A well-posed modified Myers boundary condition

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The Myers boundary condition for acoustics within flow over an acoustic lining has been shown to be illposed, leading to numerical stability issues in the time domain and mathematical problems with stability analyses. This paper gives a modification to make the Myers boundary condition well-posed, by accounting for a thin inviscid boundary layer over the lining, and correctly deriving the boundary condition to first order in the boundary-layer thickness. The modification involves two integral terms over the boundary layer. The first may be written in terms of the mass, momentum, and kinetic energy thicknesses of the boundary layer, which are shown to physically correspond a modified boundary mass, modified grazing velocity, and a tension along the boundary. The second integral term is related to the critical layer within the boundary layer.

The modified boundary condition is validated against high-fidelity numerical solutions of the Pridmore-Brown equation for sheared inviscid flow in a cylinder. Absolute instability boundaries are given for certain examples, though convective instabilities appear to always be present for any boundary layer thickness.

I. Introduction

The interaction of acoustics with an acoustic lining is specified using the impedance of the lining. Taking the fluctuating pressure within the fluid to be $p(\boldsymbol{x})\exp\{\mathrm{i}\omega t\}$ and the velocity to be $\boldsymbol{v}(\boldsymbol{x})\exp\{\mathrm{i}\omega t\}$, the impedance of the lining is $Z = p/(\boldsymbol{v}\cdot\boldsymbol{n})$, where $\boldsymbol{n}$ is the surface normal pointing out of the fluid, and Z will usually be a function of frequency ω. If the acoustics is on top of a mean flow that slips across the lining, it is well known that this boundary condition must be modified.[1–7] If the fluid velocity at the boundary is $\boldsymbol{U} + \boldsymbol{u}\exp\{\mathrm{i}\omega t\}$, then the boundary condition becomes

$$\mathrm{i}\omega\boldsymbol{u}\cdot\boldsymbol{n} = \left(\mathrm{i}\omega + \boldsymbol{U}\cdot\boldsymbol{\nabla} - (\boldsymbol{n}\cdot\boldsymbol{\nabla}\boldsymbol{U})\cdot\boldsymbol{n}\right)p/Z. \tag{1}$$

This follows from matching fluid and solid normal displacement, rather than normal velocity, and is known as the Myers[7] or Ingard[4]–Myers[7] boundary condition, although this equation (apart from the final term) was earlier given by Miles (ref. 1, equation (3.3)'. This was shown to be the correct asymptotic limit of a vanishingly-thin inviscid boundary layer, apparently independently, by Eversman & Beckemeyer[5] and Tester,[6] though the boundary needed to be extremely thin to attain this limit in some cases.[6,8]

Unfortunately, (1) applied with slipping flow leads to numerical instabilities (for example, Refs 9, 10) and mathematical illposedness.[11] In order to circumvent slipping flow, there has been recent interest in acoustics within sheared mean flow,[12,13] following the foundations set by Pridmore-Brown[14] and Mungur & Plumblee.[15] This paper will concentrate on the effect of a thin but finite-thickness inviscid boundary layer over the acoustic lining by incorporating the first- and second-order corrections in the boundary layer thickness. Such a situation has been investigated before,[6,8,16–25] at least to first order, but this is the first time to the authors knowledge that a simple asymptotically-valid Myers-type boundary condition incorporating a finite-thickness boundary layer has been proposed, compared to high-fidelity numerical simulations of the Pridmore-Brown equation, and had a stability analysis performed.

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II. Governing equations

Our governing equations are the equations of motion for an inviscid compressible perfect gas,[26]

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}) = 0, \qquad \rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\boldsymbol{\nabla}p, \qquad \frac{\mathrm{D}p}{\mathrm{D}t} = \frac{\gamma p}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t}, \tag{2}$$

where $\mathrm{D}/\mathrm{D}t = \partial/\partial t + \boldsymbol{u}\cdot\boldsymbol{\nabla}$, ρ is the density, $\boldsymbol{u}$ is the velocity, p is the pressure, and γ is the ratio of specific heats.

Although what follows is valid for any geometry, we will concern ourselves here with a circular cylinder, described by (x, r, θ) coordinates with the x-axis running along the centreline of the cylinder. We take the flow to be (for real integer m)

$$\boldsymbol{u} = U(r)\boldsymbol{e_x} + \big(\tilde{u}(r)\boldsymbol{e_x} + \tilde{v}(r)\boldsymbol{e_r} + \tilde{w}(r)\boldsymbol{e_\theta}\big)\exp\{\mathrm{i}\omega t - \mathrm{i}kx - \mathrm{i}m\theta\}, \tag{3a}$$

$$p = P + \tilde{p}(r)\exp\{\mathrm{i}\omega t - \mathrm{i}kx - \mathrm{i}m\theta\}, \tag{3b}$$

$$\rho = R(r) + \tilde{\rho}(r)\exp\{\mathrm{i}\omega t - \mathrm{i}kx - \mathrm{i}m\theta\}, \tag{3c}$$

where the tilde denotes acoustic quantities which are assumed to be small compared with the steady mean flow. We nondimensionalize distance by the radius of the cylinder, velocity by the centreline mean-flow speed of sound $\sqrt{\gamma P/R(0)}$, and density by the centreline mean-flow density $R(0)$, implying that pressure is nondimensionalized by γP where P is the mean-flow pressure. Under this nondimensionalization, the duct wall is at $r = 1$, $R(0) = 1$, $P = 1/\gamma$, $M = U(0)$ is the centreline mean-flow Mach number, and ω is the Helmholtz number.

Substituting (3a–c) into (2) and taking only terms linear in the acoustic quantities gives the linearized governing equations as

$$\mathrm{i}R(\omega - Uk)\tilde{v} = -\frac{\mathrm{d}\tilde{p}}{\mathrm{d}r}, \qquad \mathrm{i}R(\omega - Uk)\tilde{w} = \frac{\mathrm{i}m}{r}\tilde{p}, \qquad \mathrm{i}R(\omega - Uk)\tilde{u} = \mathrm{i}k\tilde{p} - R\frac{\mathrm{d}U}{\mathrm{d}r}\tilde{v},$$

$$\mathrm{i}(\omega - Uk)\tilde{\rho} = -\tilde{v}\frac{\mathrm{d}R}{\mathrm{d}r} - \frac{R}{r}\frac{\mathrm{d}(r\tilde{v})}{\mathrm{d}r} + \frac{\mathrm{i}}{r}mR\tilde{w} + \mathrm{i}kR\tilde{u},$$

$$\mathrm{i}(\omega - Uk)\tilde{p} = \frac{1}{R}\left(\mathrm{i}(\omega - Uk)\tilde{\rho} + \frac{\mathrm{d}R}{\mathrm{d}r}\tilde{v}\right),$$

which, after eliminating every acoustic variable but $\tilde{p}$, gives the Pridmore-Brown[14] equation in cylindrical form

$$\tilde{p}'' + \left(\frac{1}{r} + \frac{2kU'}{\omega - Uk} - \frac{R'}{R}\right)\tilde{p}' + \left(R(\omega - Uk)^2 - k^2 - \frac{m^2}{r^2}\right)\tilde{p} = 0, \tag{4}$$

where a prime denotes $\mathrm{d}/\mathrm{d}r$. The radial velocity $\tilde{v}$, needed for the boundary condition, is given by $\tilde{v} = \mathrm{i}\tilde{p}'/(R(\omega - Uk))$.

