ and $H^1(S, \mathcal{O}_S) = 0$ by Kodaira vanishing. Morever, $|-K_V|_{|S}$ is an ample complete linear system of Cartier divisors on S.

With a couple of exceptional cases that are easily classified and that we pass over, it follows that $|-K_V|$ is very ample. Taking another elephant $S' \in |-K_V|$

and setting $C = S \cap S'$ gives the diagram

$$\begin{array}{ccccl} V & \hookrightarrow & \mathbb{P}^{g+1} & & \text{anticanonical embedding } \varphi_{-K_V}, \\ \bigcup & & \bigcup & & \\ S & \hookrightarrow & \mathbb{P}^{g} & & \text{embedding by } |-K_V|_{|S}, \\ \bigcup & & \bigcup & & \\ C & \hookrightarrow & \mathbb{P}^{g-1} & & \text{canonical embedding } \varphi_{K_C}. \end{array}$$

Special linear systems on C interpreted in terms of geometric RR give rise to geometric properties of V. The hyperelliptic case was already passed over in what we said above; the case of C trigonal leads to V a hypersurface in a scroll, which contradicts V indecomposable for $g \geq 5$. In the same way, other Brill–Noether special linear systems on C such as g^2_5 (when C is a plane quintic) usually contradict V indecomposable.

2.4. Fano 3-folds, specific theory. None of the above is specific to 3-folds. Canonical curves and K3 surfaces continue to exist for all g, so the main result for 3-folds is thus quite remarkable.

THEOREM 2.5 (Fano, Iskovskikh, Mukai). *Indecomposable Gorenstein Fano 3-folds of genus g exist if and only if $2 \leq g \leq 10$ or $g = 12$. If $g \leq 5$, the anticanonical ring $R(V, -K_V)$ is a hypersurface or complete intersection in projective space or w.p.s. If $g = 6, \dots, 10$ then V is a complete intersection in a homogeneous projective variety. (There is also an analogous structure result for $g = 12$.)*

For example,

$$g = 2 \quad \Longrightarrow \quad V = Q_6 \subset \mathbb{P}(1,1,1,1,3),$$
$$g = 4 \quad \Longrightarrow \quad V = Q_2 \cap F_3 \subset \mathbb{P}^5.$$

As a typical case of the homogeneous projective varieties G/P,

$$g = 7 \quad \Longrightarrow \quad V = \Sigma \cap H_1 \cap \cdots \cap H_7,$$

where $\Sigma = \mathrm{OGr}^{10}(5,10)$ is the 10-dimensional spinor variety or orthogonal Grassmann variety, that is, the subset of $\mathrm{Grass}(5,10)$ consisting of maximal isotropic vector subspaces of the standard inner product $\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$ in its spinor embedding $\Sigma \subset \mathbb{P}^{15}$ (see Mukai [**Mu2**]).

As an illustration, we sketch the proof in the style of Mukai that there does not exist any Fano 3-fold $V = V_{20} \subset \mathbb{P}^{12}$ of genus 11. Write $\sigma\colon V_1 \to V$ for the blowup of a point $P \in V$, and $E \subset V_1$ for the exceptional divisor, with $E \cong \mathbb{P}^2$ and $\mathcal{O}_E(-E) \cong \mathcal{O}_{\mathbb{P}^2}(1)$. Then $K_{V_1} = \sigma^* K_V + 2E$. It follows that the anticanonical linear system $|-K_{V_1}| = |\sigma^*(-K_V) - 2E|$ is the birational transform of the linear system $|m_P^2 \cdot \mathcal{O}_V(-K_V)|$ of hyperplane sections of V containing the tangent plane $T_P V$. We assume that P does not lie on any line of V. Then since V is an intersection of quadrics, we deduce that $\varphi_{-K_{V_1}} : V_1 \to \mathbb{P}^8$ is a morphism, birational because V is indecomposable, with image V' a Gorenstein Fano 3-fold of genus 7. Now on the one hand, V' is indecomposable, so is a linear section of $\mathrm{OGr}^{10}(5,10)$ by the result for $g = 7$. On the other hand, it contains the image of E, a Veronese surface.

Finally, arguing on the geometry of $\mathrm{OGr}^{10}(5,10) \subset \mathbb{P}^{15}$ and on its universal bundle, we can deduce that the only Veronese surfaces E it contains span a 4-plane

$\mathbb{P}^4 \subset \mathrm{OGr}$; this contradicts the construction of V' as a linear section of OGr containing E.

Iskovskikh's proof (deriving from Fano) proceeds by the projection from a general line $L \subset V$; it involves proving that lines exist, and various generality statements about its projection and double projection. Mukai's proof is thus a considerable simplification even in the nonsingular case.

3. $\mathbb{Q}$-Fano 3-folds

3.1. The Mori category. The minimal model program for 3-folds (usually called MMP or Mori theory) was developed by Mori and others from the late 1970s. As in the theory of surfaces, to get one step closer to a minimal model, one makes a birational contraction, for example, contracting a copy of $\mathbb{P}^2$ on which the canonical class is negative. However, these contractions lead to singularities, so that Mori theory only works in a suitable category of singular varieties. This leads to the following definitions:

Definition 3.2. The *Mori category* consists of projective varieties with at worst $\mathbb{Q}$-factorial terminal singularities, where

(a) A variety V has *terminal singularities* if it is normal, rK_V is a Cartier divisor for some $r > 0$, and any resolution of singularities $f\colon Y \to V$ with divisorial exceptional locus $\bigcup E_i$ satisfies

$$K_Y = f^*K_V + \sum a_i E_i \quad \text{with all } a_i > 0.$$

For example, the cone over the Veronese surface (that is, the quotient singularity $\frac{1}{2}(1,1,1)$) is terminal: it is resolved by the blowup $Y \to V$ of the origin, which introduces the exceptional divisor $E \cong \mathbb{P}^2$ with $\mathcal{O}_E(-E) \cong \mathcal{O}_{\mathbb{P}^2}(2)$ and $K_{Y|E} \cong \mathcal{O}_{\mathbb{P}^2}(-1)$, so that $K_Y = f^*K_X + \frac{1}{2}E$.

(b) V is $\mathbb{Q}$-*factorial* if every Weil divisor D on V has a multiple rD that is Cartier. You should think of this as an analog in algebraic geometry of the condition that V is a $\mathbb{Q}$-homology manifold: a codimension 1 subvariety has a dual cohomology class in $H^2(V, \mathbb{Q})$. For example, the quartic hypersurface of Example 2.2 is not $\mathbb{Q}$-factorial, since it has double points on Π at which no multiple of Π is Cartier.

3.3. Terminal singularities. Terminal singularities are necessary for Mori theory: the Mori category is closed under contractions and flips, and is the smallest such category.

Terminal singularities were classified by Mori and Reid. They are made up of the following ingredients (for more details, see Reid [**YPG**]):

(1) Mild isolated hypersurface singularities, for example, double points such as $(xy = f(z,t)) \subset \mathbb{C}^4_{x,y,z,t}$. While this is a reasonably concrete class of singularities, already the special case $xy = f(z,t)$ is fairly infinite, containing all isolated plane curve singularities.

(2) The cyclic quotient singularities $\frac{1}{r}(1, a, r-a)$. The notation means the quotient $V = \mathbb{C}^3/(\mathbb{Z}/r)$, where the cyclic group $\mathbb{Z}/r$ acts by

$$(x, y, z) \mapsto (\varepsilon x, \varepsilon^a y, \varepsilon^{r-a} z).$$

Setting $x = 0$ leads to the cyclic quotient singularity $\frac{1}{r}(a, r-a)$, which is the Du Val singularity A_{r-1}

$$S : (uv = w^r) \subset \mathbb{C}^3_{u,v,w}.$$

Thus $(x = 0)$ defines a local elephant $S \in |-K_V|$.

(3) Some combinations of the above two types, typically the main Type A family $\big(xy = f(z^r, t)\big)/\frac{1}{r}(a, r-a, 1, 0)$.

Remark 3.4. Whenever we say that a point $P \in V$ is a terminal quotient singularity of type $\frac{1}{r}(1, a, r-a)$, we always assume that $0 < a < r$ and a is coprime to r. The local class group of $P \in V$ is $\mathbb{Z}/r$, with chosen generator $\mathcal{O}_V(-K_V) = \mathcal{O}(1)$, a *polarisation* of the singularity. The generators of the graded ring $R(V, A)$ that serve as orbifold local coordinates then have weights $1, a, r-a$ mod r; compare Corti, Pukhlikov and Reid [**CPR**], 3.4.6.

The quotient singularity $\frac{1}{r}(a, r-a)$ is the Du Val singularity A_{r-1}, so of course does not depend on a up to analytic isomorphism; but we work here with K3 surfaces with a chosen polarisation $\mathcal{O}(1) = \mathcal{O}(D)$, and the weights mod r of the orbifold local coordinates are uniquely determined.

Write b for the inverse of a mod r (that is $ab \equiv 1$ mod r). Choosing $\varepsilon' = \varepsilon^a$ as a new basis for the group of rth roots of unity would put the Du Val singularity in the standard A_{r-1} form $\frac{1}{r}(1, r-1)$ and the terminal 3-fold point in the form $\frac{1}{r}(b, 1, r-1)$; and $\mathcal{O}_S(D)$ is locally isomorphic to the ε'^b eigensheaf. The birational transform of the general divisor D meets the bth curve in the resolution (see Figure 4.8.1). The quantity b also appears in the orbifold RR formulas of [**YPG**], Theorem 10.2 (see Theorem 4.6 below), for essentially the same reason. Muddling up a and b in calculations is a common error, but you find out soon enough when your plurigenera turn out to be fractional; see Exercise 4.7, (3) for a practical instance.

Definition 3.5. A *Fano* 3*-fold* is a variety V for which $-K_V$ is ample. We usually add conditions to this. A *Mori Fano* 3*-fold* V is a Fano 3-fold in the Mori category and with $\operatorname{rank} \operatorname{Pic} V = 1$. As one of the possible end products of a MMP, these 3-folds are among the basic building blocks of Mori theory. V is *prime* if it is in the Mori category and $\operatorname{Cl} V = \mathbb{Z} \cdot (-K_V)$.

