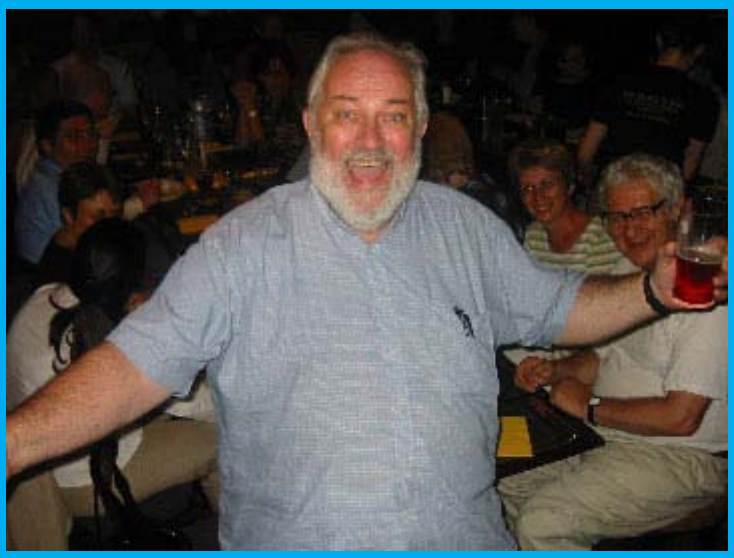
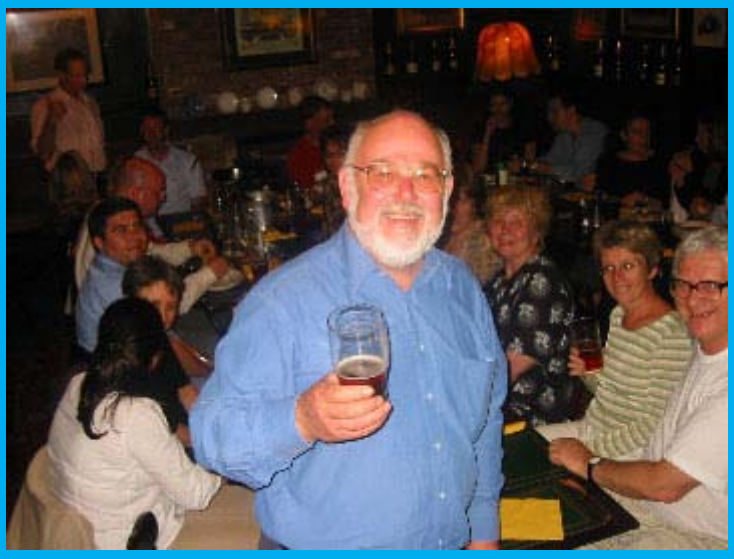


RETIREMENT
AS PROCESS AND CONCEPT

A FESTSCHRIFT FOR EDDIE GRAY AND DAVID TALL

PRESENTED AT CHARLES UNIVERSITY, PRAGUE

15-16 JULY, 2006



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Retirement as Process and Concept: A Festschrift for Eddie Gray and David Tall

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DAVID TALL AND EDDIE GRAY – HEART AND SOUL OF MATHEMATICS EDUCATION

> Go on, take the course – I'm told it's great fun and it's run by this complete bearded loony.

My first introduction to David Tall was a problem solving course I took as an undergraduate at Warwick University in the late '80s. I had become rather disillusioned with the standard definition-theorem-proof courses which (once I had had a drunken epiphany which led me to understand this 'proof' thing that the lecturers kept referring to) I was passing with consummate ease and very little understanding. A friend recommended this course to me and, to a great extent it was the bearded loony (along with *Thinking Mathematically*, the book of another of the bearded loonies of mathematics education who I was to meet later) who reignited my enthusiasm for mathematics.

Eddie, on the other hand, appeared more slowly as an influence and an immense force of kindness and support when I returned to Warwick some 10 years later as a lecturer in the Mathematics Education Research Centre. Given my own research interests, I had initially little connection with his work but became increasingly intrigued with first his papers with Demetra Pitta and later his work with David on the notion of procept. The former drew me in to some the psychological literature they had based their work on, encouraging me to read beyond the mathematics education literature into the often more developed and robust world of cognitive psychology. The latter influenced my thinking both as a theory and as a model of theory development.

The intellectual work of both Eddie and David, which has had such a profound influence on my thinking and on that of so many others, pales beside the roles they have played as mentors. More than any others in mathematics education, they have grown a family around them – their students are their academic sons and daughters and they dote on their academic grandchildren with almost as much glowing, grandfatherly pride as they dote on the astonishing number of their biological grandchildren.

In this book, you will find but a small sample of the influence they have had, and of the love and respect in which they are held. The quality of the papers and the range of topics they cover testify to their prolific generation of ideas which, in varying from counting to topology, can be said to be cover mathematics from cradle to grave.

On a personal note, there is little doubt that I would never have achieved the position that I am in without them. David's support for my thinking and positively critical challenging of my ideas are at least matched by Eddie's support for my intellectual and personal development and unstinting friendship. Warwick, which had been the home to such intellectual giants, should have been a hard place to leave, However, with the knowledge that Eddie and David were retiring and with the irreparable damage done to mathematics education at Warwick by a management ill-informed of

the achievements David and Eddie spearheaded and the regard in which they were held internationally, made my recent move easier. What will be difficult is being so far from them.

The lazy editor's quotation machine, Samuel Johnson, once said

> Don't think of retiring from the world until the world will be sorry that you retire. I hate a fellow whom pride or cowardice or laziness drives into a corner, and who does nothing when he is there but sit and growl. Let him come out as I do, and bark.

While there may be occasional hints of (justifiable) pride, no-one would dare accuse David or Eddie of cowardice or laziness, and – intellectually – both have as loud a bark as one would wish. While we wish both of you and your families well, the world will be sorry that you retire.

Adrian Simpson

ACKNOWLEDGMENTS

I would like to acknowledge the hard work of all the authors of the papers contained in this Festschrift. The conference which accompanied it and the events organised to celebrate the academic work of David and Eddie could not have gone ahead without the immense efforts of John Monaghan and, particularly, Nada Stehlikova – many thanks to both.

THE CONCEPT OF FUNCTION: WHAT HAVE STUDENTS MET BEFORE?[1]

Hatice Akkoç
Marmara University, Türkiye

Prior knowledge has important effects on the long-term cognitive development. Tall (2004) termed a current structure resulting from earlier experiences a met-before. A met-before might have positive or negative effects. In a well-designed curriculum it provides a positive foundation for successful development otherwise it might act as an epistemological obstacle. This paper investigates the effects of met-befores for the development of function concept. The data of the study was obtained from interviews with ten pre-service mathematics teachers. The analysis of data indicated that met-befores was not helpful to overcome the difficulties and had negative effects in a sophisticated context where functions are defined on a different domain other than real numbers.

INTRODUCTION

Tall (2002) claims that a smooth long-term cognitive development in which each idea builds easily on previous ones is not possible even with a well-designed curriculum. There are always discontinuities in the learning process. New contexts will always demand new ways of looking at things and often require significant cognitive reconstructions. In that sense, prior knowledge and earlier experiences have great importance for the long-term cognitive development. In a new context, earlier experiences might have negative as well as positive effects. Old experiences can cause serious conflicts. Tall (2004) calls such old experiences 'met-befores'. An example can be given in the transition from arithmetic to algebra. In arithmetic every sum has an answer, for instance, 2+3 is 5. This is a typical met-before. But in algebra, an expression such as $2+3x$ has no 'answer' unless x is known.