At $r = 0$ we require $\tilde{p}$ to be regular. The boundary condition to be applied at $r = 1$ is $\tilde{p}/\tilde{v} = Z$, giving

$$\mathrm{i}\omega Z\tilde{p}' - \omega^2 R(1)\tilde{p} = 0.$$

The Myers boundary condition (1), under the assumption that the mean-flow is constant apart from an infinitely-thin boundary layer at $r = 1$, predicts that

$$\mathrm{i}\omega Z\tilde{p}' - \big(\omega - Mk\big)^2\tilde{p} = 0.$$

It is this boundary condition that we propose to modify here.

III. Derivation of the modified boundary condition

In this section, we consider a thin boundary layer about $r = 1$ of typical width δ, outside which the mean flow is uniform, so that $U(r) = M$ and $R(r) = 1$ for the majority of the flow. (The boundary condition derived here turns out to be valid even for nonuniform mean flow outside the boundary layer.)

If the boundary layer did not exist, the solution to (4) regular at $r = 0$ would be

$$\tilde{p}(r) = E\mathrm{J}_m(\alpha r) \qquad \text{where} \qquad \alpha^2 = (\omega - Mk)^2 - k^2.$$

Accounting for the boundary layer, the asymptotics (the details of which are given in appendix A) give a composite asymptotic solution of

$$\tilde{p}(r) = E\mathrm{J}_m(\alpha r) - \alpha E\mathrm{J}_m'(\alpha)\int_0^r 1 - \frac{\big(\omega - U(r)k\big)^2 R(r)}{(\omega - Mk)^2}\,\mathrm{d}r + O(\delta^2). \tag{5}$$

This shows very good agreement with the numerics, as shown in section §IV. However, in order to derive the correction to the Myers boundary condition to $O(\delta)$, it turns out to be necessary to derive the pressure within the boundary layer to $O(\delta^2)$. This is again done in appendix A, with the result that, evaluated at the wall lining $r = 1$,

$$\tilde{p}(1) = E\mathrm{J}_m(\alpha) - \alpha E\mathrm{J}_m'(\alpha)\delta I_0 + O(\delta^2),$$
$$\tilde{v}(1) = \frac{\mathrm{i}\big(\omega - U(1)k\big)}{(\omega - Mk)^2}\left[\alpha E\mathrm{J}_m'(\alpha) - \big(k^2 + m^2\big)\delta I_1 E\mathrm{J}_m(\alpha) + O(\delta^2)\right],$$

with

$$\delta I_0 = \int_0^1 1 - \frac{\big(\omega - U(r)k\big)^2 R(r)}{(\omega - Mk)^2}\,\mathrm{d}r, \qquad \delta I_1 = \int_0^1 1 - \frac{(\omega - Mk)^2}{\big(\omega - U(r)k\big)^2 R(r)}\,\mathrm{d}r. \tag{6}$$

Note that, since the integrands of δI_0 and δI_1 are identically zero outside the boundary layer, δI_0 and δI_1 are indeed both of order δ.

We now arrive at our generalization of the Myers boundary condition by stipulating that $Z = \tilde{p}(1)/\tilde{v}(1)$, giving the boundary condition

$$\mathrm{i}\big(\omega - U(1)k\big)Z\big[\alpha\mathrm{J}_m'(\alpha) - \big(k^2 + m^2\big)\delta I_1\mathrm{J}_m(\alpha)\big] - (\omega - Mk)^2\big[\mathrm{J}_m(\alpha) - \alpha\mathrm{J}_m'(\alpha)\delta I_0\big] = 0. \tag{7}$$

This is the main result of the paper. For most purposes, no slip implies that $U(1) = 0$, although setting $U(1) \neq 0$ in order to model surface roughness[23] is also possible.

A. Interpretation of the δI_0 term

At least part of the modified Myers boundary condition (7) may be interpreted physically. Let us first define the boundary layer mass, momentum, and kinetic energy thicknesses of the boundary layer to be

$$\delta_{\text{mass}} = \int_0^1 1 - R(r)\,\mathrm{d}r, \qquad \delta_{\text{mom}} = \int_0^1 1 - \frac{R(r)U(r)}{M}\,\mathrm{d}r, \qquad \delta_{\text{ke}} = \int_0^1 1 - \frac{R(r)U(r)^2}{M^2}\,\mathrm{d}r.$$

Multiplying out δI_0 gives

$$\delta I_0 = \frac{1}{(\omega - Mk)^2}\left(\omega^2\delta_{\text{mass}} - 2\omega k M\delta_{\text{mom}} + k^2M^2\delta_{\text{ke}}\right),$$

which, when substituted into (7) gives the boundary condition

$$\alpha\mathrm{J}_m'(\alpha)\big[\mathrm{i}\big(\omega - U(1)k\big)Z + \omega^2\delta_{\text{mass}} - 2\omega kM\delta_{\text{mom}} + k^2M^2\delta_{\text{ke}}\big]$$
$$- \mathrm{J}_m(\alpha)\big[(\omega - Mk)^2 + \mathrm{i}\big(\omega - U(1)k\big)Z\big(k^2 + m^2\big)\delta I_1\big] = 0.$$

The δI_0 term may therefore be thought of as modifying the impedance from Z to Z_{mod}, where

$$\mathrm{i}\big(\omega - U(1)k\big)Z_{\text{mod}} = \mathrm{i}\big(\omega - U(1)k\big)Z + \omega^2\delta_{\text{mass}} - 2\omega kM\delta_{\text{mom}} + k^2M^2\delta_{\text{ke}}.$$

These terms have a physically significant interpretation. To see this, first consider a theoretical acoustic lining modelled as a flexible impermeable sheet, whose displacement η in the normal direction pointing out of the fluid is governed by

$$d\frac{\partial^2\eta}{\partial t^2} = \tilde{p} - K\eta - D\frac{\partial\eta}{\partial t} + T\frac{\partial^2\eta}{\partial x^2}, \tag{8}$$

where $\tilde{p}$ is a forcing term due to the acoustic pressure, d is a mass density, K is a spring constant, D is a damping constant, and T is an elastic tension in the sheet. Assuming η and $\tilde{p}$ to have $\exp\{\mathrm{i}\omega t - \mathrm{i}kx\}$ dependence gives

$$\mathrm{i}\big(\omega - U(1)k\big)Z = \frac{\tilde{p}}{\eta} = -\omega^2 d + \mathrm{i}\omega D + K + k^2 T.$$