As in the Gorenstein case, the *anticanonical ring* of a Fano 3-fold V is the graded ring

$$R(V, -K_V) = \bigoplus_{n \geq 0} H^0(V, -nK_V)). \tag{3.5.1}$$

An *elephant* of V is a general divisor $S \in |-K_V|$, just as in 2.3. If S exists, it is no longer a Cartier divisor at singularities of V of index $r > 1$. The graded rings of V and S are related by

$$R(S, A) = R(V, -K_V)/(x_0), \tag{3.5.2}$$

where $x_0 \in H^0(V, -K_V)$ is the equation of $S \in |-K_V|$. That is, $R(S, A)$ is a quotient of $R(V, -K_V)$ by a principal ideal generated by an element of degree 1. When two rings are related in this way, many basic algebraic properties of one can be inferred from the other; compare Exercise 4.7 and 5.1. This is called the *hyperplane section principle.* The most useful case is when there is an elephant S with at worst Du Val singularities; then S is a K3 surface (this is proved as in

2.3) polarised by the Weil divisor $A = -K_{V|S}$. Then via the graded rings, many properties of the Fano 3-fold V can be read from those of S.

Remark 3.6. As we see in 4.8 below, there are cases when $|-K_V| = \emptyset$, that is, no elephant exists; or when $h^0(-K_V) = 1$, and the single surface $S = |-K_V|$ happens to have slightly worse than Du Val singularities, so there is no K3 elephant. Nevertheless, the motivation arising from K3 surfaces is our main guiding principle for the study of Fano 3-folds, even in cases such as these when it is logically inapplicable. Compare Exercise 4.7.

Example 3.7. Let $x_1, \dots, x_5, y$ be coordinates on $\mathbb{P}(1^5, 2)$. We start from a $(3,3)$ c.i. $V_{3,3} \subset \mathbb{P}^5(1^5, 2)$ as described at the end of Remark 1.10, and assume that V contains the weighted projective plane

$$\Pi = \mathbb{P}(1,1,2) \quad \text{given by} \quad (x_1 = x_2 = x_3 = 0),$$

and is otherwise general. Since Π is a c.i., the two cubic equations of V are of the form

$$V : \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0, \tag{3.7.1}$$

where a_i, b_i are forms of degree 2 in $x_1, \dots, x_5, y$. We show that Π can be contracted to a quotient singularity of type $\frac{1}{3}(1,1,2)$ by a rational map $V \dashrightarrow W$, where $W \subset \mathbb{P}(1^5, 2, 3)$ is a Mori Fano 3-fold. The new coordinate z will be a rational section of $\mathcal{O}_V(-3K_V)$ with pole along Π. Thinking of (3.7.1) as 2 simultaneous equations for x_1, x_2, x_3 with coefficients a_i, b_i and solving them by Cramer's rule gives:

$$\begin{aligned} zx_1 &= a_2b_3 - a_3b_2, \\ zx_2 &= a_3b_1 - a_1b_3, \\ zx_3 &= a_1b_2 - a_2b_1. \end{aligned} \tag{3.7.2}$$

The "constant of proportionality"

$$z = \frac{a_2b_3 - a_2b_3}{x_1} = \frac{a_3b_1 - a_3b_1}{x_2} = \frac{a_1b_2 - a_1b_2}{x_3}$$

is a well defined rational form of degree 3 on V with pole along Π.

Now consider the rational map

$$\varphi \colon V \dashrightarrow W \subset \mathbb{P}(1^5, 2, 3) \quad \text{given by} \quad (x_1, \dots, x_5, y, z);$$

the image W is given by the 5 equations (3.7.1–3.7.2). φ is an morphism wherever Π is a Cartier divisor on V, contracts Π to the point $(0, \dots, 0, 1)$, and is an isomorphism outside Π. The point $(0, \dots, 0, 1)$ is the $\frac{1}{3}(1,1,2)$ singularity $\mathbb{C}^3_{x_4,x_5,y}/(\mathbb{Z}/3)$, because when $z = 1$, (3.7.2) gives x_1, x_2, x_3 as implicit functions. V is quasi-smooth at points of Π where the matrix in (3.7.1) has rank 2, and Π is a Cartier divisor there. We can assume that this matrix has rank ≥ 1 everywhere on Π, hence the blowup of Π is a small morphism. It makes Π a Cartier divisor so that φ is a morphism on the blowup.

Notice that the graded ring $R(V, -K_V + \frac{1}{3}\Pi) = R(W, -K_W)$ is generated by $(x_1, \dots, x_5, y, z)$ with relations (3.7.1–3.7.2). These can be put together as the five

4×4 Pfaffians of the skew matrix

$$\begin{pmatrix} z & a_1 & a_2 & a_3 \\ & b_1 & b_2 & b_3 \\ & & x_3 & -x_2 \\ & & & x_1 \end{pmatrix} \quad \text{of degrees} \quad \begin{pmatrix} 3 & 2 & 2 & 2 \\ & 2 & 2 & 2 \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}.$$

See Remark 1.8 for our conventions on Pfaffians.

At the end of Remark 1.10, we noted that the Fano 3-fold $V_{3,3} \subset \mathbb{P}(1^5, 2)$ has as its linear section the graded ring $R(C, K_C + \frac{1}{2}P)$ corresponding to the orbifold canonical class of a curve of genus 3 together with an orbifold point P of order $r = 2$. The construction of this example restricted to C is the graded ring $R(C, K_C + \frac{2}{3}P)$ corresponding to the orbifold canonical class $K_C + \frac{2}{3}P$ (compare Example 1.9).

4. Numerical data and Hilbert series

4.1. The aim. Several hundred families of K3 surfaces S and Mori Fano 3-folds V are known. We now develop methods to guarantee gratification from this cornucopia. The cases treated in Examples 1.9 and 3.7 are pretty tame, and more complicated things like Example 4.2 below are more typical. Since we study a polarised variety X, A in terms of the graded ring $R(X, A)$ of (1.1.1), the natural numerical invariants of X to take is the *list of ingredients* that go into its Hilbert series $P_X(t)$. We explain this in Theorem-Definition 4.6, but first we work through an example.

Example 4.2. Consider the weighted hypersurfaces:

$$S_{44} \subset \mathbb{P}(4, 5, 13, 22) \quad \text{and} \quad V_{44} \subset \mathbb{P}(1, 4, 5, 13, 22); \tag{4.2.1}$$

this is Fletcher [**Fl**], List 13.3, No. 91, Reid1(91) in the Magma database. We sketch how to calculate with these hypersurfaces. For more details, see [**Fl**], Section 13 (compare Dolgachev [**D**]). Write $x_0, \dots, x_4$ or x, y, z, t, u for variables of weights $a_0, \dots, a_4 = 1, 4, 5, 13, 22$, and let $f_{44}(x, y, z, t, u)$ be a general weighted polynomial of degree 44 in them. Set

$$\begin{aligned} &R[V] = \mathbb{C}[x, y, z, t, u]/\big(f_{44}(x, y, z, t, u)\big), \\ &\mathcal{C}V = \operatorname{Spec} R[V] : \big(f_{44} = 0\big) \subset \mathbb{C}^5, \text{ and} \\ &V = \operatorname{Proj} R[V] : \big(f_{44} = 0\big) \subset \mathbb{P}(1, 4, 5, 13, 22). \end{aligned}$$

Then $\mathcal{C}V$ is the affine cone over V, and $V = \operatorname{Proj} R[V]$ the quotient of $\mathcal{C}V$ by the $\mathbb{C}^*$ action $x_i \mapsto \lambda^{a_i} x_i$ for $\lambda \in \mathbb{C}^*$ with the given weights a_i. By a combination of Bertini's theorem at general points and explicit calculations at the coordinate strata, one checks that $\mathcal{C}V$ is nonsingular outside the origin for general f_{44}. We say that V is *quasi-smooth*. Then the only singularities of $V \subset \mathbb{P}(1, 4, 5, 13, 22)$ arise from the fixed points of the $\mathbb{C}^*$ action; these are the points of the x_i-axis, fixed by the cyclic group $\mu_{a_i} \cong \mathbb{Z}/(a_i)$ of a_ith roots of 1, and the points of the y, u-plane, which are fixed by $\mu_2 = \{\pm 1\}$, because $2 = \operatorname{hcf}(4, 22)$. Thus in general V has the following cyclic quotient singularities:

$$\begin{cases} \frac{1}{2}(1, 1, 1) & \text{at } (f = 0) \text{ on the } yu \text{ line (one point)}; \\ \frac{1}{5}(1, 3, 2) & \text{at the } z \text{ point } (0, 0, 1, 0, 0); \\ \frac{1}{13}(1, 4, 9) & \text{at the } t \text{ point } (0, 0, 0, 1, 0). \end{cases} \tag{4.2.2}$$

For example, along the t axis of $\mathcal{C}V$, we can assume that f_{44} has the monomial zt^3, which means that $\frac{\partial f}{\partial z} \neq 0$ when $t \neq 0$, so that z is an implicit function of the

other variables. It follows that $\mathcal{C}V$ is nonsingular there and that V has a quotient singularity of type $\frac{1}{13}(1,4,22) = \frac{1}{13}(1,4,9)$ at $(0,0,0,1,0)$. At each singular point, $\mathcal{O}_V(1) = \mathcal{O}_V(-K_V)$ is a generator of the local class group.

Definition 4.3 (Hilbert series). Let $R = \bigoplus_{n\geq 0} R_n$ be a graded ring. We assume that R is generated by finitely many elements x_i of positive degree over $R_0 = \mathbb{C}$. Its Hilbert series $P(t)$ is defined by setting

$$P_n = \dim_{\mathbb{C}} R_n \quad \text{and} \quad P(t) = \sum_{n\geq 0} P_n t^n.$$

In cases of interest, P_n is given by a formula of orbifold RR type (see [**YPG**], Chapter 3), possibly with some corrections for low values of n when cohomology is still present.