Nogueira & Tall (2006) gives another example of a met-before in the context of equations. Students' earlier experiences of equations are 'based on arithmetic met-befores, where the equal sign is conceived as "something to do" to get the solution' (p. 7). In their study, they found that the students did not use embodied meanings such as balancing both sides of an equation.

MET-BEFORE FOR THE FUNCTION CONCEPT

With the "new mathematics" movement, function concept has been seen as the underlying concept for the whole curriculum. This might only be possible for an expert who is able to see the role of function concept throughout the whole of mathematics (Tall, to appear). On the other hand, the story is different for a student.

[1] Inspired from the title of one of David Tall's joint papers called 'The concept of equation: What have students met before?' (Nogueira & Tall, 2006).

A student needs to construct new ideas on previously constructed met-befores. For instance, students first meet functions in the form of a linear assignment e.g. y=2x+1 where the value of y is found from x by doubling x and adding 1. By having such experiences, students develop met-befores that identify functions as formulas in which values of x are entered to calculate the value of y (Tall, to appear).

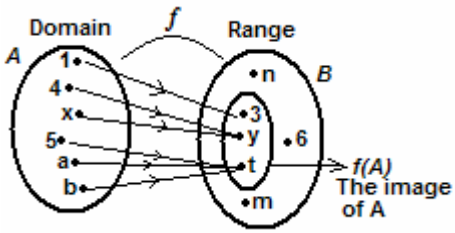
BACKGROUND OF THE STUDY

In the Turkish curriculum, the function concept is introduced in grade 1 (15 year-old students) in high school. After the introduction of relations, equivalence and ordered relations, the function concept is given by the following colloquial definition:

Let *A* and *B* be two non-empty sets. A relation from *f* from *A* to *B* is called a function if it assigns every element in *A* to a unique element in *B*. or A function *f* defined from *A* to *B* assigns: 1. All elements in *A* to elements in *B*, 2. Every element in *A* to a unique element in *B*.

Table 1: Colloquial definition of function

This colloquial definition is followed by a visual representation as seen in figure 1, together with the introduction of notation as follows:



If $x \in A$ and $y \in B$ and if a function *f* from *A* to B assigns x to y then it is denoted by $f : A \to B$, $x \to y = f(x)$.

'$y = f(x)$' is read as '*y* is equal to *f* of *x*'.

Figure 1: Set correspondence diagram and the definition of function

In the newly announced curriculum, the function concept is introduced with a function box after giving the concept of relation (Orta Öğretim Matematik Sınıflar Dersi Öğretim Programı, 2005). However, participants of this study have not met function box in high school.

At the undergraduate level, defining properties of the function concept are given symbolically as shown in Table 2.

The colloquial definition is also given besides the formal symbolic definition at this level. Therefore, the colloquial definition and the set correspondence diagrams are two met-befores explicitly given in the Turkish context. Studies such as Akkoç (2002; 2005) which were conducted in that context indicated that students preferred and successfully used the colloquial definition and set correspondence diagrams when they needed to focus on the defining properties of functions. These met-befores

Let X and Y be two non-empty sets. A relation defined from X to Y is called a function if it satisfies the following conditions:

$\forall x \in X$, $\exists y \in Y$ such that $f(x) = y$

If $x_1 = x_2$ then $f(x_1) = f(x_2)$

Table 2: Formal symbolic definition of function

are purposefully introduced to students by the curriculum designers and by the teacher who implements this curriculum in the classroom.

METHODOLOGY

This is a qualitative study which aims to answer the following research questions:

- RQ1: What kinds of met-befores do participants use in a more sophisticated context for the function concept?
- RQ2: What kinds of effects do these met-befores have in the new context?

The participants of the study are ten pre-service mathematics teachers who completed all the required mathematics courses from a mathematics department in a university in Istanbul, Turkey. Among forty-five pre-service teachers, ten of them were selected considering a spectrum of performance on a question in a written exam. Five of them were successful and five of them were less successful in answering the question. The question required them to identify and work with a function in a more sophisticated context (Breidenbach *et al*., 1992) as shown in table 3.

Let H be the set of all functions whose domain and range are the set of all real numbers. Let M be the operation that acts on a function, say f, in H and transforms it to the function g where $g(x)=f(-2x)$.

a) Is M a function? Explain.

b) If f is the function in H defined by $f(x) = x^3$, what is $M(f)(3)$?

c) Describe the inverse of M.

Table 3: Interview question

This question was chosen to reveal the effects of previously constructed met-befores for the function concept. Therefore, a function was chosen in a more sophisticated context where the elements of the domain and range are not numbers and where the function is not given by a formula. Participants were interviewed on this question using semi-structured interviews. Interviews were recorded and transcribed.

DATA ANALYSIS

The data from the interviews were analysed using the principles of open coding (Strauss & Corbin, 1990). The data obtained from the interviews was open coded using line-by-line and paragraph-by-paragraph analysis. The line-by-line analysis

was used to examine the effects of met-befores inherent in the language participants used during the interview. For the paragraph-by-paragraph analysis, responses to each part of the question were treated as a paragraph. Labels emerged from this analysis were used to create categories as presented below:

	Is M a function? Why?	$M(f)(3) = ?$	Describe the inverse of M
A1	CD-C PE (SD) SD used to explain the CD	OP-C $M(f)(3) = g(3) = f(-2.3)$ $= f(-6) = -216$	Inv-C, $M(f(x)) = g(x) = f(-2x)$ $M^{-1}(g(x)) = f(x)$, $M^{-1}(f(-2x)) = f(x)$
A2	CD-P, PE (SD)	OP-C, $g(x) = -8x^3$, $g(3) = -216$	Inv-W, $g(x) = f(-2x)$, $M^{-1}(g(x)) = f(-2x)$
A3	CD-W	OP-C, $g(x) = -8x^3$ $M(f)(3) = -8.3^3 = -216$	Inv-C, $M : f \to g$, $M^{-1} : g \to f$ $f(x) \to f(-2x)$ $f(-2x) \to f(x)$
A4	CD-W, PE (*f*, *g*, *h* functions in the set containers)	OP-W, $M(f)(3) = M(27)$ $M(x^3) = -8x^3$ $M(x) = -8x$ $M(27) = -8.27$	Inv-C, $M(f) = g$, $M^{-1}(g) = f$ $M^{-1}(f(-2x)) = f(x)$
A5	CD-P and PE (*f*, *g* functions in the set containers	OP-C, $Mf(3) = g(3)$ $f(-6) = -216$	Inv-C, $M(f(x)) = g(x)$, $M^{-1}(g(x)) = f(x)$ $M^{-1}(f(-2x)) = f(x)$
B1	CD-W and PE (SD)	OP-C: $M(f)(3) = g(-\frac{3}{2})$	Inv-W: $M^{-1}(g(x)) = f(-2x) = -8x^3$ $M(g(f(x_n))) = f(x_i)$
B2	CD-C	OP-W, $M(f)(3) = M(27) =$ $gof(27) = f(-2.27^3)$	Inv-W, $M = foh$, $M^{-1} = g^{-1}of^{-1}$
B3	CD-St PE (Function machine)	OP-C, Cmp, $Mf(x) = -8x^3$ $Mf(3) = -216$, $foh = g$	Inv-W, PE (SD) $M = foh = g$ $M^{-1} = g^{-1}$
B4	CD-St, Cmp, $foM = g$ (because f $f(-2x) = g(x)$)	Cmp, OP-W, $(Mof)(3) =$ $M(27) = -54$	Inv-W $M = -2x$ $M^{-1} = -x/2$
B5	CD-St, PE (SD)	OP-W $f(3) = g(-3/2) = 27$	Inv-W, $f^{-1}(x) = \sqrt[3]{x}$, $g^{-1}(x) = f^{-1}(-2x) = \sqrt[3]{-2x}$

Table 4: Categories of responses[2].