The modified impedance Z_{mod} based on this theoretical boundary model is therefore

$$\mathrm{i}\big(\omega - U(1)k\big)Z_{\mathrm{mod}} = \frac{\tilde{p}}{\eta_{\mathrm{mod}}} = -\omega^2(d - \delta_{\mathrm{mass}}) + \mathrm{i}\omega D + K + k^2(T + M^2\delta_{\mathrm{ke}}) - 2\omega k M\delta_{\mathrm{mom}},$$

which could be interpreted as a second theoretical boundary with displacement η_{mod} and governing equation

$$(d - \delta_{\mathrm{mass}})\left(\frac{\partial}{\partial t} - \frac{M\delta_{\mathrm{mom}}}{d - \delta_{\mathrm{mass}}}\frac{\partial}{\partial x}\right)^2 \eta_{\mathrm{mod}} = \tilde{p} - K\eta_{\mathrm{mod}} - D\frac{\partial \eta_{\mathrm{mod}}}{\partial t} + \left(T + M^2\left(\delta_{\mathrm{ke}} + \frac{\delta_{\mathrm{mom}}^2}{d - \delta_{\mathrm{mass}}}\right)\right)\frac{\partial^2 \eta_{\mathrm{mod}}}{\partial x^2}. \quad (9)$$

The modified boundary layer term δI_0 therefore represents three physical effects:

1. The mass deficit δ_{mass} in the boundary layer causes the effective boundary to be lighter.
2. The momentum deficit $M\delta_{\mathrm{mom}}$ in the boundary layer causes the advection (accounted for in the Myers boundary condition at a velocity M) to be corrected by the effective velocity deficit of the boundary layer, $M\delta_{\mathrm{mom}}/(d - \delta_{\mathrm{mass}})$.
3. The kinetic energy deficit $\frac{1}{2}M^2\delta_{\mathrm{ke}}$ in the boundary layer, together with the momentum deficit, instantiate themselves as a tension along the boundary.

The effect of tension along the boundary is particularly interesting, since it suggests there may be unforced travelling waves permitted along the boundary (which would be damped if $D \neq 0$). Note that if $T = 0$ the uncorrected boundary displacement (8) is local and does not support or prevent travelling waves, whereas the modified boundary (9) is always nonlocal and does support travelling waves. In the limit $\delta \to 0$, this added tension disappears and, at fixed frequency, the wavenumbers for these travelling waves tend to infinity as $\delta \to 0$. This may well correspond to the illposedness and instability of the Myers boundary condition at arbitrarily short wavelengths.[11]

B. Interpretation of the δI_1 term

The δI_1 term is important, in that it is this term that appears to be responsible for the wellposedness of the modified boundary condition. The interpretation of the δI_1 term, though, is more tricky, and no physical explanation of it is given here. It is worth noting that the presence or absence of a critical layer at $r = r^*$, for which $\omega - U(r^*)k = 0$, may have a significant affect on δI_1, especially if $(1 - r^*)/\delta \ll 1$.

Asymptotic values of δI_1 may be calculated, and it can be shown that, for $U(1) = 0$,

$$\delta I_1 \sim \int_0^1 1 - 1/R(r)\,\mathrm{d}r + \frac{2kM}{\omega}\int_0^1 1 - \frac{U(r)}{MR(r)}\,\mathrm{d}r \qquad \text{for} \qquad \omega/k \gg 1$$

$$\delta I_1 \sim \frac{-M^2 k}{\omega R(1) U'(1)} \qquad \text{for} \qquad \omega/k \ll 1$$

Moreover, for a constant density $R(r) \equiv 1$ and a linear boundary layer profile,

$$U(r) = \begin{cases} M(1 - r)/\delta & (1 - r) < \delta \\ M & (1 - r) > \delta \end{cases}, \quad (10)$$

δI_1 may be directly calculated to give $\delta I_1 = \delta M k/\omega$.

C. The modified boundary condition in the time domain

In order to apply the new modified boundary condition in the time domain, (7) needs some coercing into a suitable form. One possible way to do this would be to assume that the density is uniform and the boundary layer profile linear (10) with a given momentum thickness $\delta_{\mathrm{mom}} = \delta/2$. Then (7) may be rearranged to give

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} + M\frac{\partial}{\partial x}\right)\tilde{v} + 2M\delta_{\mathrm{mom}}\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \theta^2}\right)\tilde{p} = \left(\frac{\partial}{\partial t} + M\frac{\partial}{\partial x}\right)^2\tilde{v}_b,$$

where $\tilde{v}_b = \tilde{p}/Z_{\mathrm{mod}}$ is the velocity of the boundary using a modified boundary model incorporating the δI_0 terms from §IIIA above. For example, if the desired boundary model in the time domain were (8), then $\tilde{v}_b$ would be given by $\tilde{v}_b = \partial\eta_{\mathrm{mod}}/\partial t$ with η_{mod} satisfying (9).

IV. Comparison with numerics

In order to validate the above asymptotics, the asymptotic solution will be compared with a numerical solution to the full Pridmore-Brown equation (4). The numerical solutions were generated using a 12th order symmetric finite-difference discretization with unevenly spaced collocation points, so that the collocation points could be clustered within the boundary layer. Typically 8000 points were used, with at least 400 points within the boundary layer irrespective of the width of the boundary layer. The $r = 1$ boundary condition used was $\tilde{p}(1) = 1$. A Newton–Raphson iteration was then used to find roots of the dispersion relation $\tilde{p}/\tilde{v} = Z$.

The boundary layer profile used for most examples given here was chosen to be the tanh profile of Rienstra & Vilenski,[24]

$$U(r)/M = \tanh\left(\frac{1-r}{\delta}\right) + \big(1 - \tanh(1/\delta)\big)\left(\frac{1+\tanh(1/\delta)}{\delta}r + (1+r)\right)(1-r), \qquad (11)$$

with constant density $R(r) \equiv 1$.

For this profile with $\delta = 2\times 10^{-4}$, two comparisons between the numerical solution to (4), the first-order-accurate composite solution (5) and the uniform solution are given in figure 1. Figure 2 gives a comparison between duct modes satisfying the impedance boundary condition using the numerics, the Myers boundary condition, and the modified boundary condition (7). Despite this being for such a thin boundary ($\delta = 2\times 10^{-4}$), the highly cutoff modes and the surface mode in the upper-right quadrant are noticeably affected by the finite thickness of the boundary layer.