Exercise 4.4. Use (1.5.1) to show that the canonical ring $R(C, K_C)$ of a curve C of genus g has Hilbert series

$$P(t) \;=\; \frac{1 + (g-2)t + (g-2)t^2 + t^3}{(1-t)^2}\,.$$

Now write $A = K_C + \sum \frac{r-1}{r} P$ for the orbifold canonical class of Remark 1.10; the sum takes place over a basket $\mathcal{B} = \{P, \frac{r-1}{r}\}$ of orbifold points P of order r. In degree n, only the rounded down integral divisor $\lfloor nA \rfloor = nK_C + \sum \lfloor \frac{n(r-1)}{r} \rfloor P$ moves. Deduce that the Hilbert series of the orbifold canonical ring $R(C, A)$ is given by

$$\begin{aligned} P(t) &= \sum_{\mathcal{B}} h^0\Big(nK_C + \sum \lfloor \frac{n(r-1)}{r} \rfloor P\Big) t^n \\ &= \frac{1 + (g-2)t + (g-2)t^2 + t^3}{(1-t)^2} + \sum_{\mathcal{B}} \frac{t(t + \cdots + t^{r-1})}{(1-t)(1-t^r)} \qquad (4.4.1) \\ &= \frac{1 - (g-1)t - t^2}{1-t} + \frac{t}{(1-t)^2} \deg A - \sum_{\mathcal{B}} \frac{1}{(1-t^r)} \sum_{i=1}^{r-1} \tfrac{r-i}{r} t^i. \end{aligned}$$

The first expression (4.4.1) shows how much the integral divisor $\sum \lfloor \frac{n(r-1)P}{r} \rfloor$ contributes to each $H^0(C, nA)$. The second expression is a transparent case of an orbifold RR formula: we write $\deg A = \frac{1}{r} \deg(rA)$ where rA is a Cartier divisor (A itself does not make sense as a sheaf); in each term, the effect of the denominator $1 - t^r$ is to multiply by $1 + t^r + t^{2r} + \cdots$, so that the fractional part $\{\frac{i(r-1)}{r}\} = \frac{r-i}{r}$ discarded in the rounddown is repeated periodically with period r.

This formula for orbifold curves can serve as a model for the formulas of [**YPG**], Chapter 3. In fact, a result of Becky Leng's thesis [**Leng**] derives the orbifold RR formula for 3-folds ([**CPR**], Theorem 10.2) via a reduction to the curve case.

4.5. Hilbert series for K3 surfaces and Fano 3-folds. The analogous formulas for the Hilbert series of the graded ring over a K3 surface S or a Fano 3-fold V are contained in Altınok [**A**] and [**A1**]. The proofs are based on the results of [**YPG**], Chapter 3.

THEOREM-DEFINITION 4.6. *Let S be a K3 surface with Du Val singularities, and D a Weil divisor. The Hilbert series $P_S(t) = \sum_{n\geq 0} h^0(S, nD)t^n$ is given by*

$$P_S(t) \;=\; \frac{1+t}{1-t} + \frac{t(1+t)}{(1-t)^3}\,\frac{D^2}{2} - \sum_{\mathcal{B}} \frac{1}{(1-t^r)} \sum_{i=1}^{r-1} \frac{\overline{bi}(r-\overline{bi})t^i}{2r}\,, \tag{4.6.1}$$

where the sum takes place over a basket $\mathcal{B} = \{\frac{1}{r}(a, -a)\}$ *of cyclic quotient singularities. In each term, b is the inverse of a mod r, as in Remark 3.4, and $\overline{bi}$ denotes the minimal nonnegative residue mod r.*

We define the genus $g = g(S, D)$ *by the formula:*

$$P_1(S, D) = h^0(\mathcal{O}_S(D)) = g + 1.$$

The numerical data *of S, D are the genus g and the basket $\mathcal{B} = \{\frac{1}{r}(a, -a)\}$. Calculating $P_1(S, D)$ as the coefficient of t in* (4.6.1) *gives a formula for D^2 in terms of g and $\mathcal{B}$:*

$$D^2 = 2g - 2 + \sum_{\mathcal{B}} \frac{b(r-b)}{r}\,.$$

Let V be a Fano 3-fold and write $A = -K_V$. Then its anticanonical ring has Hilbert series

$$\begin{aligned} P_V(t) \;&=\; \sum_{n\geq 0} h^0(V, nA)t^n \\ &=\; \frac{1+t}{(1-t)^2} + \frac{t(1+t)}{(1-t)^4}\,\frac{A^3}{2} - \sum_{\mathcal{B}} \frac{1}{(1-t)(1-t^r)} \sum_{i=1}^{r-1} \frac{\overline{bi}(r-\overline{bi})t^i}{2r}\,. \end{aligned} \tag{4.6.2}$$

where the sum takes place over a basket $\mathcal{B} = \{\frac{1}{r}(1, a, r-a)\}$ of terminal quotient singularities. We define the genus $g = g(V)$ *by*

$$P_1(V, A) = h^0(V, A) = h^0(\mathcal{O}_V(-K_V)) = g + 2.$$

Then, as in the K3 case, the numerical data *of V are the genus g and the basket $\mathcal{B}$; calculating P_1 from* (4.6.2) *gives essentially the same formula as above:*

$$A^3 = 2g - 2 + \sum_{\mathcal{B}} \frac{b(r-b)}{r}\,.$$

EXERCISE 4.7. (1) If a Fano 3-fold V has a K3 elephant $S \in |-K_V|$, the numerical data of V and S are related in the obvious way, and (4.6.1) and (4.6.2) only differ by the extra factor $(1-t)$ in the denominator. Use the hyperplane section principle (3.5.2) to derive (4.6.2) from (4.6.1).

(2) In the same way, if S, C is a polarised K3 surface having only singularities of type $\frac{1}{r}(1, r-1)$ (with C passing through each singular point as a nonsingular curve in the 1 eigenspace), derive (4.6.1) from the orbifold canonical curve formula (4.4.1). Singularities of type $\frac{1}{r}(1, r-1)$ account for the large majority of all cyclic quotient singularities in K3 baskets.

(3) The varieties in Example 4.2 have $g = 0$ and the basket (4.2.2). Check that

$$D^2 = -4 + \frac{1}{2} + \frac{2\cdot 3}{6} + \frac{3\cdot 10}{13} = \frac{44}{4\cdot 5\cdot 13\cdot 22} = \frac{1}{130}$$

(note the shift from $a = 4$ in $\frac{1}{13}(4, 9)$ to its inverse $b = 10$ in the fractional contribution $\frac{3\cdot 10}{13}$). Check that $P_S(t)$ given by (4.6.1) satisfies

$$(1 - t^4)(1 - t^5)(1 - t^{13})(1 - t^{22})P_S(t) = 1 - t^{44}.$$

4.8. Numerical data of K3s and Fano 3-folds.

4.8.1. *Invariants and inequalities.* As we saw in Theorem-Definition 4.6, the numerical data of a K3 surface or a Fano 3-fold consist of an integer genus g and a basket $\mathcal{B}$ of fractional expressions. For a K3 surface in characteristic 0 we have

$$g \geq -1, \quad 2g - 2 + \sum \frac{b(r-b)}{r} > 0 \quad \text{and} \quad \sum_{\mathcal{B}} (r-1) \leq 19. \tag{4.8.1}$$

The last inequality comes from the fact that the singularities of the basket contribute $\sum(r-1)$ exceptional -2-curves to $\operatorname{Pic} S$, that form a negative definite set; in characteristic p, the result would be $\sum(r-1) \leq 21$.

There are approximately 6,640 possible baskets satisfying (4.8.1), and countably many possible values of g for each basket. A small number of extreme cases are excluded because the polarisation and the basket cannot fit inside the K3 lattice Λ_{K3} (see 4.8.4 below); for example, a K3 surface cannot contain ≥ 17 disjoint -2-curves. However, apart from these, all other g and $\mathcal{B}$ really occur. Most give rise to graded rings of high codimension, and our methods based on graded ring are no longer appropriate for studying them; the same applies to canonical curves or K3 surfaces of genus $g \gg 0$, for which explicit equations and graded ring methods give little information, and other ideas are needed, such as the period space and Torelli for K3 surfaces.

For a Fano 3-fold V we have

$$g \geq -2, \quad 2g - 2 + \sum \frac{b(r-b)}{r} > 0 \quad \text{and} \quad \sum_{\mathcal{B}} \left(r - \frac{1}{r}\right) < 24. \tag{4.8.2}$$

The last inequality comes from [**YPG**], Corollary 10.3, the orbifold analog of the RR formula $\frac{1}{24}c_1c_2 = \chi(\mathcal{O}_V) = 1$; compare also Kawamata [**Ka1**], 2.2 and [**Ka2**]. In fact the argument of [**Ka2**] using Bogomolov stability gives

$$\sum \mathcal{B}\left(1 - \frac{1}{r}\right) < 24 - 8(-K_V)^3,$$

a slightly stronger inequality.

Note that (4.8.2) is stronger than (4.8.1) if the number of singularities is large; for example, (4.8.2) allows only 15 singularities. On the other hand, (4.8.2) is weaker if there are few singularities – for example, it would allow a cyclic singularity of index 24, or a singularity of index 22 plus one of index 2. Since we expect there to be rather few families of Fano 3-folds with $h^0(-K_V) = 0$, and those with $h^0(-K_V) > 0$ to have a K3 elephant after deformation, it seems unlikely that any of these extra Fano 3-fold cases really occur.

4.8.2. *Negative genus.* On a K3 surface, $g = -1$ means that the polarising divisor D is ineffective; or we can say that a curve of genus -1 does not exist (pretty reasonable, uh?). A Fano 3-fold with $g = -2$ has $|-K_V| = \emptyset$; we can think of $g = -2$ as saying that two things do not exist: V does not have a canonical curve section $S_1 \cap S_2$, and moreover, it does not even have an elephant $S \in |-K_V|$. Only three or four families of Fano 3-folds with $h^0(-K_V) = 0$ are known; the simplest of these, and the first to be discovered, is the codimension 2 weighted c.i.

$V_{12,14} \subset \mathbb{P}(2,3,4,5,6,7)$ due to Fletcher (see [**Fl**], List 16.7, No. 60 and compare Reid [**Ki**], Example 9.14).