A more detailed account of how participants responded to the question, not just categories but also parts of the participants' responses, is given in table 4 above. Successful participants are indicated by the letter A and less successful participants are indicated by the letter B.

[2] CD: Colloquial Definition; CD-St: Colloquial definition stated but not used to explain why M is a function; CD-C: Colloquial definition correctly used to explain why M is a function; CD-P: Colloquial definition partially used to explain why M is a function; CD-W: Colloquial definition wrongly used to explain why M is a function; OP-C: Order of paranthesis correctly used; OP-W: Order of paranthesis wrongly used; Cmp: Establishing a composition between M, f and g e.g. foM=g or M=fog; Inv-C: Finding the inverse of M correctly; Inv-W: Finding the inverse of M incorrectly; PE: Use of physical embodiment such as drawing set containers, set correspondence diagrams or function machines; SD: Drawing a set diagram.

RESULTS

Results from the responses to why M is a function

None of the participants explained why M is a function using the formal symbolic definition. Line-by-line analysis indicated that participants used the words such as "assigned to", "goes to" or "corresponds to" which are used in the colloquial definition although the word "transform" is used in the question. As seen in table 4, all of the ten participants used the colloquial definition. Two of them (A1 and B2) correctly and two of them (A2 and A5) partially used it. For instance, A2 correctly explained that for every function *f* in *H*, a function *g* can be found in *H* such that $f(-2x) = g(x)$. However, he could not correctly explain why *f* is assigned to a unique *g*:

A2: *g(x)=f(–2x)*. *g* is unique. If it was $f(x^2)$ then there would be two roots. It is *–2x* so we can find a unique *g*

Similarly, A5 could not explain why *g* is unique. Three participants (A3, A4 and B1) incorrectly used the colloquial definition. Instead of *M*, they applied it to *f* and/or *g* which are defined on *R*. One participant (B4) gave other reasons to explain why *M* is a function. She defined *M* as $M(x) = -2x$ by expressing it as *foM=g*. Two participants (B3 and B5) just stated the colloquial definition without applying it to *M*.

Seven participants used physical embodiment as indicated by PE in table 4. Most of these physical embodiments were set correspondences between two set containers (A1, A2, B1 and B5). However, these drawings became too complex and they could not handle them. For instance, B1 interpreted the equation $g(x) = f(-2x)$ as 'the images of *f* are transformed under the function *g*'. She used set correspondence diagram to illustrate the transformation of the elements as seen in figure 2 below:

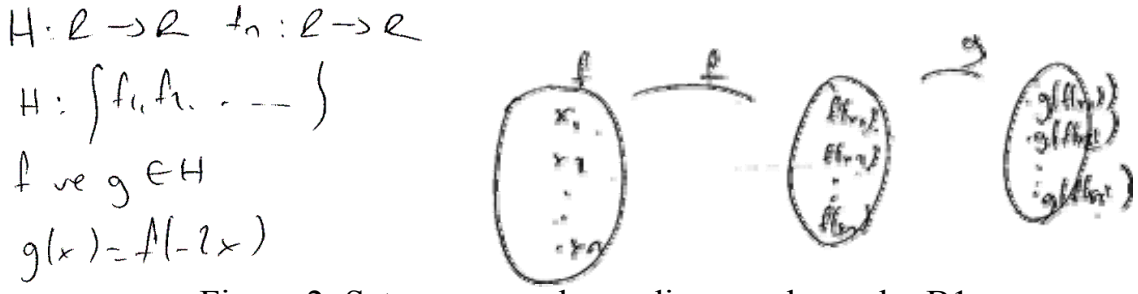


Figure 2: Set correspondence diagram drawn by B1

She then tried to represent *M* using the set correspondences as shown in figure 3. However, it was not helpful for her. She transformed real numbers to real numbers instead of transforming functions to functions.

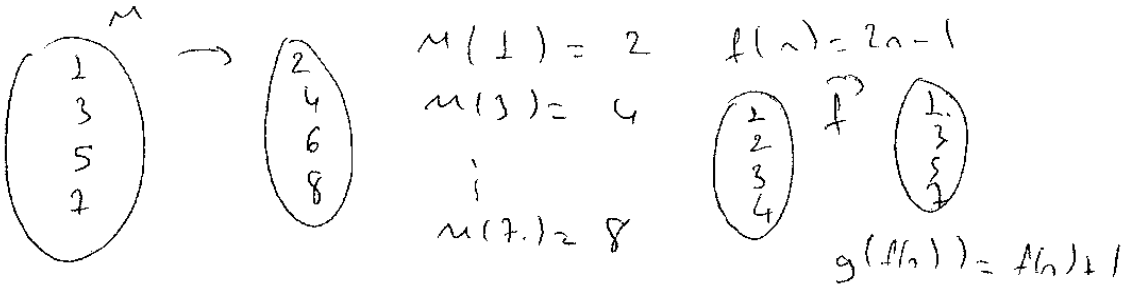


Figure 3: Set correspondence diagram drawn by B1

One participant (A1) used the set correspondence diagram just to explain the colloquial definition but did not use it to explain why *M* is a function. One participant B3 expressed *M* as a function box where the inputs *f*'s are dirty cloths and the outputs *g*'s are clean cloths as shown in Figure 4.

However, she could not use it to explain why *M* is a function.

Results from the responses to *M(f)(3)=?*

Participants were more successful with this part of the question. The main difficulty encountered by the participants when they were evaluating $M(f)(3)$ is related to the order of parenthesis.

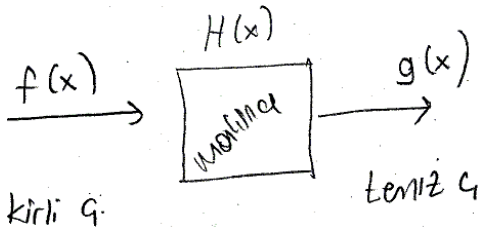


Figure 4: Function box drawn by B3

Four out of ten participants (A4, B2, B4 and B5) first calculated *f(3)=27* and found *M(27)* which is not defined since *M* transforms functions to functions. Six participants (A1, A2, A3, A5, B1 and B3) correctly calculated $M(f)(3)$ which is $M(f)(3) = g(3) = -8.3^3 = -216$. One participant (B3) considered *g* as a composition of *f* and *h* (*foh=g* where $h(x) = -2x$) and found $M(f)(3) = -216$.