V. Stability and well-posedness

One major need for a generalization of the Myers boundary condition is that the Myers boundary condition is illposed,[11] manifesting itself as numerical instability,[9,10] which we hope to alleviate using this modified boundary condition (7). Whatever the boundary condition at $r = 1$, it yields either allowable values of ω if k is specified, or allowable values of k if ω is specified; these are here referred to as modes. For the problem to be wellposed, there should be a lower bound to $\mathrm{Im}\big(\omega(k)\big)$ for real k. For further explanation, see Ref. 11.

Figure 3 shows $\omega(k)$ as k is varies with $\mathrm{Im}(k) = 0$. The parameters used are the same as Rienstra & Vilenski.[24] Note that for both the numerical solution and the modified boundary condition, $\mathrm{Im}\big(\omega(k)\big)$ is bounded below (say by -5), while the Myers solution is unbounded. Hence, the numerical and modified boundary condition solutions are wellposed. Since they are wellposed, we may apply the Briggs–Bers stability analysis[27,28] to these cases. As there are real values of k with $\mathrm{Im}\big(\omega(k)\big) < 0$, there is a convective instability present. However, we must also look for absolute instabilities, where the system chooses its own preferred frequency and disturbances at that frequency grow exponentially in time. Absolute instability occurs for values of ω with $\mathrm{Im}(\omega) < 0$ for which two modes collide in the k-plane giving a double root. Only if this collision is between a mode originating for large $\mathrm{Im}(\omega)$ from the lower-half k-plane and one originating from the upper-half k-plane does this then signify an absolute instability (see Ref. 11 for details). For this example, such a pinch does occur (as shown in figure 3), but for $\mathrm{Im}(\omega) > 0$, so that in this case no absolute instabilities are present. However, as the boundary layer thickness δ is reduced, this double root moves into the lower-half ω-plane and produces an absolute instability. The critical value of δ for which $\mathrm{Im}(\omega) = 0$ for

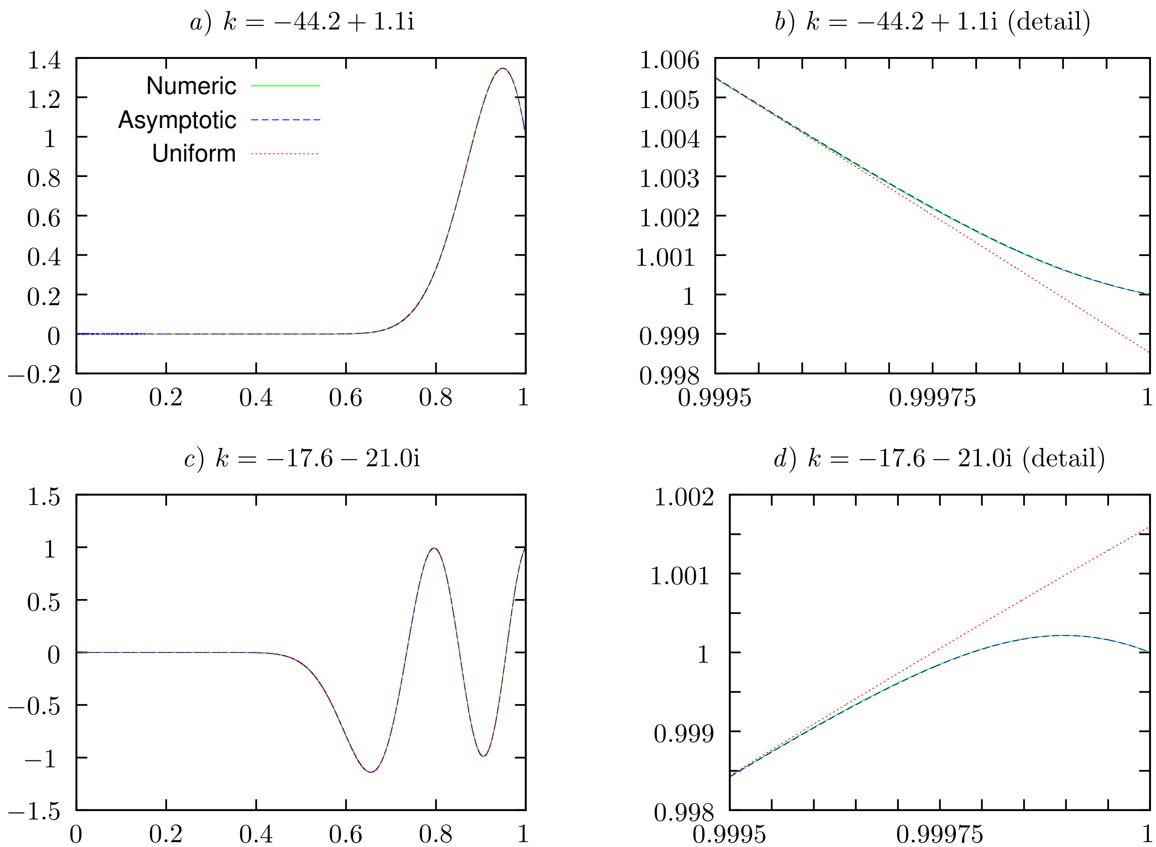

Figure 1. Plots of $\mathrm{Re}(\tilde{p}(r))$ against r, comparing the numerical solution to the Pridmore-Brown equation (4), the first-order asymptotic solution (5), and the uniform-flow solution. $\omega = 31$, $m = 24$, and $U(r)$ is given by (11) with $M = 0.5$ and $\delta = 2 \times 10^{-4}$. The values of k used correspond to a cuton upstream-propagating mode (*a* and *b*) and a partially cutoff downstream-propagating mode (*c* and *d*) if the lining has impedance $Z = 2 + 0.6\mathrm{i}$.