4.8.3. *$\mathbb{Q}$-smoothing and the general elephant.* A Mori Fano 3-fold V is allowed to have general terminal singularities. Or it might happen that $|-K_V| \neq \emptyset$, but every $S \in |-K_V|$ has an essential singularity (worse than Du Val); this is not very surprising if $P_1(V) = 1$. For example, one can construct a surface $S_{14} \subset \mathbb{P}(2,2,3,7)$ having an elliptic singularity, but contained in a quasi-smooth $V_{14} \subset \mathbb{P}(1,2,2,3,7)$. (Explicit construction: start from the scroll $\mathbb{F}_3$, with fibre A and negative section B; then S_{14} is the double cover ramified in $B + C_1 + C_2$ where $C_1 \in |3A + B|$ is the general hyperplane section of the cone $\overline{\mathbb{F}}_3$, and $C_2 \in |7A + 2B|$ has a tacnode on C_1, giving $C_1 + C_2$ an infinitely near triple point.)

It is known that a terminal 3-fold singularity has a $\mathbb{Q}$-*smoothing*, that is, a small deformation with only cyclic quotient terminal singularities (see [**YPG**], 6.4 for explicit equations). This is the idea behind the basket appearing in (4.6.2), and one step in its proof ([**YPG**], proof of Theorem 10.2). It seems reasonable to conjecture that every prime Fano 3-fold V also has a $\mathbb{Q}$-smoothing; compare Namikawa [**N**] and Minagawa [**Mi1**], [**Mi2**] for partial results. One might hope to prove this in terms of deformation theory and Hodge theoretic consequences of $-K_V$ ample, by analogy with work on smoothing Calabi–Yau 3-folds (see Gross [**G**] and Namikawa and Steenbrink [**NS**]). Moreover, if $g \geq -1$, we also conjecture that V together with its elephant $S \in |-K_V|$ has a *simultaneous $\mathbb{Q}$-smoothing*, that is, a small deformation such that the pair $S \subset V$ has only standard cyclic singularities $\frac{1}{r}(a, r-a) \subset \frac{1}{r}(1, a, r-a)$. If in addition $g \geq 0$ and the basket of V consists only of $\frac{1}{r}(1,1,r-1)$, then after a small deformation, V would have a curve section $C = S_1 \cap S_2$ that is an orbifold canonical curve as in Remark 1.10.

4.8.4. *Numerical data and the K3 lattice.* The numerical data of a polarised K3 surface S corresponds to a sublattice

$$L\Big(g, \mathcal{B} = \Big\{\frac{1}{r}(a, r-a)\Big\}\Big) \subset \Lambda_{\mathrm{K3}}$$

of the standard K3 lattice. We assume that, after a $\mathbb{Q}$-smoothing, the singularities of S are exactly the cyclic quotient singularities of the basket. Let $f\colon T \to S$ be the minimal resolution of singularities, and consider the sublattice $L \subset \operatorname{Pic} T$ generated by the exceptional -2-curves of f together with $f^*(ND)$, where N is the global index of D. The lattice $L = L(g, \mathcal{B})$ is based by the *quasistellar graph* $\Gamma(g, \mathcal{B})$ of Figure 4.8.1. The central vertex is a divisor B of self-intersection $B^2 = 2g - 2$

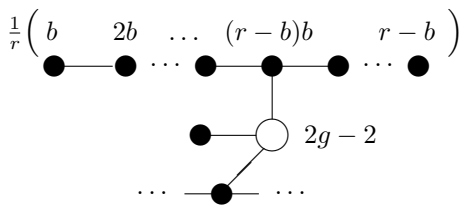

FIGURE 4.8.1. The quasistellar graph $\Gamma(g, \mathcal{B})$.

(ineffective if $g = -1$, that is, $B^2 = -4$); for each term $\frac{1}{r}(a, r-a) \in \mathcal{B}$, it meets the bth curve in a chain of $r-1$ exceptional -2-curves, where $ab \equiv 1 \bmod r$. (Compare Belcastro [**Be**] for the lattice of the famous 95 families of K3 hypersurfaces.)

The polarising $\mathbb{Q}$-divisor $D = f^*D = B + \sum m_i E_i$ has exceptional curves in each chain weighted by the monotone sequence of arithmetic progressions

$$\frac{1}{r}\Big(b, 2b, \dots, b(r-b-1), b(r-b), (b-1)(r-b), \dots, 2(r-b), r-b\Big),$$

giving

$$D \cdot E_i = 0, \quad \text{and} \quad D^2 = D \cdot B = 2g - 2 + \sum \frac{b(r-b)}{r}.$$

The lattice $L = L(g, \mathcal{B})$ has signature $(+1, -\sum(r-1))$ and discriminant $(\prod_{\mathcal{B}} r)D^2$. Although L is determined up to isomorphism by the numerical data g and $\mathcal{B}$, its embedding into the standard K3 lattice

$$L \hookrightarrow H^2(T, \mathbb{Z}) = \Lambda_{\mathrm{K3}}$$

is not in general completely determined by g and $\mathcal{B}$, and is an additional topological invariant of S, D. For example, if S is nonsingular then $L = \mathbb{Z} \cdot D = \langle 2g-2 \rangle$ is the lattice with one generator of square length $2g-2$, but it can happen that D is divisible in $\operatorname{Pic} S$ by some integer n with $n^2 \mid g-1$, and then $L \subset \Lambda_{\mathrm{K3}}$ is not a primitive sublattice. If L has high rank, it can have several inequivalent primitive embeddings $L \hookrightarrow \Lambda_{\mathrm{K3}}$, having nonisomorphic orthogonal complements.

The Hilbert series of a Fano 3-fold V is controlled by the same combinatorics: the lattice $L(g, \mathcal{B})$ is still there (even if $g = -2$). For simplicity, assume that V has only terminal quotient singularities $\frac{1}{r}(1, a, r-a)$. Make the economic resolution $Y \to V$ of these singularities by successive $(1, a, r-a)$ weighted blowups. These are the Kawamata blowups of Corti, Pukhlikov and Reid [**CPR**], 3.4.2 (see also [**YPG**], 5.7; but note that the chain of Kawamata blowups is a unique "best" choice of economic resolution that came on line after [**YPG**]). The lattice $L(g, \mathcal{B})$ is the Picard lattice $\operatorname{Pic} Y$, with bilinear product

$$(D_1, D_2) \mapsto (-K_Y) \cdot D_1 \cdot D_2.$$

If $g \geq -1$ and we assume that V (or its deformation) has a K3 elephant, then $L(g, \mathcal{B})$ is a sublattice of Λ_{K3}. If $g = -2$, so that $B^2 = -6$, we have to imagine that we are looking at a nonexistent K3 surface having a polarising divisor D with $h^0(D) = -1$. In this case, there is no obvious a priori relation between $L(g, \mathcal{B})$ and Λ_{K3}; however, as we said in 4.8.1, there are probably only a few cases, and it is not a substantial restriction for a lattice in this range to be a sublattice of Λ_{K3}, so who knows?

5. From K3s to Fano 3-folds

Just as the general theory of 2.3, what we said in Section 4 applies to K3 surfaces and to Fano 3-folds alike. We said in 4.8.1 that K3 surfaces go on for ever and for ever, with many baskets continuing to exist for infinitely many values of g. However, as in the more specific 3-fold theory of 2.4, there are results of Kawamata [**Ka1**]–[**Ka2**] saying that Fano 3-folds are bounded. By analogy with Theorem 2.5, we guess that only a couple of thousand families exist, and we eventually aspire to a precise classification. Most Fano 3-folds have $H^0(-K_V) \neq 0$, and, as a first attempt, we can take those having a K3 elephant as typical, and allow ourselves extra generality assumptions on V such as $\mathbb{Q}$-smoothing, Brill–Noether general behaviour of linear systems (nonhyperelliptic, etc.).

5.1. Takagi's results. In his lecture at the Singapore conference [**T1**], Takagi sketched a preliminary classification of prime $\mathbb{Q}$-Fano 3-folds of genus $g \geq 2$ having at worst terminal quotient singularities and a K3 elephant. The assumption that $|-K_V|$ contains a K3 is a strong condition: it guarantees that there is a resolution on which $-K_{V'}$ is nef. In this case, the anticanonical system $|-K_{V'}|$ is a big linear system of K3 surfaces, and defines a birational model of V as an indecomposable Gorenstein Fano 3-fold X (Fano would appreciate this picture at once!). Note that the anticanonical system $|-K_V|$ itself (but not $|-2K_V|$, $|-3K_V|$, etc.) is invariant under passing to the economic resolution of points of index $r \geq 2$. Mukai's Theorem 2.5 applies to X, thus reducing the study of $\mathbb{Q}$-Fano 3-folds to problems in projective geometry concerning sections of homogeneous spaces with special configurations of planes and singularities. In particular, Takagi proves that if V has at least one terminal quotient singularity then $g \leq 8$; the cases $g = 7$ and $g = 8$ is accessible since $X = \varphi_{-K_V}(V)$ is a linear section of a Grassmann or orthogonal Grassmann variety. Takagi's thesis [**T**] also settled all cases with only $\frac{1}{2}(1,1,1)$ singularities.

5.2. Low codimension. We study Fano 3-folds in terms of the (pluri-) anticanonical ring

$$R = R(V, -K_V) = \mathbb{C}[x_0, \dots, x_N]/I_V,$$

where $x_0, \dots, x_N$ (more usually $x, y, \dots, u$, etc.) are homogeneous generators of weights $a_0, \dots, a_N$ and I_V is the ideal of all relations. Since $-K_V$ is ample, $\operatorname{Proj} R$ is the anticanonical model of V in w.p.s.:

$$V = \operatorname{Proj} R \subset \mathbb{P}^N(a_0, a_1, \dots, a_N).$$

Here we take the complete anticanonical ring $R(V, -K_V)$, so that $V \subset \mathbb{P}^N$ is projectively normal; it is known that V is projectively Gorenstein, that is, R is a Gorenstein ring (see for example Goto and Watanabe [**GW**]).

The *codimension* of R means the codimension $N - 3$ of the anticanonical ring $R(V, -K_V)$. It is a measure of the difficulty of studying V directly. In codimension 2, a Gorenstein ring is a c.i. (Serre), and in codimension 3, it is given by the $2k \times 2k$ diagonal Pfaffians of a skew $(2k+1) \times (2k+1)$ matrix (Buchsbaum and Eisenbud); only 5×5 occurs here, with the single exception of the classic case $Q_1 \cap Q_2 \cap Q_3 \subset \mathbb{P}^6$.