Results from the responses to the inverse of *M*

Four participants (A1, A3, A4 and A5) correctly defined M^{-1} which assigns g to f as seen in table 4. Six participants (A2, B1, B2, B3, B4 and B5) could not find M^{-1} correctly. One participant (A2) assigned $g(x)$ to $f(-2x)$ which is $g(x)$ itself. One participant (B5) found the inverse of $f(x)$ for a special case of $f(x) = x^3$. One participant (B4) defined *M* as $M(x) = -2x$ since she interpreted $f(-2x) = g(x)$ as *foM=g*. Therefore, she found the inverse as $M^{-1} = -x/2$.

DISCUSSION AND CONCLUSION

As indicated by the data in this study, a few students were successful with functions in a sophisticated context. In response to research question 1, it can be concluded that

when identifying *M*, two met-befores came into play. First one is the colloquial definition which emphasizes the assignment expressed in a colloquial language. A second met-before is the set correspondence diagram which emphasizes the assignment of the elements pictorially. Both of these met-befores are explicitly presented in the curriculum and given by the teachers in the classroom. Colloquial definition is preferred over the formal symbolic definition. Set correspondence diagrams feature the defining properties of functions. Therefore, it is also used to explain one-to-one-ness and being onto.

In response to the second research question, it can be concluded that the met-befores for the function concept were not helpful for the participants to overcome the difficulties in the new context. The two met-befores, the colloquial definition and the set correspondence diagrams were explicitly given to students assuming that they can help them understand the properties of an abstract concept. However, both met-befores have negative effects in a more sophisticated context. First of all, both of them feature a static assignment rather than a dynamic input-output process. Second, we can only put finite number of elements in a set container. Third, if one tries to represent a function which transforms functions to functions then one should draw set correspondence diagrams inside a set container which becomes too complicated to handle.

The third met-before which identifies functions as formulas is not explicitly given to students. However, students' concept images are dominated by functions given by a formula similar to the students in various studies (Tall & Vinner, 1981; Vinner, 1983). As indicated by the data in this study, this met-before had negative effects. Participants had difficulties in explaining why *M* is a function and finding the inverse of *M*. In previous experiences with functions which are mainly given by formulas, there is an algorithm to follow to find the inverse (e.g. for $y=f(x)$ find x in terms of y and substitute x with y). However, in a new context where there is no formula there is no algorithm to find the inverse.

IMPLICATIONS FOR TEACHING AND FUTURE STUDIES

Long term cognitive development is not smooth and students need to do reconstructions along the way. Therefore, one of the aims of mathematics teaching should be to provide appropriate met-befores to help students to achieve these reconstructions more easily. One might ask the question: Is it always possible to find such met-befores? If not, we should try to avoid the negative effects of met-befores.

The participants in this study have not met the 'function box' either in high school or in the mathematics courses at undergraduate level. The idea of a function box was introduced to them in the mathematics teaching method course at their last year of pre-service education. DeMarois, McGowen & Tall (2000a, 2000b) suggest that the function box can act as a cognitive root for the function concept. Function box as a met-before might have positive effects in a more sophisticated context. Further research is needed to investigate the function box as a more appropriate candidate of

a met-before. This can be investigated by a teaching experiment where students experience function box as a met-before.

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QUALITY OF INSTRUCTION AND STUDENT LEARNING: THE NOTION OF CONSTANT FUNCTION

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This paper examines two experienced teachers' instructions of the constant function and their students' learning of this notion. The teachers differ remarkably in their approaches to the essence of the concept and although they use epistemologically the same tasks their students' understanding of the concept is qualitatively different. A process-oriented teaching approach appears to encourages students' development of a process conception of the constant function whilst action oriented practices appear to largely constrain students' understanding to an action conception of constant function. The evidence suggests that to help students construct an epistemologically correct and conceptually rich knowledge of the constant function teachers should prioritise the concept itself.

INTRODUCTION

There has recently been a growing interest in examining the impact of teaching practices on student learning (Askew et al, 1996; Fennema et al, 1996). Prompting this interest is the belief that teachers play an active and direct role in their students' knowledge construction.

Studies that deal with the relationship between teaching and learning can be seen in two categories: 'simple process-product research' and 'qualitative process-product research'. The former focuses mainly upon directly observable variables in teacher's instruction and relate them to the students' achievement as measured through standard tests. Brophy & Good (1986) document development of this tradition. Qualitative process-product research employs in-depth qualitative inquiry to gain a better understanding of the psychological and pedagogical aspects of teaching and learning and the interaction between the two (see, for example, Pirie & Kieren, 1992; Cobb et al, 1997). Pirie and Kieren conceptualise teaching as the continuing act of creating learning opportunities. Learning is seen as an individual's mental processing of the knowledge offered by those opportunities.

This paper takes this interest further by examining two Turkish teachers' instruction of the constant function and relates it to their students learning of the concept. The teachers' pedagogic content knowledge, their practices within the classroom and the quality of their students understanding, are drawn together to consider the notion of constant function in a perspective that is associated with action and process conceptions of the concept.

THEORETICAL FRAMEWORK

The Turkish mathematics curriculum introduces the constant function through a definition:

> A function is called a constant function if it matches every element in the domain to one and the same element in the co-domain (Çetiner, Yıldız, & Kavcar, 2000; p. 86).

This is a specific form of the concept definition which addresses an 'all-to-one' transformation and, in a mathematical sense, brings simplicity for learners. However, previous studies have reported that many students develop serious difficulties and misconceptions with the notion. Markovits et al (1986) indicated that very few high school students demonstrated a full understanding of an 'all-to-one' transformation in a situation where $g(x)=4$. Approximately half of the participants (college students) in Tall & Bakar's (1992) study rejected the idea that a line parallel to the x-axis represents a function. The students largely claimed that y couldn't be independent of the value of x. It was suggested that this misinterpretation arises because of an implicitly existing idea in school mathematics: *"the notion of function has variables, and if a variable is missing, then the expression is not a function of that variable"* (p.109). It is conjecture within this paper that learners need a process conception of function to overcome such difficulties and misconceptions. Thus, the paper considers two teachers' instructional practice and their students' consequent learning from the perspective of action and process conceptions of function (Dubinsky & Harel, 1992).

An action conception of function embraces the transformation of an element (an input) through the step-by-step procedure indicated by an explicit algebraic formula (Breidenbach et al, 1992). It is conjectured that those whose understanding is restricted to such a conception may recognise an algebraic or a graphical form of a constant function from memory but fail to recognise the 'all-to-one' transformation in the situation. A process conception of function entails the mental ability to talk about a function process in terms of inputs-outputs (ibid) but it also requires the cognitive capability to interpret a function process in the light of the concept definition without losing the sight of univalence condition. With this level of understanding one would recognise an 'all-to-one' transformation in both algebraic and graphical representations without the disruption caused by the absence of explicit formulas in the situations.

Arising from these notions, action-oriented teaching is considered to constrain students' understanding to an action conception of function whilst process-oriented teaching would encourage students' to develop a process conception of function. As we shall see, the essence of the former emphasises step-by-step application of an algorithmic procedure whilst that of the latter projects a higher level of sophistication in that not only are characteristics associated with an action conception illustrated but these are rationalised in terms of the concept definition. These aspects of teaching orientations are considered through the data presentation which is drawn together to conclude that the quality of the learner's understanding of the constant function is related to the pedagogical indications if the teacher.