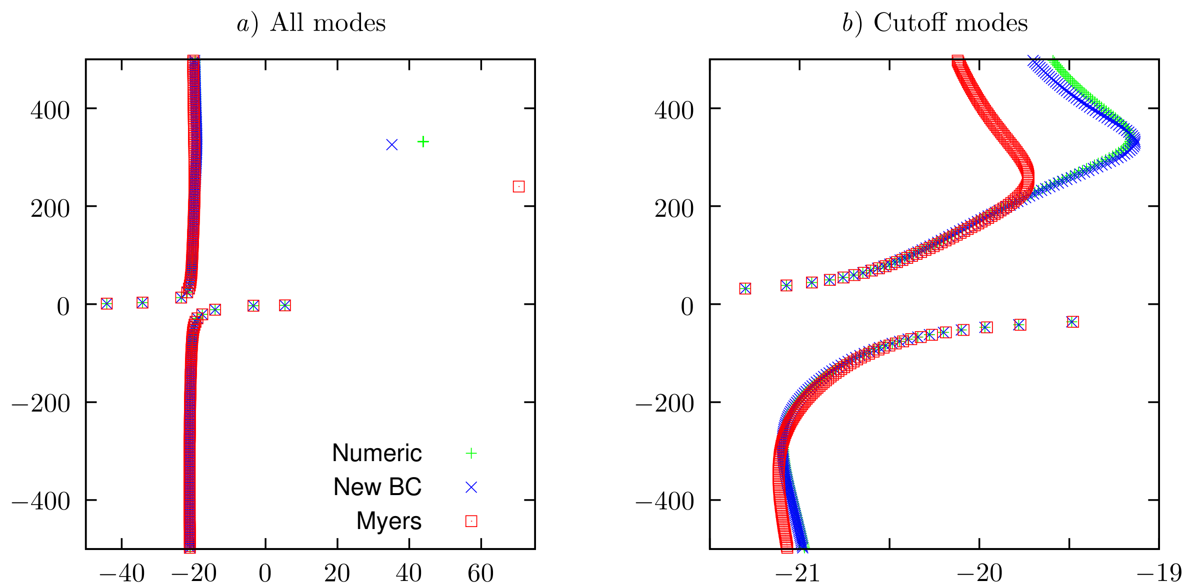

Figure 2. Comparison of axial wavenumbers k, plotted in the k-plane, for fully numerical solutions and solutions of the modified (7) and original Myers boundary conditions. $\omega = 31$, $m = 24$, $Z = 2 + 0.6\mathrm{i}$, and $U(r)$ is given by (11) with $M = 0.5$ and $\delta = 2 \times 10^{-4}$.

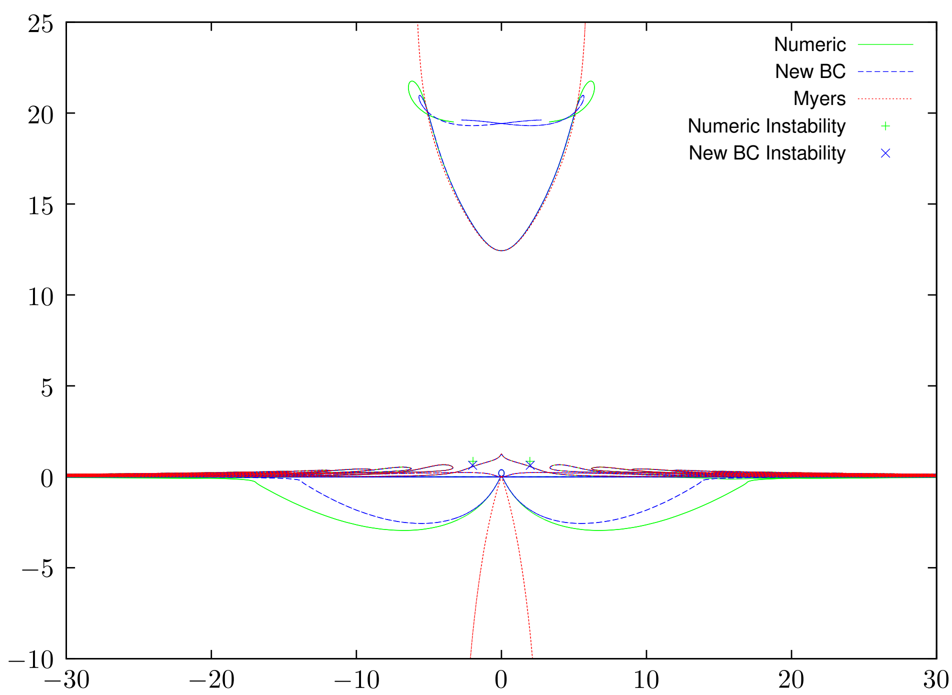

Figure 3. Plotted in the ω-plane are trajectories of $\omega(k)$ as k is varied with k real, for the fully numerically-calculated roots, solutions to the modified Myers boundary condition (7), and the original Myers boundary condition. $\times$ and $+$ denote values of ω for which two k-roots coincide (a double root). $m = 0$, $Z = 3 + 0.15\mathrm{i}\omega - 1.15\mathrm{i}/\omega$, with $U(r)$ given by (11) with $M = 0.5$ and $\delta = 2 \times 10^{-3}$.

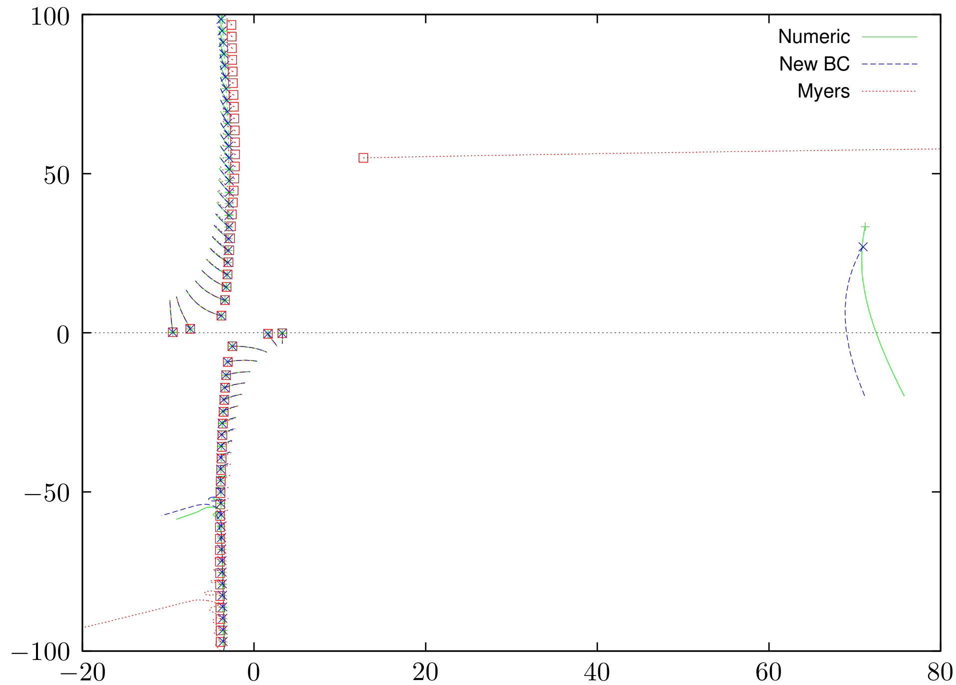

Figure 4. Trajectories of modes in the k-plane as ω is varied from 5 to $5 - 5\mathrm{i}$, with other parameters as for figure 3. The points are for $\omega = 5$. Since the modes on the far right cross the real k-axis as $\mathrm{Im}(\omega)$ is varied, they correspond to downstream-propagating instabilities.

this double root is found numerically to be 8.6×10^{-4} for the numerics, occurring at $\omega = 4.3$, $k = 19.8 + 95.3\mathrm{i}$, and $\delta = 9.7 \times 10^{-4}$ using the new modified boundary condition, occurring at $\omega = 3.7$ and $k = 22.8 + 82.1\mathrm{i}$.