The cases when V is a hypersurface or codimension 2 c.i. are settled in Fletcher [**Fl**], and the codimension 3 cases in Altınok [**A**]. There are 95 families of Fano 3-fold hypersurfaces, 84 codimension 2 c.i.s (including Fletcher's example mentioned in 4.8.2 with $H^0(-K_V) = 0$), and 70 families of codimension 3 Pfaffians. These varieties are given by explicit equations, that we can write down without reference to K3s, and depend on an irreducible parameter space. Their construction is unobstructed, and every family of K3 surfaces extends to a family of Fano 3-folds.

5.3. Weighted Grassmannian. In codimension 3, the five Pfaffians are a weighted form of the equations of the Grassmannian $\mathrm{Grass}(2,5)$ (see 1.8). Thus a codimension 3 K3 surface $S \subset \mathbb{P}(a_0, \dots, a_5)$ has Hilbert series

$$1 - \sum t^{b_i} + \sum t^{k-b_j} - t^k,$$

where $k = \sum_0^5 a_i$ and $2k = \sum_1^5 b_i$, so that the skew matrix in 1.8 has entries of weight $\operatorname{wt} x_{ij} = k - b_i - b_j$ (compare Reid [**Ki**], Example 3.8). We can view the

x_{ij} as coordinates on $\bigwedge^2 U \otimes L$, where U is a weighted $\mathbb{C}^5$ with weights $c - b_i$, and $L = \mathbb{C}$ with weight $k - 2c$ (for some convenient c); then the equations of S are the pullback of the weighted Grassmann $\mathrm{wGr} \subset \mathbb{P}(\bigwedge^2 U)$ (tensors of rank 2) by a weighted projective map $\mathbb{P}(a_0, \dots, a_5) \dashrightarrow \mathbb{P}(\bigwedge^2 U)$. In the spirit of Mukai's work discussed in 2.4, the interesting question is to construct directly an exceptional (orbi-) vector bundle $\mathcal{E}$ on S whose Serre module $H^0_*(\mathcal{E})$ is generated by 5 sections $u_i \in \mathcal{E}(c - b_i)$, giving the embedding $S \hookrightarrow \mathrm{wGr} \subset \mathbb{P}(\bigwedge^2 U)$.

The point here is to get away from the algebraic treatment of the equations of S to a more geometric understanding of what they mean; that is, eventually, to replace the appeal to Buchsbaum and Eisenbud's algebraic result on codimension 3 Gorenstein rings by arguments in the style of Mukai in terms of exceptional vector bundles on the K3 section. There are preliminary results in this direction due to Corti and Reid [**CR**]. This raises important unsolved problems in higher codimension: weighted versions of Mukai's symmetric spaces make sense, but seem to give few examples of varieties of small coindex.

5.4. Gorenstein unprojection. In codimension ≥ 4, there is no analogous structure theorem for Gorenstein rings; our current strategy for K3s and Fano 3-folds in codimension 4, 5, etc., is based (in most cases) on *Gorenstein projection.* A model case is the projection $S_d \dashrightarrow S_{d-1}$ of a del Pezzo surface $S_d \subset \mathbb{P}^d$ from $P \in S_d$ (for simplicity, think of $d \geq 4$). Here both S_d and S_{d-1} are anticanonical: the blowup $S' \to S_d$ of P has discrepancy 1, so that the anticanonical map $S' \to S_{d-1}$ given by $|-K_{S_{d-1}}| = |-K_{S_d} - E|$ is the same thing as the linear projection from P.

Although the study of a Fano 3-fold $V \subset \mathbb{P}^N(a_0, a_1, \dots, a_N)$ is a biregular problem in the first instance, we can use birational methods to attack it. Fano's linear projection of $V = V_{2g-2} \subset \mathbb{P}^{g+1}$ with $g \geq 7$ from a line can also be interpreted as making a blowup $V_1 \to V$, then recalculating the anticanonical ring $R(V_1, -K_{V_1})$ as a subring of $R(V, -K_V)$, thus deducing a birational map $V \dashrightarrow V' = \operatorname{Proj} R(V_1, -K_{V_1})$. In Fano's study, V is the unknown, and is hard to work with because it has high codimension, whereas its projection V' has smaller codimension, so it, together with the exceptional divisor of the projection $F \subset V'$, may be more tractable.

For us, the key point is that the birational relation between V and V' (geometrically a projection) can be handled in terms of inclusions

$$R(V', -K_{V'}) \subset R(V, -K_V)$$

between Gorenstein rings. This area has recently been clarified by Papadakis and Reid's treatment of the inverse birational map $V' \dashrightarrow V$ as *Gorenstein unprojection* or "constructing big Gorenstein rings from small ones" (see [**PR**] and [**Ki**], and compare Kustin and Miller [**KM**]); in [**PR**], the inverse is constructed in terms of the adjunction formula for the Grothendieck–Serre dualising sheaf.

5.5. Type I unprojection. We discuss here the most straightforward case of projection, corresponding to Kustin and Miller unprojection, which already applies to the majority of K3s in codimension ≥ 4. Let

$$S \subset \mathbb{P}(a_1, \dots, a_N) \quad \text{or} \quad V \subset \mathbb{P}(1, a_1, \dots, a_N)$$

be a K3 surface or Fano 3-fold. We only treat the Fano case here, leaving the reader to make the obvious modifications in the K3 case.

A coordinate point $P_k = (0, \dots, 1, \dots, 0) \in V$ is a *Type I centre* if $P_k \in V$ is a terminal quotient singularity $\frac{1}{r}(1, a, r-a)$, where $\operatorname{wt} x_k = r$, and global coordinates of the projective space x_0, x_i, x_j of weight equal to $1, a, r-a$ provide local orbifold coordinates at P_k. Let $V_1 \to V$ be the Kawamata blowup, that is, the weighted blowup of P_k with weights $1, a, r-a$; then the exceptional locus $E \cong \mathbb{P}(1, a, r-a)$ has minimal discrepancy $\frac{1}{r}$, and, following [**CPR**], we write

$$A = -K_V \quad \text{and} \quad B = -K_{V_1} = A - \frac{1}{r}E.$$

Now because $\operatorname{wt}(x_0, x_i, x_j) = (1, a, r-a)$, and x_0, x_i, x_j vanish along E with multiplicity $\frac{1}{r}, \frac{a}{r}, \frac{r-a}{r}$ (see [**CPR**], Proposition 3.4.6), they belong to the subring $R(V_1, B)$. Thus $R(V_1, B)$ is the subring $k[x_0, \dots, \widehat{x_k}, \dots, x_N] \subset R(V, A)$ obtained by eliminating x_k only. Then

$$V' = \operatorname{Proj} R(V_1, B) \subset \mathbb{P}(1, a_1, \dots, \widehat{a_k}, \dots, a_N)$$

is a weak Fano 3-fold containing the plane $\Pi = \mathbb{P}(1, a, r-a)$. Be warned that V' is not a Mori Fano 3-fold, since the Weil divisor Π is not in $\mathbb{Z} \cdot (-K_{V'})$; one usually expects V' to be the *midpoint* of a Sarkisov link, compare [**CPR**], 4.1, (3).

A simple example of a Type I unprojection was treated in Example 3.7. Type I projections include the construction of the 64 quadratic involutions of [**CPR**], 4.4 (or rather, of the first half $X \dashrightarrow Z$, up to the midpoint of the link).

Exercise 5.6. In the numerical data $g, \mathcal{B}$ for a K3 surface (Section 4), replace an element $\frac{1}{r}(a, r-a)$ of $\mathcal{B}$ by two elements $\frac{1}{a}(r, -r)$ and $\frac{1}{r-a}(r, -r)$, and assume that D^2 remains positive. Study how this numerical projection affects the rhs of (4.6.1): show that it subtracts $\frac{t^r}{(1-t^r)(1-t^a)(1-t^{r-a})}$ from the Hilbert series $P_{S,D}(t)$, and reduces D^2 by $\frac{1}{ra(r-a)}$. Compare [**PR**], Exercise 2.7, and see Example 6.1, page 23 for a numerical instance.

The effect of a Type I projection as in Exercise 5.6 on the numerics of Section 4 gives the set of numerical data of K3 surfaces the structure of a directed graph, with comparatively few connected components. With few exceptions, families of K3 surfaces in codimension ≥ 2 have projections to smaller codimension, most commonly of Type I. Of the 142 codimension 4 K3s in the K3 database, 116 have a numerical Type I centre. All but 2 of the remaining cases are covered by the higher types of projection discussed in Reid [**Ki**], Section 9 and [**T4**]. These higher projections, and the small core of exceptions not admitting any projections, are interesting and demand further study; we suspect that these more complicated K3s are unlikely to extend to Fano 3-folds.

5.7. Tom and Jerry unprojections to codimension 4. When applicable, a Type I projection as described in 5.5 reduces the study of a Fano V in w.p.s. $\mathbb{P}^N$ to a variety V' in w.p.s. $\mathbb{P}^{N-1}$, but specialised to contain a weighted projective subspace:

$$\mathbb{P}(1, a, r-a) \subset V' \subset \mathbb{P}^{N-1}. \tag{5.7.1}$$

Thus Fano 3-folds whose numerical data admits a codimension 4 candidate can often be studied via projections. However, setting up the unprojection data (5.7.1) is still a difficult problem; complicated features of V, such as obstructed equations or reducible moduli spaces, must be faithfully reproduced in the unprojection data.

In (5.7.1), the w.p.s. $\mathbb{P}(1, a, r-a)$ is a codimension 4 c.i., and V' is a codimension 3 variety given by the Pfaffians of a 5×5 skew matrix. Thus to obtain codimension 4 Fanos with this type of projection, the problem is how to put a codimension 4 c.i. inside a 5×5 Pfaffian. Similarly for K3s.