RESEARCH METHOD

The paper builds upon a Ph.D. study (Bayazit, 2005) that explores the relationship between teaching and learning the concept of function. A qualitative case study (Merriam, 1988) was used to interpret teaching orientations and their possible impact on students' learning. Two teachers participated in the study: Ahmet who had 25 years teaching experience and Burak had 24 years. The students of both were in the 9th grade students (age 15). A purposeful sampling strategy (Merriam, 1988) was used to involve teachers who used different approaches to teach functions, to control the students' initial levels of understanding, their socio-economic backgrounds, and other school-related factors such as instructional facilities.

Each teacher was observed teaching all aspects of function but within this paper we concentrate on the constant function. Lessons were tape-recorded and annotated field notes were taken to record the teachers' pedagogical indications and visual attributes not detected by the audiotape. Data associated with student understanding was gathered through pre- and post tests that included an opportunity for students to provide reasons for their answers. Clarification interviews with three students from each class were carried out after each test. The students were selected on the basis their achievement and quality of explanations within the pre-test.

Descriptive statistics were used to analyse the students' test results whilst discourse and content analysis (Philips & Hardy, 2002) were used to consider the transcripts of lessons and student interviews. The objective of the discourse analysis was not to interpret a specific instructional act in its own context but to construe that act within its surrounding conditions. Cross-case analysis (Miles & Huberman, 1994) was used to establish the relationship between the teaching orientations and the students' learning.

RESULTS

The results are presented in two ways. First we consider the teachers' instructions of the constant function, and secondly we examine the students' understanding of this concept.

An analysis of preliminary interviews with the two teachers (**A**hmet: Class **A** and **B**urak: Class **B**) established that the two teachers had similar mathematical backgrounds and teaching experience, possessed a strikingly similar ability to diagnose their students' difficulties and misconception and possessed similar awareness of the way in which they thought their students would think about the function concept. However, they differed considerably in suggesting ways in which they would overcome difficulties and misconceptions. Whilst Ahmet recommended a pedagogical approach distinguished by the use of multi-representations and the encouragement of visual thinking, Burak indicated that he would prioritise verbal descriptions and offer every-day analogies to contextualise concepts. He thought, "*using two of more representations together would confuse the students*". Thus,

though the two teachers possessed similar pedagogical content knowledge, their beliefs about teaching suggested a divergence in their pedagogical treatment of the function concept. To illustrate how these differences may be seen in practice, an example is presented from each teacher.

Ahmet presented the following to his students:

> What are the values of 'a' and 'b' for which $f: R \rightarrow R, f(x)=(a-2)x^2+(b+1)x+5$ represents a constant function?

After bringing the definition of constant function to the students' attention Ahmet continued (Episode A**):**

Ahmet:	This expression involves something that does not allow the transformation of all the real numbers to one and the same element. What should we do so that this function produces the same element irrespective of whatever we put into the *x*?
Student:	The value of 'a' is 2 and the value of 'b' is -1.
Ahmet:	How did you find out?
Student:	The expression must involve just 5 so that it matches all the values of *x* to 5. Therefore, I equalised the coefficients of the other terms to zero.
Ahmet:	If the rule of a function involves an independent variable like *x*, that function produces different outputs. We should fix the value of *y*, the image. We ensure it as we remove the terms containing *x*'s. So, we have to equalise the coefficients of x^2 and *x* to 0. [Ahmet manipulates the expression and obtains the function $f(x)=5$]. No matter what we put into the *x*, say -5, 0, 4…, all goes to 5 [under this function]…

It is clear that Ahmet used the definition of constant function as a cognitive tool to establish the solution to the problem. Note also that he uses a guided discovery teaching method to encourage his students to figure out what they had to do (eliminating terms with *x* from the expression) and why they had to do it.

In contrast, though Burak emphasised that the term with *x* must be removed from a similar expression he did not clarify 'why it should be removed'. His students were asked to:

> Work out the precise form of the constant function $f(x)=(4n-2)x+(2n+3)$; and sketch its graph.

Handling the first part of the problem Burak explained **(**Episode B):

Burak:	We described it like a fixed minded person… Whatever we say; he never changes his mind. …no matter whatever we put into the *x* we come up with the same image. … Let's remember the algebraic form of the constant function; it will help us so much… … In general we represented it as $f(x)=a$, *a* in *R*. So, could we say that…a constant function involves just a number; this number would be an integer, a natural number… … In this expression [$f(x)=(4n-2)x+(2n+3)$] there are two terms; one is the constant term, $2n+3$, and the other is a term involving *x*, $(4n-2)x$. …So,

first of all we should get rid of the term containing x; because if this is the constant function…it must not involve x. How can we do that…?

Student: We would equalise the coefficient of x to 0

Burak: Yes! Exactly! This is what we must do here. We should equalise the coefficient of x, $4n-2$, to 0.

[Burak manipulated the expression to obtain the function $f(x)=4$].

Here we see his use of analogy but this was but not linked to the targeted concept. He suggested that the human mind as a constant function receiving all the ideas (inputs) proposed, processing (reasoning) them, and reaching a single conclusion (an output).

Through these two examples, which were typical of their teaching, we can see that the two teachers were substantially different in their approaches to the essence of the constant function. Ahmet usually employed a process-oriented teaching approach and consistently engaged his students with the idea of constant function by using several relevant strategies, the most prominent of which utilised the definition of the constant function as a cognitive tool to provide a basis for process-oriented language. In contrast, Burak used mostly action-oriented teaching and, engaging his students with the visual properties of algebraic and graphical forms of the constant function, placed little emphasis on the connections between their underlying meaning but emphasised the acquisition of factual knowledge.

The key features of each teachers approach are summarized in Table 1.

Ahmet	Burak
Introduces the concept through an operational definition.	Introduces the concept through a structural (an algebraic) description.
Establishes connections between representations and between ideas.	Does not establish connections between representations and between ideas.
Uses clear concept-driven language to elucidate the idea of constant function.	Uses concise but vague language to explain the concept but clear and explicit language to establish a procedure.
By solving particular problems in different ways creates new opportunities to maintain students' engagement with the concept.	Offers everyday analogies, but does not link them to the concept.
Uses set-diagrams and ordered pairs like scaffolding to support students' accession to a function process in the algebraic and graphical context.	Makes use of the students' previous knowledge to teach the procedure not the concept.
Encourages students' visualisation of the graph of constant functions.	
Implements a guided discovery teaching method.	Implements a focused-questioning teaching approach.

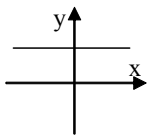
Table 1: Key features of the teachers' instructions of the constant function.

LEARNING OUTCOME

Pre-tests given to the students of each class indicated that there was almost no difference in their conceptual knowledge (for example, an understanding of dependence between two varying quantities, the ability to interpret an implicit relation within a set of ordered pairs) and in their procedural skill in manipulating algebraic expressions.

However after the instructional treatment the groups differed considerably in their understanding of constant function. To illustrate this we consider their responses to two situations.

> **Situation 1:** Does this graph represent a function from R to R? Give your answer with the underlying reasons.