For values of δ sufficiently large that there is no absolute instability, the system here is convectively unstable, with a downstream-propagating mode that has $\mathrm{Im}(k) > 0$. This can be seen by applying the Briggs–Bers[27,28] criterion, which says that no convective instabilities are present provided $\mathrm{Im}(\omega)$ is sufficiently negative; i.e. below all $\omega(k)$ for real k. We may therefore ascertain the stability of a mode by tracking modes as ω is varies from sufficiently imaginary to real. This is done for $\mathrm{Re}(\omega) = 5$ and $\mathrm{Im}(\omega) \in [-5, 0]$ in figure 4. All modes but one are seen to be stable. The unstable mode on the far right of figure 4 is the mode previously suggested by Rienstra[29] as a possible hydrodynamic instability mode. Note that the original Myers boundary condition fails to correctly predict the behaviour of this mode.

There are several different models for the impedance Z. For single-frequency simulations a common assumption is that Z is constant. However, even in these situations the dependence of Z on ω is important for stability. The same situation as figure 3, but with Z fixed at a constant value of $3 + 1.39\mathrm{i}$, is shown in figure 5. This demonstrates that the new boundary condition also regularizes this problem even if the impedance is taken to be constant, although in this case an absolute instability is now present, as seen in the figure.

Recently, Rienstra & Darau[25] performed a similar analysis for a flat liner subjected to incompressible flow, using a linear velocity profile (10) and a Helmholtz resonator model for the impedance, $Z = D + \mathrm{i}d\omega - \mathrm{i}\cot(\omega L)$. Nondimensionalizing their parameters under the assumption of a 0.2 meter diameter cylindrical duct (chosen to exaggerate any effect of the cylindrical geometry), the corresponding stability graph of $\omega(k)$ for real k is given in figure 6. In this case, Rienstra & Darau predict absolute instability for $\delta < 3.6 \times 10^{-4}$. Here, the modified boundary condition and the numerics both predict a critical value for δ of 3.7×10^{-4}, which is in good agreement with Rienstra & Darau considering they were considering a flat surface and incompressible flow, and here the duct is cylindrical and the flow compressible.

So far, all of these results have assumed an axisymmetric mode with azimuthal order $m = 0$. However, the new boundary condition multiplies the I_1 term by $k^2 + m^2$, and so large values of m may lead to different behaviour. However, similar simulations to those shown here demonstrate that even in this case the problem is well posed.

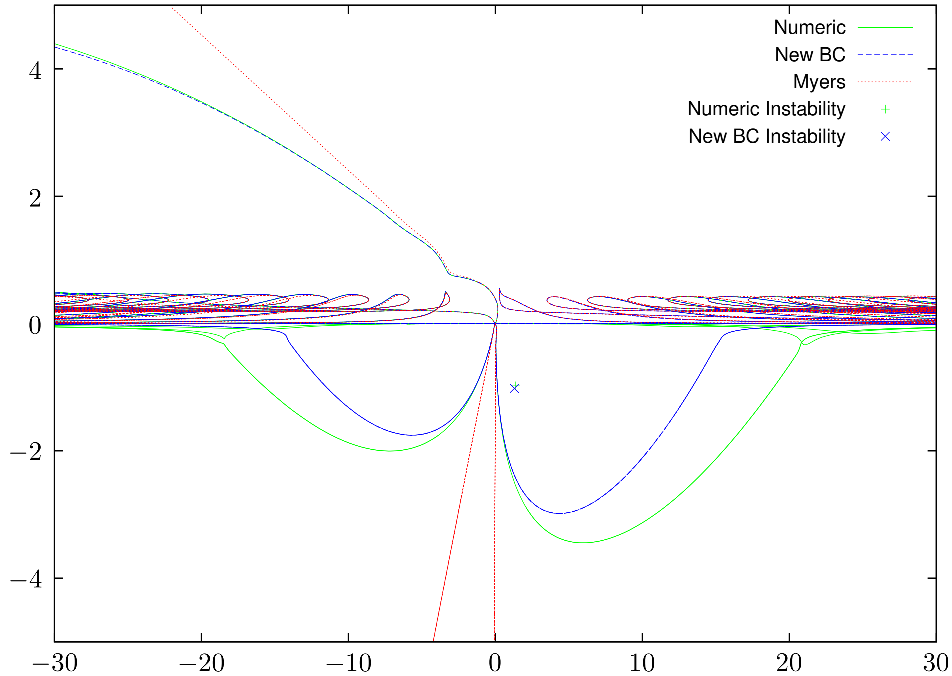

Figure 5. Same plot as figure 3 but with $Z = 3 + 1.39\mathrm{i}$.

$U(r)$ **profile**	M	m	Z	δ **numerical**	δ **asymptotic**
tanh (11)	0.5	0	$3 + 0.15\mathrm{i}\omega - 1.15\mathrm{i}/\omega$	8.6×10^{-4}	9.7×10^{-4}
tanh (11)	0.5	0	$3 + 1.39\mathrm{i}$	3.0×10^{-2}	8.2×10^{-2}
tanh (11)	0.5	24	$2 + 0.8\mathrm{i}\omega - \cot(0.06\omega)$	1.7×10^{-3}	1.7×10^{-3}
linear (10)	0.176	0	$2 + 3.9\mathrm{i}\omega - \cot(0.3\omega)$	3.7×10^{-4}	3.7×10^{-4}

Table 1. Table of critical boundary layer thicknesses (δ) for several different situations. For smaller values of δ, the system is absolutely unstable. For larger values of δ, , the system is at most is convectively unstable.

The critical values for δ found to date are given in table 1. They show a good agreement between the modified boundary condition and the numerics, with very good agreement for thin boundary layers (as is to be expected). In all these cases, the system was still convectively unstable for a boundary layer of width $\delta = 0.1$, though as δ is made progressively larger the frequencies for which instability can occur become smaller. For very small values of δ the solution approximates the Myers solution until k becomes sufficiently large that the $O(\delta)$ term in the modified boundary condition become important and the problem remains regularized. Setting $\delta = 0$ yields the Myers boundary condition and the problem once again becomes illposed.

VI. Conclusion

The main conclusion of this paper has to be that the new modified Myers boundary condition, derived asymptotically for thin boundary layers and given in (7) (with δI_0 and δI_1 given in equation (6)), solves the illposedness problem associated with the standard Myers boundary condition. It can therefore be expected to alleviate the numerical instabilities associated with simulations using the Myers boundary condition in the time domain (applied, for example, as suggested in §IIIC), as well as allowing a rigorous stability analysis (which is helpful for numerical simulations in the frequency domain).