There are two different solutions to this problem, called *Tom* and *Jerry*, treated in detail in Papadakis [**P**]–[**P1**] (see also [**Ki**], Examples 6.4 and 6.8, and Section 8). In these two cases, we specialise the skew 5×5 matrix defining V' to

$$M_{\mathrm{Tom}} = \begin{pmatrix} x_{12} & x_{13} & x_{14} & x_{15} \\ & a_{23} & a_{24} & a_{25} \\ & & a_{34} & a_{35} \\ & & & a_{45} \end{pmatrix} \quad \text{or} \quad M_{\mathrm{Jerry}} = \begin{pmatrix} a_{12} & a_{13} & a_{14} & a_{15} \\ & a_{23} & a_{24} & a_{25} \\ & & x_{34} & x_{35} \\ & & & x_{45} \end{pmatrix}$$

where the 6 entries a_{ij} in the bottom left 4×4 block of M_{Tom} specialise to lie in a codimension 4 complete intersection ideal $(y_1, \dots, y_4)$ (and ditto for the 7 entries a_{ij} in the first two rows and columns of M_{Jerry}). In either case, the theoretical construction of [**PR**] or [**KM**] gives the unprojection $V \subset \mathbb{P}^N$ and its anticanonical ring, and Papadakis [**P1**], Section 5 gives an explicit presentation of the ring.

Tom and Jerry occur in hundreds of constructions of Gorenstein codimension 4 rings with 9×16 presentation, and seem to be related to the respective cones over $\mathbb{P}^2 \times \mathbb{P}^2$ and over $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and their weighted homogeneous deformations; but exactly what this means remains to be elucidated. See [**Ki**], Section 8 for a more detailed discussion.

To deal with our codimension 4 K3s and Fanos, we have to determine whether Tom and Jerry matrixes can be set up to give V having only Mori category singularities. There are a few hundred problems here, and we have only just started working systematically on the Fanos. For each case where the numerical data of the K3 admits a Type I projection, we can ask for a Tom or Jerry matrix – from the experience of Altınok's thesis [**A**] and [**CPR**], we suspect that in each case at least one of Tom or Jerry exists, and that when both exist, the two families give isomorphic polarised K3s, so do not correspond to different irreducible components of moduli.

When we ask the same question for Fanos 3-folds, one of several things may happen: either of the Tom and Jerry matrixes could exist to give a Mori Fano 3-fold, or either could fail. We suspect that each of these possibilities happens in many numerical cases. When both Tom and Jerry exist, the resulting unprojected varieties may give essentially different different Fano 3-folds: see [**Ki**], Examples 6.4 and 6.8, based on Takagi's thesis, where Tom and Jerry unprojections with the same numerical data correspond to Fano 3-folds that are not biregular, and not obviously birational. Since we have several hundred cases to settle, and much of the calculation comes down to checking that monomials of suitable degree exist, we hope that most of the calculation can eventually be entrusted to the computer.

6. The K3 database in Magma

The program outlined in Sections 4–5 envisaged listing the many thousand possible values $g, \mathcal{B}$ for the numerical data of families of K3s or Fano 3-folds. For each $g, \mathcal{B}$ in the list, we calculate the Hilbert series and try to deduce a plausible shape for the anticanonical ring and its possible projections. In some cases, we can establish the existence of a quasi-smooth surface, extension to Fano 3-fold, and

connected components of moduli – these are all questions that become nontrivial when $R(S, D)$ has codimension ≥ 4. These calculations are akin to traditional work on embedding curves and surfaces in Italian algebraic geometry, although rational contributions from quotient singularities add a certain spice, and unlimited possibilities for errors of arithmetic. With patience, any particular calculation can be done by hand, but automation has obvious advantages when doing these calculations on an industrial scale.

Several items of this program have already been carried out for polarised K3 surfaces. Hilbert series methods apply to many other situations in algebraic geometry involving graded rings or modules; current work in progress by students at Warwick includes Suzuki [**S**] on Fano 3-folds of Fano index $f \geq 2$, Keenan's project [**Ke**] on subcanonical curves and Buckley's study [**Bu**] of polarised Calabi–Yau 3-folds with strictly canonical singularities. Nonetheless, rings over K3 surfaces have several advantages that make them an ideal target for a computer study: the numerical data and the RR statement are simple to state, there is no cohomology, and most K3 surfaces have projections to smaller codimension. Moreover, work over the last 20 years (Fletcher [**Fl**], Altınok [**A**] and Corti, Pukhlikov and Reid [**CPR**]) already provides us with several hundred worked out examples of what to expect.

We have programmed these calculations as routines in John Cannon's computer algebra system Magma [**Ma**]. The results of all the calculations in codimension ≤ 4 are assembled as a database containing 391 families of K3 surfaces. We describe here a little of what our code does and how to use it. Although more sophisticated upgrades are projected (and implemented in part, see the Graded Ring Database Website [**GRDW**]), we restrict ourselves to the first working version of our routines, in the widely available export 2.8 of Magma [**Ma**]. Please see [**GRDW**] for more recent work; this includes a prototype database of K3s up to codimension 10, and will document upgrades as they come on line.

Example 6.1 (making a single surface). Suppose we want to study a polarised K3 surface S, D with ineffective D (that is, $g = -1$, see 4.8.2), and having basket of singularities $\mathcal{B} = \{\frac{1}{2}(1,1), \frac{1}{5}(1,4), \frac{1}{13}(3,10)\}$. As usual, > at the beginning of a line is the Magma prompt. Anything after this prompt is user input, while anything else is Magma output (occasionally subjected to a little editing for legibility). We have formatted the Magma input and output along the lines of our mathematical writings, and you should not have too much trouble interpreting the language.

```
> Q := Rationals();
> R<t>:=PolynomialRing(Q); // Omit these lines at your peril.
> B := [ [2,1], [5,1], [13,3] ];
> S := K3Surface(-1,B); // We input genus -1 and Basket B.
> S;
Codimension 4 K3 surface with data
  Weights: [ 3, 4, 5, 6, 7, 10, 13 ]
  Numerator: t^48 - t^36 - t^35 - t^34 - t^33 - t^32 - t^31 - t^30
    + t^27 + 2*t^26 + 2*t^25 + 2*t^24 + 2*t^23 + 2*t^22 + t^21
    - t^18 - t^17 - t^16 - t^15 -  t^14 - t^13 - t^12 + 1
  Basket: [ 2, 1 ], [ 5, 1 ], [ 13, 3 ]
```

The computer has calculated the Hilbert series, and, after experimenting with a number of possibilities, has found

$$(1-t^3)(1-t^4)(1-t^5)(1-t^6)(1-t^7)(1-t^{10})(1-t^{13})$$

as a plausible denominator, corresponding to generators for the ring $R(S,D)$, or an embedding of S in w.p.s. $\mathbb{P} = \mathbb{P}^6(3,4,5,6,7,10,13)$. The printout lists the conclusion: a candidate family of codimension 4 K3 surfaces $S \subset \mathbb{P}$ with the stated Hilbert numerator. We should be clear that in general

> *there is no a priori guarantee that this or any other candidate surface proposed by our Magma functions exists as a subvariety in the indicated w.p.s. with the indicated properties.*

After all, at this stage the computer has done nothing more than a formal game with denominators for Hilbert series. Having said that, assume for a moment that $S \subset \mathbb{P}^6(3,4,5,6,7,10,13)$ really exists, polarised by the divisorial sheaf $\mathcal{O}_S(D) = \mathcal{O}_S(1)$. Clearly D is not an effective divisor, since no x_i has weight $a_i = 1$, but its multiples do exist: in particular, S has a single effective divisor in $|3D|$, $|4D|$, $|5D|$, a pencil $|6D|$, etc.

The `Numerator` is a polynomial $p = P(t)\prod(1-t^{a_i})$, where $P(t)$ is the Hilbert series of $R(S,D)$

$$P_S(t) = \sum_{n\geq 0} h^0(S,nD)t^n = \frac{p}{\prod(1-t^{a_i})}\,.$$

Here the product is taken over the weights a_i of the w.p.s. At first sight, the 7 negative terms $-t^{12} - t^{13} - \cdots$ in the numerator output by Magma might suggest that $S \subset \mathbb{P}$ could have 7 equations in degrees 12, 13, 14, 15, 16, 17, 18. Gorenstein rings with 7×12 resolution certainly exist, so this is not completely excluded, but this structure appears very rarely (compare Papadakis and Reid [**PR**], 2.8); perhaps this candidate appears more exotic than it actually is. The 9×16 format is ubiquitous, and experience suggests that this is really a 9×16 resolution in disguise.

How can we prove that this surface actually exists? The method of Altınok's thesis [**A**] is to look for a codimension 3 K3 surface that is already known to exist, and could be a projection of the desired surface. Then we could go on to show that the codimension 3 surface can be made to contain an *unprojection divisor* as in 5.5 that can be contracted or *unprojected*, to give the codimension 4 surface. This second stage is the hard part, and we pass over it for the present (see [**PR**], [**P1**] and [**Ki**]), although in time the computer will have something useful to add here too. One point is that as we write there are many kinds of projection for which the unprojection is still something of a mystery (see [**Ki**], Section 9). That is, the *theory* of unprojection is not quite complete enough yet to handle all the cases we might need.