25% of Class B and 18% of class A either did not respond to the question or suggested that the graph did not represent a function. Two thirds of Class A established the function process from the *x* to the *y*-axis and identified the 'all-to-one' matching over the graph whilst less than half of the students in Class B did so. Most of the remaining students from Class A simply identified the situation as 'function' because of the vertical line test but for every one of these in class A there were two in Class B whilst yet others in Class B gave the rule provided by Burak during his teaching:

> Yes, we could say, in general, that every line parallel to the x-axis represents a constant function. **I suggest you to write down this note on your notebook**… (Burak)

This was the reason that Belgin and Serap, two of the students from Class B who were interviewed about their response to situation 1, gave for identifying the graph as a function.

> … I mean, we solved similar problems in the lessons; there are similar examples in my notebook…umm…we learned that every parallel line to the x-axis is a graph of function.
>
> (Belgin, Class B)

> …as far as I remember every line parallel to the x-axis represents a function…
>
> (Serap, Class B)

Belgin's argument suggests that she was acting with her concept image —she had a notion that a graph of function transforms elements from the *x* to the *y*-axis but she could not indicate how or why this transformation occurred and during the interview she revealed a misconception arising from her interpretation of the situation:

> …this element [where the graph intersects the y-axis] has not been assigned to any element in the domain… (Belgin, Class B).

Such comments were in marked contrast to those given during the interviews by two of the Class A students of which Oken's is typical.

> It matches every value of x on the x-axis to this point, say 8 (the intersection point on the y-axis). I mean, this function matches infinite numbers in R to 8 therefore it is a graph of constant function. (Oken, Class A)

The second situation reported within this paper is the students' understanding of the constant function in the algebraic situation.

> **Situation 2:** Given the functions $f:R\rightarrow R$ $f(x)=5$ and $g:R\rightarrow R$ $g(x)=3,$ what is the value of $(fg)(7)$? Give the reason to your answers.

Interestingly, 82% of both classes provided a correct response to this question. However, here the similarity ended — concept driven explanations dominated the reasons given by Class A students (90% of correct responses) and exceeded those given by students from Class B in a ratio of 2:1. Half of Class B either produced no answer or moved away from the problem once they had satisfied the composition protocol such that $(fg)(7)=f(g(7))$.

Class differences in this task cannot be simply explained by difference in students' understanding of the composite function, because in the same post-test, 89% of Class A and 93% of Class B worked out the image of 2 when the functions $f(x)=2x+1$ and $g(x)=x^2-1$ were given

Interviews with six students (3 from Class A and three from Class B) provided detail that complimented the results from the questionnaire. The three students from Class A (Okan, Demet, and Erol) indicated a strong process conception of constant function whereas only one student in Class B (Aylin) did so. These four all commented on the 'all-to-one' transformation from x to the y-axis in situation 1. Their responses to situation 2, also articulated the idea that a constant function transforms every input to one and the same output. Aylin's response is typical:

> …I have to find, first, the value of $g(7)$… $g(x)$ matches all the elements to 3. Why? Because it is a constant function; therefore it takes 7 to 3. …$f(x)$ matches all the values of x to 5; again the reason is the same; it is a constant function…so $f(x)$ matches 3 to 5…
>
> (Aylin, Class B)

Serap, (Class B), appeared to be in transition from an action to a process conception of function in both the graphical and the algebraic contexts. She was able to interpret the implicit processes in the algebraic situations and manipulate the inputs (7 and 3) through the process of $g(x)$ and $f(x)$ respectively but her conception missed the essence of the constant function as an 'all-to-one' transformation. We saw previously that she suggested that every line parallel to the x-axis represented a function. When asked to explain the underlying meaning of this she indicated:

> Let me think…if I say this graph matches the elements on the x-axis to this point [where the graph intersects the y-axis], may I make a mistake…[silence]…I am not sure; but this is the only reason I would give… (Serap, Class B)

Though Serap accurately describes 'all-to-one' transformation she suggests uncertainty in what she says. It is conjectured that this shows that she had not fully attained a process conception of the constant function.

Belgin, the final student from Class B was considered to be at the action level in both the graphical and the algebraic situations. She did not attempt the latter because from her perspective she needed algebraic formulas to carry out the composition.

Table 2 summarises the interviewees' development of the constant function and indicates class differences identified through the post-test questionnaire.

	Class A			Class B		
Representation	Okan	Demet	Erol	Aylin	Serap	Belgin
Graphical Situation (I)	P	P	P	P	A→P	A
Algebraic Expressions (2)	P	P	P	P	A→P	A

Table 2: Summary of the interviewees' development of the constant function in the graphical and algebraic situations[1].

Table 2 suggests three students in Class A and one in Class B attained a process conception of constant function, whilst the other two in Class B either indicated progress towards a process conception of constant function or remained at the action level.

DISCUSSION AND CONCLUSION

Teaching is a social and cognitive skill delivered in an ill-structured environment (Leinhardt, 1988). Learning is a cumulative process that an individual develops through interacting with external stimuli. The mediating process between the two is open to the influence of many factors that may include the individuals' cognitive ability and attitude to mathematics and external factors such as teaching. The difficulty in controlling all of these influences does not permit an explanation of the relationships between teaching and learning in the sense of a cause-effect relationship. However, our evidence suggests that teaching practices that differ in their approaches to the essence of a concept are likely to produce qualitatively different learning outcomes. Ahmet's process-oriented teaching approach appears to have encouraged his students to develop a process conception of constant function whilst Burak's action oriented practices largely constrained his students' understanding to an action conception of constant function.

The development in each group of students' understanding cannot be explained by any one particular aspect of the teachers' instruction; instead it can be best construed

[1] Abbreviations: A - An action conception of function. P - A process conception of function. A→P - Transition towards a process conception of function

as the full impact of teaching inputs that make up action-oriented and process-oriented teaching approaches (Table 1). Additionally, what the students had learned in other lessons on functions might have positively or negatively affected their acquisition of the constant function.

Having said this, to illustrate the distinction in teachers' approaches to the essence of the concept and its consequent learning outcomes we consider again the teaching discourses upon the algebraic problems. As has been seen in Episode A, the focus of Ahmet's instruction is on the concept of constant function. The definition of constant function is used as a cognitive tool throughout the problem solution. Unlike Burak, he does not set up an easily accessible goal (equalise the coefficients of x^2 and x to zero and calculate the values of 'a' and 'b'). Rather, he acts as a facilitator and prompts the students' thinking through indirect but concept-driven explanations: *"…there is something that does not allow the transformation of all the real numbers to one and the same…"*. The effectiveness of this approach can be seen in the students' data. 85% of his class composed two constant functions at the point 7 with a clear articulation that a constant function does an 'all-to-one' transformation — a feature confirmed through the Class A interviews.

In contrast, Burak, when solving almost the same problem (Episode B), engaged his students with the visual properties of the algebraic form of the constant function. He brought the algebraic description ($f(x)=a$, a in R) to the students' attention and stressed time and again factual knowledge: a constant function does not involve x. Taking this idea as a referent, Burak sets up the goal that the term with x must be removed from the expression, but he does not illustrate nor does he encourage the students to find an answer to the question 'why should it be removed?'. We can see the negative impact of this action-oriented teaching approach on his students' learning. In response to the questionnaire, less than half of his class indicated a full understanding of the constant function in an algebraic form, a feature replicated during the interviews when only one of his students did so.