The stability analyses conducted on the examples given here indicate that these examples are absolutely unstable for sufficiently thin boundary layers, and are convectively unstable otherwise. The only unstable mode found is the one predicted as being a hydrodynamic instability by Rienstra.[29]

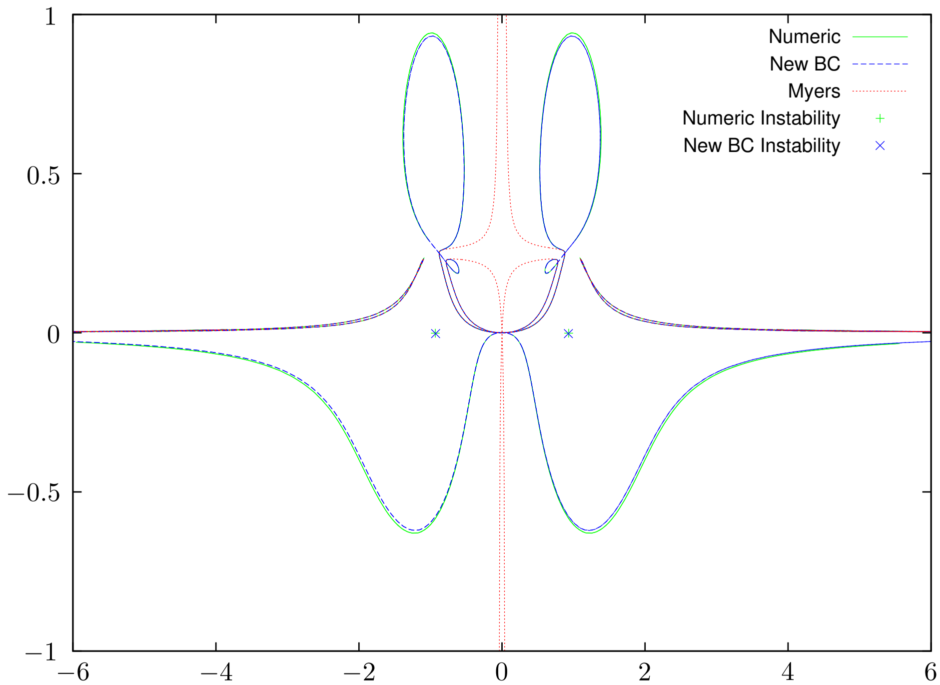

Figure 6. The same plot as figure 3, but with $Z = 2 + 3.9\mathrm{i}\omega - \mathrm{i}\cot(0.3\omega)$, with $U(r)$ given by (10) with $M = 0.176$ and $\delta = 3.6 \times 10^{-4}$.

The new terms in the boundary layer are encapsulated with the δI_0 and δI_1 terms, which are integrals over the boundary layer and are valid for any boundary layer profile. The δI_0 term may be calculated knowing only the mass, momentum, and kinetic energy thicknesses of the boundary layer. These three terms can be physically interpreted as a change in boundary mass, change in convected boundary speed, and a tension along the boundary. The δI_1 term is more difficult to classify, but expressions are given in §IIIB in the high- and low-frequency limits, and an exact expression is given for δI_1 for a linear boundary layer profile. It is the δI_1 term that is responsible for regularizing the Myers boundary condition, though it should be stressed that both terms are important in order that the new boundary condition be asymptotically accurate.

The asymptotics have been derived under the assumption that $O(\delta^2)$ quantities may be neglected in the boundary condition; this is only valid provided k, m and ω remain $O(1)$. For example, once k becomes sufficiently large that a wavelength becomes comparable to the boundary layer thickness, the asymptotics described in this paper can be expected to be inaccurate.

As well as regularizing the Myers boundary condition, the new boundary condition may be used to gain insight into the effect of a boundary layer on the modal wavenumbers. This effect is small (i.e. $O(\delta)$), and is typically of little interest for most of the acoustic modes. However, for surface modes, or when surface modes and acoustic modes interact, the effect on modal wavenumbers may be important.

For all examples considered here, $R(r) \equiv 1$, and so $\delta_{\mathrm{mass}} = 0$. However, cases when $\delta_{\mathrm{mass}} \neq 0$ are both realizable in practice and potentially interesting mathematically, especially for $\omega \gg 1$, since δI_0 includes an $\omega^2 \delta_{\mathrm{mass}}$ term. Such situations will be investigated further in future.

A. Derivation of the boundary layer asymptotics

In this appendix, we derive the acoustic field within a thin boundary layer of width δ correct to $O(\delta^2)$. For those not interested in the derivation, a summary of the major results derived here is given in the final paragraph of this appendix.

Our governing equation is the cylindrical form of the Pridmore-Brown equation (4) for the acoustic pressure in a sheared flow,

$$\tilde{p}'' + \left(\frac{1}{r} + \frac{2kU'}{\omega - Uk} - \frac{R'}{R}\right)\tilde{p}' + \left(R(\omega - Uk)^2 - k^2 - \frac{m^2}{r^2}\right)\tilde{p} = 0,$$

with the radial velocity given by $\tilde{v} = \mathrm{i}\tilde{p}'/(R(\omega - Uk))$. Here, $U(r)$ is the mean-flow velocity and $R(r)$ is the mean-flow density, nondimensionalized so that $U(0) = M$, the Mach number, and $R(0) = 1$. We consider a thin boundary layer about $r = 1$ of typical width δ, outside of which the mean flow is uniform, so that $U(r) = M$ and $R(r) = 1$ for r outside the boundary layer.

If the boundary layer did not exist, the solution to (4) would be $\tilde{p}(r) = \tilde{p}_o(r)$,

$$\tilde{p}_o(r) = E\mathrm{J}_m(\alpha r) \qquad\qquad \alpha^2 = (\omega - Mk)^2 - k^2.$$

Expanding this for $r = 1 - \delta y$ gives

$$\tilde{p}_o(1 - \delta y) = E\mathrm{J}_m(\alpha) - \delta y \alpha E\mathrm{J}_m'(\alpha) - \tfrac{1}{2}\delta^2 y^2 E\Big[\alpha \mathrm{J}_m'(\alpha) + \big(\alpha^2 - m^2\big)\mathrm{J}_m(\alpha)\Big] + O(\delta^3). \tag{12}$$

This is our outer expansion, that the inner expansion within the boundary layer must match with.