However, the K3 database is very good at finding projections of the types we already understand, such as the Type I projections of Section 5. These occur when we project from a cyclic quotient singularity $\frac{1}{r}(a, r-a)$ whose local orbifold coordinates come from generators of $R(S,D)$ of weight $a, r-a$; a singularity with this property is a *Type I centre*. In our example, we can see a possible Type I centre, the singularity $\frac{1}{13}(3,10)$. The point is that the local weights 3 and 10, a priori only defined mod 13, can arise as the global generators corresponding to the weights 3 and 10 of the w.p.s. All the other singularities fail this test, since the w.p.s. has no weight 1.

```
> DB := K3Database("t"); /* Loads the DB, requiring "t" as
 variable in Hilbert series. This takes several seconds. */
> Centres(~S,DB); // Searches DB for projection centres of S.
> S;
Codimension 4 K3 surface with data
  Weights: [ 3, 4, 5, 6, 7, 10, 13 ]
  Numerator: t^48 - t^36 - t^35 - t^34 - t^33 - t^32 - t^31 - t^30
 + t^27 + 2*t^26 + 2*t^25 + 2*t^24 + 2*t^23 + 2*t^22 + t^21
 - t^18 - t^17 - t^16 - t^15 -  t^14 - t^13 - t^12 + 1
  Basket: [ 2, 1 ], [ 5, 1 ], [ 13, 3 ]
  Centre 1: [ 5, 1, 4 ] has Type 2 projection to 10 in codim 1
  Centre 2: [ 13, 3, 10 ] has Type 1 projection to 42 in codim 3
```

This says that the singularity $\frac{1}{13}(3,10)$ is a Type I centre as expected. It has also found a Type II centre, that we pass over for the moment. But it does more: Magma applies the calculus of Exercise 5.6 to predict the weights and basket of the image of a Type I projection, and searches the database for surfaces with the right properties, finding the K3 surface numbered 42 in the database as a plausible image of the projection from this centre. (The internal numbering of items in the database is arbitrary, and may differ from session to session.) We ask what it is:

```
> K3Surface(DB,42);
Codimension 3 K3 surface, number 42, Altinok3(63), with data
  Weights: [ 3, 4, 5, 6, 7, 10 ]
  Numerator: -t^35 + t^23 + t^22 + t^21 + t^20 + t^19
 - t^16 - t^15 - t^14 - t^13 - t^12 + 1
  Basket: [ 2, 1 ], [ 3, 1 ], [ 5, 1 ], [ 10, 3 ]
```

Now if we know that the image codimension 3 surface $S' \subset \mathbb{P}(3,4,5,6,7,10)$ exists, and contains the stratum $\mathbb{P}(3,10) \subset \mathbb{P}$ as unprojection divisor, then it has the required geometric properties to be unprojected, and we conclude from [**PR**] that the original surface exists. Indeed, since we know a lot about unprojection, we can even write down the equations of the codimension 4 surface S based on those of the codimension 3 surface S' and its unprojection divisor $\mathbb{P}(3,10)$. (Since $\mathbb{P}(3,10)$ is the c.i. defined by $x_4 = x_5 = x_6 = x_7 = 0$, and the unprojection variable s has weight 13, we verify that S also needs equations in degree $19, 20$, so our intuition that S has a 9×16 resolution was right.)

The basic existence part is easy. The structure theorem for Gorenstein rings of codimension ≤ 3 makes it easy for us to figure out equations. Altınok goes further to check that a surface containing the unprojection divisor can be made. Using Altınok's numbering in [**A1**], Tables 5.1 and 5.2, the codimension 3 surface is $\text{Altınok}_3(63)$, as the database tells us, while the codimension 4 surface is $\text{Altınok}_4(104)$.

Of course, the projection calculus is easy to do by hand: we could simply have searched the lists of [**Fl**] and [**A**] to find this surface (as we always did in the dark ages before the great Graded Ring Database Revolution). For that matter, we could have just called for a surface having the desired basket and again found this one without trouble. At this stage, we have not gained much by having the results of these calculations in a database. The next section contains a calculation for which the database is essential.

Example 6.2 (searching the database). As databases go, our current database of 391 K3s is tiny. Even so, its key advantage over a printed list is that we can search it quickly and accurately. Suppose, for example, that we are interested in finding out whether our lists contain a K3 having a singularity of index 17. In this context, 17 is a large number, taking the sublattice $L(g, \mathcal{B}) \subset \operatorname{Pic} S$ close to the maximum that can live in the K3 lattice Λ_{K3} (see 4.8.4). To do this search in Magma, we first summon the database as usual:

```
> DB := K3Database("t");
> #DB; // # returns the number of elements of a list.
391
```

Singularities are denoted `p = [r,a]`, with a coprime to the index `r = p[1]` and $a < r/2$. So we must search the database for surfaces having a singularity $[r, a] \in \mathcal{B}$ having $r = 17$.

```
> surfaces := [ S : S in DB | &or[ p[1] eq 17 : p in Basket(S) ] ];
> #surfaces;
2
```

Note that `&or` taken over a list of Boolean values is equivalent to "there exists p in $\mathcal{B}$ with $p[1]$ equals 17". The last line says that there are just two surfaces in DB with a singularity of index 17. We ask to see them.

```
> for S in surfaces do print S;
> end for;
Codimension 2 K3 surface, number 1, Fletcher2(82), with data
  Weights: [ 3, 4, 7, 10, 17 ]
  Numerator: t^41 - t^21 - t^20 + 1
  Basket: [ 2, 1 ], [ 17, 7 ]
Codimension 4 K3 surface, number 250, Altinok4(79), with data
  Weights: [ 2, 3, 5, 5, 7, 12, 17 ]
  Numerator: t^51 - t^41 - t^39 - t^37 - t^36
   + t^29 + t^27 + t^26 + t^25 + t^24 + t^22
   - t^15 - t^14 - t^12 - t^10 + 1
  Basket: [ 17, 5 ]
```

Again, we want to know that these surfaces exist. The codimension 2 surface is Fletcher [**Fl**], List 13.8, no. 82, and it is easy to write down its equations.

For the codimension 4 surface, the technique of the previous example locates an image of projection from the Type I centre $\frac{1}{17}(5, 12)$ in the database, and we can construct S by unprojection as before. This codimension 4 example S, called Altınok$_4$(79) is interesting in that it has exactly the same Hilbert series as the codimension 4 c.i. $T_{10,12,14,15}$; however, none of the equations of the c.i. T can involve the last variable u_{17}, so that T is a kind of weighted cone over a curve. Deformations of polarised K3 surfaces are unobstructed, as are c.i.s, so that having proved that S exists, we have found two different irreducible components of the Hilbert scheme, one containing S, the other containing T.

6.3. K3 database functions in Magma. As with much computer algebra, just a few internal functions do the work, and a lot of the rest is cosmetic renaming to simplify the user's access to these. We give a brief and possibly inadequate description of some core functions to give some idea of what is happening inside

the computer; these remarks can also serve as a first introduction to the code for anybody wishing to modify it for use in other contexts.

6.3.1. *Listing all baskets.* The function

```
> Baskets(n);
```

returns a list of all possible baskets $\mathcal{B} = \{\frac{1}{r}(a, r-a)\}$ for K3 surfaces of singular rank $\sum(r-1) < n$. For example, the following commands generates a list of all baskets with $\sum(r-1) = 11$, and asks for the 44th basket in the list.

```
> BB, gg := Baskets(12); /* BB is the list of baskets,
   gg the parallel list of minimal genera */
Actual number: 329 ... Checking degrees ...
> BB11 := [ B : B in BB | not (B eq [])
    and &+[p[1]-1 : p in B] eq 11 ];  // &+ is sum over set.
> #BB11;
109
> b := BB11[44];
> b, gg[Index(BB,b)];
[ [ 4, 1 ], [ 5, 2 ], [ 5, 2 ] ]
0
> Degree(-1,b);
-17/20
> Degree(0,b);
23/20
```

Recall from 4.8.1 that the possible baskets $\mathcal{B}$ on a K3 surface are limited by two inequalities on the pairs r, a. The first is given by the rank of the Picard group: $\sum(r-1) < n = 20$. The argument n of `Baskets(n)` is the bound (we eventually set $n = 20$ for complete lists). The other inequality, saying that the polarised surface S, D has degree $D^2 > 0$, involves the genus g of S. If $g = -1$ or 0, a basket may give $D^2 \leq 0$. The second return value of `Baskets` (called gg in the first line above) is a parallel sequence of minimal genera that give $D^2 > 0$. In the above example, we checked that $g = -1$ is illegal for the 44th basket, because it gives $D^2 = -17/20$.

The numerical data $g, \mathcal{B}$ with $D^2 > 0$ determines a Hilbert series $P(t) = \sum P_n t^n$. If it actually corresponds to a K3 surface, then vanishing implies that $P_n \geq 0$ for all $n \geq 1$. However, this does not follow from the inequality $D^2 > 0$. Indeed, there exist numerical data $g, \mathcal{B}$ with a negative coefficient P_n; curiously, there are just three of these eccentrics (compare [**CPR**], 7.9). The function `Baskets` also checks that $P_n \geq 0$ for the first 20 coefficients before confirming the minimum genus g. When $n = 20$, the result is all 6640 possible baskets, together with, for each, the minimum genus that makes the Hilbert series positive.

6.3.2. `HilbertSeries(g,B)`, *etc.* The Magma function

```
> HilbertSeries(g,B);
```

has the formula (4.6.1) built in, and simply evaluates it at the data $g, \mathcal{B}$ as a rational function in t. For example,

```
> R<t>:=RationalFunctionField(Q); // Require "t" as variable.
> P:=HilbertSeries(-1,[[ 3, 1 ], [ 4, 1 ], [ 11, 2 ]]);
> P*(1-t^2)*(1-t^3)*(1-t^4)*(1-t^11);
-t^20 - t^15 - t^13 - t^11 + t^9 + t^7 + t^5 + 1
```

suggests the candidate surface $S \subset \mathbb{P}(2,3,4,5,7,9,11)$ with the stated g and $\mathcal{B}$. (To make this work for other types of graded rings, you need to figure out what Hilbert series you want to use, and program it in as a substitute for our formula (4.6.1).)

In the above example, we put in the denominator by hand, and this turned out to be a lucky guess. The heart of the whole package is a suite of functions to make a reasonable analysis of this Hilbert series, using a few tricks based on the experience of Altınok's thesis [**A**]. One expects generators corresponding to positive terms early on in the Hilbert series, but this ceases to be logically reliable once relations appear. For each singularity $\frac{1}{r}(a, r-a)$ in the basket, there must be generators of weight divisible by r to cancel the periodicity in the Hilbert series, and a generator of weight $\equiv a$ and $-a$ to provide local orbifold coordinates at the singularity. We usually expect a generator in each degree r as in the above example, but sometimes a GCD of two different r can account for two at one go.

Deploying these tricks is something of an art, and there is no point in being precise here about how we do it. Experience suggests that they need to be combined in different ways in different situations (we find different and better ways of putting in the generators systematically each time we extend the database). But however we use the numerical tricks on the Hilbert series, there will usually be work left to do. Particular features of the geometry may demand extra generators not predicted by the numerical data. For K3 surfaces in higher codimension, the existence of particular linear systems often forces us to include extra generators. One reason for stopping at codimension 4 with the current version of the Magma database is that there is little extra work in that case.

6.3.3. `MakeK3Database(BB,gg,cmax)`. This function takes as its arguments a list of baskets BB with list of minimal genus gg, and an integer `cmax`. It takes each basket and genus pair in turn and applies any analysis of the Hilbert series that is implemented. If the codimension gets larger than `cmax`, the result is discarded (in the current implementation — this is handled in a more useful way in the prototype future implementation). Once done, the result is wrapped in cosmetics and available for computer study.