In conclusion, the evidence suggests that teachers have a considerable role to play in students' knowledge construction. They perform this role by creating opportunities in which individuals construct their own knowledge. Provision of analogies and the use of students' previous knowledge systems cannot facilitate meaningful learning unless they are (re)organised and presented in a way that communicates the essence and the components of the targeted concept to the learners. The evidence suggests that to help students construct epistemologically correct and conceptually rich knowledge of constant function teachers should prioritise the concept itself. They should allow the students to experience the concept across the representations, utilise pedagogically powerful representation to encourage the students' accession to the idea in the implicit ones, establish connections between the ideas and between the representations, and provide concept-driven clear and explicit language illustrating the notion of constant function, particularly, in the algebraic and graphical context.

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CONCEPT IMAGE REVISITED

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Concept image and concept definition is an important construct in mathematics education. Its use, however, has been limited to cognitive studies. This paper revisits concept image in the context of socio-cultural research on undergraduate students' understanding of the derivative. The literature, mainly on concept image and concept definition but also on theories of learning, is considered before outlining the research study and results which inform considerations of concept image. The discussion section, which pays particular regard to the evidence for claims made, addresses three themes: students' developing concept images of the derivative; the relationship between teaching and students' developing concept images; students' developing concept images and their departmental affiliation.

INTRODUCTION

Concept image and concept definition (hereafter referred to as 'CI & CD') is now an old construct in mathematics education but it has weathered the years well and continues to be cited in the literature, e.g. Przenioslo (2004). It is, in our opinion, an important construct. It appears that when it was first introduced there was a widespread belief that if mathematics teachers/lecturers got their definitions right, then the concepts behind the definitions would, by careful tutor explanation and student diligence, become transparent to the student. If this interpretation is correct, then the authors of the construct contributed to our current understanding that while a tutor's definition of a concept may evoke correct associations for some students, many students will generate, amongst some intended associations, unintended *concept images*. Whatever the interpretation, students' concept images became an object of study.

The construct CI & CD was born in an era where individual and cognitive theories of learning dominated English language mathematics education publications and it was a *child of its time*: it focused on individual student mathematical constructions and the theories of learning alluded to/developed by early authors were all cognitive theories of learning. This is an observation, not a criticism. What is surprising is that as social theories of learning have gained ascendancy, CI & CD has remained the preserve of cognitive theorists. This need not be the case and, as we shall argue in this paper, the construct can be used in interpreting data in what may be called a *socio-cultural* study.

The story behind this paper is that we embarked on a study of first year Mechanical Engineering and Mathematics undergraduate students' understanding of the derivative. The data from the end of semester 1 post-test was, frankly and in our opinion, amazing – Mechanical Engineering students were thinking in terms of *rate of change* whilst Mathematics students were thinking in terms of *tangents*. We explored this further, examining our lecture observation data and by constructing new

questions for students to answer in semester 2 which allowed them to exhibit their preferences for forms of knowledge about the derivative. These data suggested reasons for the *cognitive shift* with regard to the derivative concept we noticed in the two groups of students: the teaching *privileged* specific forms of thinking about the derivative in each of the groups of students; many of the students *positioned* themselves to their perceptions of their departments' ways of thinking about the derivative. But the ulterior motive for this paper is to pay homage to contribution of David Tall and Shlomo Vinner: the cognitive/socio-cultural divide is not so much a divide as a development and we acknowledge the contribution of 'past masters'.

The next section considers the literature; mainly the literature on CI & CD but also literature on social theories of learning, on teaching and on students' understanding of the derivative. We then briefly outline the study that informs this paper and provide summary results from data collected. The discussion section has three themes: students' developing concept images of the derivative; the relationship between teaching and students' developing concept images; students' developing concept images and their departmental affiliation. The conclusion concludes the paper.

LITERATURE REVIEW

We mainly focus on the CI & CD literature but also consider social theories of learning, teaching, and students' understanding of the derivative. All four areas are widely researched and our review is, for the sake of brevity and focus, selective.

CI & CD emerged as a public construct in a PME paper (Vinner & Herschowitz, 1980) and in the next year the first journal paper on the construct, Tall & Vinner (1981), appeared. This paper introduced all the current terminology around the construct used today and is by far the most referenced paper on the construct. A saying of David Tall is "You don't take an idea, you **steal** it, make it your own". This, presumably, applied to CI & CD! Concept definition is straightforward to describe, it is the form of words/symbols used by the tutor/course notes/textbook to define a mathematical concept. We are not really interested in concept definition and include it merely because it historically accompanies concept image. The notion of concept image is less straightforward to describe. Tall & Vinner (1981) describe it as the total cognitive structure associated with a concept in an individual's mind. It includes *mental pictures*, associated properties and processes as well as strings of words and symbols. It is a dynamic entity that develops, differentially over students, through a multitude of experiences. Some of these will, from a mathematical viewpoint, be incorrect, e.g. squaring a number could be defined as "multiplying a number by itself" and an associated property of squaring, grounded in students' experiences natural numbers, might be "squaring makes the number bigger". It is unlikely that knowledge that forms a concept image (mental pictures, associated properties and processes) will be simultaneously brought to bear by students in their actions in mathematical tasks. Tall & Vinner call the knowledge that is brought to

bear by a particular student, at a particular time, and on a particular task, the *evoked concept image*. Of course, as for most important constructs in mathematics education, it is extremely difficult to gain insight into students' evoked concept images. All we can do is to find what we regard as valid ways to observe students and interpret the data.

The 1981 paper focused on limits and continuity and the construct was taken up by mathematics educators, who met at PME conferences, who were interested in *advanced mathematical thinking* (AMT). This set the tone for CI & CD to be used by researchers interested in higher mathematics and/or older academic stream students. This is clearly a potentially fruitful area of application as definitions are arguably more widely used in high school and undergraduate mathematics than they are in the early years of schooling, but the construct has been applied to early learners too (Gray, Pitta, & Tall, 2000). It should be noted that it is unlikely that the application of this concept in AMT studies *just happened*; it is more likely that the construct grew from discussions within the informal PME AMT group.

CI & CD is a robust construct with regard to theories of learning in that it basically states that students bring all sorts of ideas to bear when they work on mathematical tasks, and no one disputes that. Prior to this year (2006) CI & CD had not, to our knowledge, been taken up by researchers of a socio-cultural persuasion but there appears to be no principled reason for this; indeed, the construct can be viewed in terms of Vygotsky's (1934/1986) *complexes*, a "phase on the way to concept formation" (ibid., p.112). The nearest we have found in the mathematics education literature to social theory of learning authors referring to this construct is Yackel, Rasmussen & King's (2000) paper on sociomathematical norms in an undergraduate differential equations class, and they simply note Tall & Vinner (1981) in a list of authors for whom their work complements. At the time of writing, however, Pinto & Moreira (to appear) examine students', in vocational schools, concept images of tangent lines with regard to participation in communities of practice. Given the honorary nature of this conference, we view it as 'nice' that the only 'social theories of learning' papers on CI & CD include authors who are ex students of David Tall.