To consider the boundary layer, it is helpful to first rearrange the governing equation to give

$$\left(\frac{r\tilde{p}'}{(\omega - Uk)^2 R}\right)' + \left(r - \frac{k^2 r + m^2/r}{(\omega - Uk)^2 R}\right)\tilde{p} = 0. \tag{13}$$

Substituting $r = 1 - \delta y$ into (13) and using a subscript y to denote $\mathrm{d}/\mathrm{d}y$ gives

$$\left(\frac{\tilde{p}_y}{(\omega - Uk)^2 R}\right)_y = \delta\left(\frac{y\tilde{p}_y}{(\omega - Uk)^2 R}\right)_y - \delta^2\left(1 - \frac{k^2 + m^2}{(\omega - Uk)^2 R}\right)\tilde{p} + O(\delta^3).$$

In a slight but obvious abuse of notation, $U(y)$ is used to represent $U(r)$ with $r = 1 - \delta y$, and similarly for R. We pose the solution $\tilde{p} = \tilde{p}_0 + \delta\tilde{p}_1 + \delta^2\tilde{p}_2 + O(\delta^3)$.

To leading order,

$$\tilde{p}_0 = A_0 + B_0 y - B_0 \int_0^y 1 - \frac{\big(\omega - U(y')k\big)^2 R(y')}{(\omega - Mk)^2}\,\mathrm{d}y'.$$

Matching with the outer solution (12) at leading order gives $A_0 = E\mathrm{J}_m(\alpha)$ and $B_0 = 0$. Since $B_0 = 0$, at first order we similarly find

$$\tilde{p}_1 = A_1 + B_1 y - B_1 \int_0^y 1 - \frac{\big(\omega - U(y')k\big)^2 R(y')}{(\omega - Mk)^2}\,\mathrm{d}y',$$

while matching with the outer solution (12) gives $B_1 = -\alpha E\mathrm{J}_m'(\alpha)$ and $A_1 = B_1 I_0$, with

$$I_0 = \int_0^\infty 1 - \frac{\big(\omega - U(y)k\big)^2 R(y)}{(\omega - Mk)^2}\,\mathrm{d}y.$$

At second order we find

$$\begin{aligned}\tilde{p}_2 = A_2 + B_2 \int_0^y \frac{\big(\omega - U(y')k\big)^2 R(y')}{(\omega - Mk)^2}\,\mathrm{d}y' + B_1 \int_0^y y'\frac{\big(\omega - U(y')k\big)^2 R(y')}{(\omega - Mk)^2}\,\mathrm{d}y' \\ - A_0 \int_0^y \big(\omega - U(y')k\big)^2 R(y') \int_0^{y'} 1 - \frac{k^2 + m^2}{\big(\omega - U(y'')k\big)^2 R(y'')}\,\mathrm{d}y''\,\mathrm{d}y'.\end{aligned}$$

Rewriting this in terms of bounded integrals gives (to help with matching with the outer solution) gives

$$\tilde{p}_2 = A_2 + B_2 y - B_2\int_0^y 1 - \frac{\big(\omega - U(y')k\big)^2 R(y')}{(\omega - Mk)^2}\,\mathrm{d}y' + B_1\int_0^y y'\left(\frac{\big(\omega - U(y')k\big)^2 R(y')}{(\omega - Mk)^2} - 1\right)\mathrm{d}y' + \tfrac{1}{2}B_1 y^2$$
$$- A_0\int_0^y\left(\frac{\big(\omega - U(y')k\big)^2 R(y')}{(\omega - Mk)^2} - 1\right)\int_0^{y'}(\omega - Mk)^2 - \big(k^2+m^2\big)\frac{(\omega - Mk)^2}{\big(\omega - U(y'')k\big)^2 R(y'')}\,\mathrm{d}y''\,\mathrm{d}y'$$
$$- A_0\big(k^2+m^2\big)\int_0^y\left(\int_0^{y'} 1 - \frac{(\omega - Mk)^2}{\big(\omega - U(y'')k\big)^2 R(y'')}\,\mathrm{d}y'' - I_1\right)\mathrm{d}y'$$
$$- A_0\tfrac{1}{2}\Big((\omega - Mk)^2 - k^2 - m^2\Big)y^2 - A_0 I_1(k^2+m^2)y,$$

where all the integrals are bounded provided

$$I_1 = \int_0^\infty 1 - \frac{(\omega - Mk)^2}{\big(\omega - U(y)k\big)^2 R(y)}\,\mathrm{d}y$$

We now match this to the outer solution (12). The $O(y^2)$ terms match identically using our definitions of A_0 and B_1 above. Matching the $O(y)$ terms gives $B_2 = A_0 I_1(k^2+m^2)$, while matching the $O(1)$ terms gives

$$A_2 = B_2 I_0 + B_1\int_0^\infty y\left(1 - \frac{\big(\omega - U(y)k\big)^2 R(y)}{(\omega - Mk)^2}\right)\mathrm{d}y$$
$$+ A_0\int_0^\infty\left(\frac{\big(\omega - U(y)k\big)^2 R(y)}{(\omega - Mk)^2} - 1\right)\int_0^y(\omega - Mk)^2 - \big(k^2+m^2\big)\frac{(\omega - Mk)^2}{\big(\omega - U(y')^2\big)R(y')}\,\mathrm{d}y'\mathrm{d}y$$
$$+ A_0(k^2+m^2)\int_0^\infty\left(\int_0^y 1 - \frac{(\omega - Mk)^2}{\big(\omega - U(y')^2\big)R(y')}\,\mathrm{d}y' - I_1\right)\mathrm{d}y.$$

Fortunately, for the modified Myers boundary condition we do not need to evaluate A_2!

A. Summary

Now that we have the inner solution through the boundary layer, we may calculate $\tilde{p}$ and $\tilde{v}$ at the wall $r = 1$, $y = 0$. Using $\tilde{v} = \mathrm{i}\tilde{p}'/(R(\omega - Uk))$, we find that

$$\tilde{p}(1) = E\mathrm{J}_m(\alpha) - \alpha E\mathrm{J}_m'(\alpha)\delta I_0 + O(\delta^2),$$
$$\tilde{v}(1) = \frac{\mathrm{i}\big(\omega - U(1)k\big)}{(\omega - Mk)^2}\left[\alpha E\mathrm{J}_m'(\alpha) - \big(k^2+m^2\big)\delta I_1 E\mathrm{J}_m(\alpha) + O(\delta^2)\right],$$

where we may write δI_0 and δI_1 in terms of integrals over r as

$$\delta I_0 = \int_0^1 1 - \frac{\big(\omega - U(r)k\big)^2 R(r)}{(\omega - Mk)^2}\,\mathrm{d}r, \qquad \delta I_1 = \int_0^1 1 - \frac{(\omega - Mk)^2}{\big(\omega - U(r)k\big)^2 R(r)}\,\mathrm{d}r.$$

To first order, we may form the composite solution for the pressure,

$$\tilde{p}_c(r) = \mathrm{J}_m(\alpha r) - \alpha\mathrm{J}_m'(\alpha)\int_0^r 1 - \frac{\big(\omega - U(r)k\big)^2 R(r)}{(\omega - Mk)^2}\,\mathrm{d}r. + O(\delta^2).$$

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