Of course, as we learn more tricks for analysing Hilbert series, we can run them through the database, modifying candidate surfaces as we go. But we believe that the current Magma database of 391 K3 surfaces is reliable and complete. (For that matter, we are fairly confident of the codimension 5 list in [**GRDW**], which has $N_5 = 162$ elements.)

6.3.4. `Centres(~DB)`, *etc.* There are several functions used to study the database by hand. The procedure

```
> DB := K3Database("t");
> Centres(~DB); // Takes some minutes.
```

performs the projection calculus of Exercise 5.6 to compute all possible projections between the surfaces of the database `DB`; the tilde indicates that the command is *procedural*, that is, it actually modifies `DB` by writing in the centres it computes (the next version of the database will have the centres ready for use on loading). Once the centres are in, we can look at the projections from any surface. For example, we can find the surface S of 6.1 in the database and then look for all possible iterated projections from S:

```
> S := K3SurfaceFromWeights(DB,[3,4,5,6,7,10,13]);
> pc := ProjectionChains(S,DB); #pc;
```

```
4
> pc[1];
[
 [ [ 254, 4 ], [ 5, 1, 4 ], [ 10 ] ],
 [ [ 10, 1 ], [ 13, 3, 10 ], [ 40 ] ],
 [ [ 40, 1 ], [ 10, 3, 7 ], [ 79 ] ],
 [ [ 79, 1 ] ]
]
```

This output is opaque (Leitmotif: please buy the upgrade, with its many major enhancements), but you can probably see what is going on. There are 4 chains of projections starting with S, and `pc[1]` only asks for the first. If we denote the nth surface in the database by S_n, then $S = S_{254}$ and the first chain of projections, returned by `pc[1]`, is

$$S \dashrightarrow S_{10} \dashrightarrow S_{40} \dashrightarrow S_{79}.$$

The output records the codimension of each surface. In this case the final three surfaces are all codimension 1: the projections between them are the quadratic involutions of [**CPR**]. The second column gives the type of centre `[r,a,r-a]` of each projection.

The fourth chain of projections is

```
> pc[4];
[
 [ [ 254, 4 ], [ 13, 3, 10 ], [ 42 ] ],
 [ [ 42, 3 ], [ 10, 3, 7 ], [ 81 ] ],
 [ [ 81, 2 ], [ 7, 3, 4 ], [ 107 ] ],
 [ [ 107, 1 ] ]
]
```

It consists purely of Type I projections. In terms of homogeneous coordinate rings, at each stage a single variable (of weight 13, 10, 7 respectively, the index of the corresponding centre) is eliminated from the ring. A basic exercise in Magma makes this very clear:

```
> [ Weights(DB[i]) : i in [254,42,81,107] ];
[
    [ 3, 4, 5, 6, 7, 10, 13 ],
    [ 3, 4, 5, 6, 7, 10 ],
    [ 3, 4, 5, 6, 7 ],
    [ 3, 4, 5, 6 ]
]
```

terminating with the surface S_{107} which is

```
> DB[107];
Codimension 1 K3 surface, number 107, Reid1(39), with data
  Weights: [ 3, 4, 5, 6 ]
  Numerator: -t^18 + 1
  Basket: [ 2, 1 ], [ 3, 1 ], [ 3, 1 ], [ 3, 1 ], [ 4, 1 ], [ 5, 1 ]
```

Exercise 6.4. Write down a hypersurface $S_{18} \subset \mathbb{P}(3,4,5,6)$ that contains $\mathbb{P}(3,4)$ as the unprojection divisor of the final projection, and singularities on it to provide the inverses of the successive unprojections.

References

[A] S. Altınok, Graded rings corresponding to polarised K3 surfaces and $\mathbb{Q}$-Fano 3-folds, Univ. of Warwick PhD thesis, Sep. 1998, 93 + vii pp., get from www.maths.warwick.ac.uk/∼miles/doctors/Selma

[A1] S. Altınok, Hilbert series and applications to graded rings, submitted

[Be] Sarah-Marie Belcastro, Picard lattices of families of K3 surfaces, to appear in Communications in Algebra, preprint math.AG/9809008, 21 pp.

[Br] Gavin Brown, Datagraphs in algebraic geometry, submitted to Proceedings of SNSC '01 (Linz), Ed. F. Winkler

[GRDW] Gavin Brown, Magma algebraic geometry database website, see www.maths.warwick.ac.uk/∼gavinb/grdb.html

[Bu] Anita Buckley, Graded rings over polarised Calabi–Yau 3-folds, work in progress

[CPR] A. Corti, A. Pukhlikov and M. Reid, Birationally rigid Fano hypersurfaces, in Explicit birational geometry of 3-folds, A. Corti and M. Reid (eds.), CUP 2000, 175–258

[CR] A. Corti and M. Reid, Weighted Grassmannians, first draft submitted to Proc. Francia memorial conference (Genova, 2001), currently 12 pp.

[D] I. Dolgachev, Weighted projective varieties, in Group actions and vector fields (Vancouver, B.C., 1981), LNM**956**, pp. 34–71

[Fl] A.R. Iano-Fletcher, Working with weighted complete intersections, in Explicit birational geometry of 3-folds, CUP 2000, pp. 101–173

[GW] S. Goto and K-i Watanabe, On graded rings. I, J. Math. Soc. Japan **30** (1978) 179–213

[G] M. Gross, Deforming Calabi-Yau threefolds, Math. Ann. **308** (1997) 187–220

[I1] V. A. Iskovskikh, Fano threefolds. I, Izv. Akad. Nauk SSSR Ser. Mat. **41** (1977) 516–562 = Math. USSR–Izv. **11** (1977) 485–527

[I2] V. A. Iskovskikh, Fano threefolds. II, Izv. Akad. Nauk SSSR Ser. Mat. **42** (1978) 506–549 = Math. USSR–Izv. **12** (1978) 469–506

[Ka1] KAWAMATA Yujiro, On the plurigenera of minimal algebraic 3-folds with $K \approx 0$, Math. Ann. **275** (1986) 539–546

[Ka2] KAWAMATA Yujiro, Boundedness of $\mathbb{Q}$-Fano threefolds, in Proc. Internat. Conference on Algebra (Novosibirsk, 1989), Contemp. Math. **131**, AMS, 1992, Part 3, pp. 439–445

[Ke] Adam Keenan, Riemann–Roch, subcanonical divisors on curves and graded rings, work in progress

[KM] A. Kustin and M. Miller, Constructing big Gorenstein ideals from small ones, J. Algebra **85** (1983) 303–322

[Leng] R. Leng, McKay correspondence and equivariant Riemann–Roch, Warwick PhD thesis in preparation

[Ma] Magma (John Cannon's computer algebra system): W. Bosma, J. Cannon and C. Playoust, The Magma algebra system I: The user language, J. Symb. Comp. **24** (1997) 235–265. See also www.maths.usyd.edu.au:8000/u/magma

[Mi1] MINAGAWA Tatsuhiro, Deformations of $\mathbb{Q}$-Calabi–Yau 3-folds and $\mathbb{Q}$-Fano 3-folds of Fano index 1, math.AG/9905106, J. Math. Sci. Univ. Tokyo Tokyo **6** (1999) 397-414

[Mi2] MINAGAWA Tatsuhiro, Deformations of weak Fano 3-folds with only terminal singularities, math.AG/9905107, Osaka J. Math. **38** (2001), to appear

[MM1] MORI Shigefumi and MUKAI Shigeru, Classification of Fano 3-folds with $B_2 \geq 2$, Manuscripta Math. **36** (1981/82) 147–162

[MM2] MORI Shigefumi and MUKAI Shigeru, On Fano 3-folds with $B_2 \geq 2$, in Algebraic varieties and analytic varieties (Tokyo, 1981), Adv. Stud. Pure Math. **1**, Kinokuniya and North-Holland, 1983, pp. 101–129

[Mu1] MUKAI Shigeru, Curves and Grassmannians, in Algebraic geometry and related topics (Inchon, 1992), Internat. Press, Cambridge MA 1993, pp. 19–40

[Mu2] Mukai Shigeru, Curves and symmetric spaces. I, Amer. J. Math. **117** (1995) 1627–1644

[N] NAMIKAWA Yoshinori, Smoothing Fano 3-folds, J. alg geom **6** (1997) 307–324

[NS] NAMIKAWA Yoshinori and J.H.M. Steenbrink, Global smoothing of Calabi-Yau threefolds, Invent. Math. **122** (1995) 403–419

[P] Stavros Papadakis, Gorenstein rings and Kustin–Miller unprojection, Univ. of Warwick PhD thesis, Aug 2001, vi + 72 pp., get from www.maths.warwick.ac.uk/∼miles/doctors/Stavros

[P1] Stavros Papadakis, Kustin–Miller unprojection *with* complexes, submitted, math.AG/0111195, 23 pp.

[PR] Stavros Papadakis and Miles Reid, Kustin–Miller unprojection without complexes, J. algebraic geometry (to appear), preprint math.AG/0011094, 18 pp.

[YPG] M. Reid, Young person's guide to canonical singularities, in Algebraic geometry (Bowdoin, 1985), Proc. Sympos. Pure Math. **46** Part 1, AMS 1987, pp. 345–414,

[R1] M. Reid, Projective morphisms according to Kawamata, Warwick preprint, 1983, unpublished, get from www.maths.warwick.ac.uk/∼miles/3folds

[Ki] M. Reid, Graded rings and birational geometry, in Proc. of algebraic geometry symposium (Kinosaki, Oct 2000), K. Ohno (Ed.), 1–72, get from www.maths.warwick.ac.uk/∼miles/3folds

[S] K. Suzuki, Univ. of Tokyo PhD thesis, in preparation

[T4] M. Reid, Examples of Type IV unprojection, preprint math.AG/0108037, 16 pp.

[T] TAKAGI Hiromichi, On the classification of $\mathbb{Q}$-Fano 3-folds of Gorenstein index 2. I, II, RIMS preprint 1305, Nov 2000, 66 pp.

[T1] TAKAGI Hiromichi, work in progress

[W] WATANABE Kei-ichi, Some remarks concerning Demazure's construction of normal graded rings, Nagoya Math. J. **83** (1981) 203–211

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