Most papers on CI & CD focus on the individual student but Vinner (1992) notes that the teacher plays an important role in the formation of students' concept images, though he does not explore this interaction/development empirically. We do not dispute this, indeed, it appears as obvious to us that in all but a few instances where a student is learning completely alone (if this is possible) that the tutor will, intentionally or inadvertently, suggest associations which different students will explicitly or implicitly appropriate. We further believe that a teacher is not a *lone agent* but, knowingly or unknowingly, adapts her/his teaching to the class and/or institution s/he is teaching in. We consider a teacher-calculus and an institution-calculus study shortly but first briefly attend to studies on students' understanding of the derivative.

In the history of mathematics education, studies on students' understanding of calculus ideas are comparatively recent but there was a *boom* of papers in the 1980s. The original CI & CD authors wrote many of these papers, e.g. Tall (1985), Vinner & Dreyfus (1989), and Vinner (1982). Papers dealt with limits (Davis & Vinner, 1986), infinitesimals (Tall, 1980) and graphic (especially computer graphic) approaches (Tall, 1986) amongst other things. Differentiation was dealt with alone (Orton, 1983a) or jointly with integration (Orton, 1983b) and/or differential equations (Tall & West, 1986). The upshot of cognitive studies in the 1980s and early 1990s was that calculus was a very difficult subject (Tall, 1991), that there were many ways to view it (Schwarzenberger, 1980), that students generated many unexpected interpretations (Tall, 1991), and that computer graphics (and other computer software) could be used to transform learning and teaching from being procedure-dominated to being more conceptually-orientated (Leinbach et al., 1991; Steen, 1987).

The 1990s saw the rise of mathematics education studies which adopted social theories of learning. The theories represented, however, have many differences and range from social constructivist, e.g. Yackel & Cobb (1996), who focus on verbal interactions between teachers and learners to socio-cultural studies, e.g. Hershkowitz & Schwarz (1999), who view student activity in terms of artefact-mediated actions. We end this literature review by considering two studies which focus on social aspects of learning calculus: Kendal & Stacey (2001) and Maull & Berry (2000). These are interesting studies to focus on, for our reasons in this paper, because they deal with students' understanding of calculus, pay essential regard to the conditions that learning takes place under and because all four authors appear to have come to vaguely socio-cultural positions as a result of research rather than commencing their research with a particular social theory of learning in mind.

Kendal & Stacey (2001) use the term 'privileging' to describe the impact of teachers' emphases on students' learning differentiation. They borrow this term from Wertsch (1991) who described how different forms of mental functioning dominate in different contexts. They examine how two teachers (Teacher A and Teacher B) taught differentiation using a hand held computer algebra system, which made numerical, graphical and symbolic representations of the derivative readily available to their Year 11 classes. Although the teachers planned the lessons together they made differing pedagogical choices regarding aspects of differentiation to emphasise: Teacher A privileged rules and exhibited a strong preference for symbolic representation whilst teacher B privileged conceptual understanding and student construction of meaning; Teacher A privileged graphical-numerical connections whilst Teacher B privileged graphical-symbolic connections.

Kendal & Stacey link these instructional differences to students' performance on differentiation items. Students of Teacher A, who privileged routines, did better on items concerned with formulating the problem in terms of differentiation at a point. Students of Teacher B, who privileged conceptual understanding, did better on items concerned with interpreting the derivative. The results show that students'

performances were strongly influenced by the aspect of the derivative privileged by their teachers.

Maull & Berry (2000) examined first and final year (mechanical) engineering and mathematics undergraduates alongside postgraduate students and professional engineers through a questionnaire which sought to elicit concept images attached to key mathematical concepts including the derivative. They found that students showed similar patterns of responses at the entry but, by the final year, the groups' responses diverged: mathematics (respectively, engineering) students displayed preferences for tangent (respectively, rate of change) aspects of the derivative over rate of change (respectively, tangent) aspects of the derivative (engineering students also displayed a greater preference for tangent aspects of the derivative than mathematics students). They did not, however, provide reasons for this divergence and called for further research:

> There is evidence in the literature that engineering students are socialised into ways of thinking and behaving, and we may ask whether the difference found stems from socialisation, from the interactions between students and their peers, lecturers and other professional contacts, or whether there is also a second acculturation process through their discovery of what is useful in the context of their study and work (ibid, p.916).

They suggest exploring whether differences between mathematics and engineering students' developing concept images are related to parameters of the department to which they belong. Our work goes a little way towards addressing this issue.

THE RESEARCH

Our research investigated first year Mechanical Engineering (ME) and Mathematics (M) students' conceptual development of the derivative with particular reference to rate of change and tangent aspects, and examined contextual influences of students' departments on their knowledge development (Bingolbali, 2005). Most, but not all, students had been introduced to the derivative in High School. The research was conducted in a large university in Turkey. Data were collected by a variety of means: quantitative (pre-, post- and delayed post- tests), qualitative (questionnaires and interviews) and ethnographic (lesson observations and 'coffee house' discussions). The study adopted a 'naturalistic' approach (Lincoln & Guba, 1985) – situations were not manipulated nor were outcomes presumed. We report on a subset of this data in this paper; the Appendix shows the questions, Q1 – Q6, we report on.

We present a number of test results in this paper. An important set of tests were pre-, post- and delayed post-tests. Prior to administering the pre-test students sat a basic test on the derivative – as mentioned not all students studied the derivative in High School and these students were excluded from further testing. Some students missed one or other of the three tests. This left 50 ME and 32 M student who completed all three tests, in October, January and April/May. Test questions addressed 'rate of change' and 'tangent' aspects of the derivative in graphic, algebraic and application formats. The pre-test showed no significant difference between ME and M students'

performance. In the post-test both groups improved their performance but ME students did better than M students on all forms of rate of change questions whilst M students did better than ME students on all forms of tangent questions. In the period between these two tests, the calculus modules were observed and copies of students' notes were made to see how the topic 'the derivative' was taught in each department.

We found that 'overall, ME students did better than M students on all forms of rate of change questions whilst M students did better than ME students on all forms of tangent questions' in data analysis in February. This suggests a fundamental shift in cognition over a one semester period of time. We regarded it as an important result and set about designing data collection tools to explore this matter further. We designed further interviews with lecturers and several 'student preference' questions. Space does not permit us to detail all of these tools in this paper. We have selected one student preference question, Q6, which we believe sheds light on ME and M students' developing concept images of the derivative.

RESULTS

The results are presented in three sections. First, results related to rate of change and tangent items from pre-, post- and delayed post-tests are presented. Second, results particularly concerned with concept images are provided. Finally, the results related to both departments' calculus courses are presented. For reasons of space we err on the side of brevity in presenting and commenting on these results.

Questions 1 to 4

We report on two tangent (Q1 & Q2) and two rate of change (Q3 & Q4) questions. We first present a brief descriptive analysis of the questions based on correct/incorrect responses and then provide the results of statistical analysis conducted on these questions. Table 1 shows the pre-, post- and delayed post-test results for Q1-Q4.

Before commenting on these results we wish to point out that the situation was not quite as 'bad' as these figures suggest. We have our collapsed

		Pre-test		Post-test		Delayed post-test	
		ME	M	ME	M	ME	M
Q1 a	C	22	22	32	42	42	56
	IC	78	78	68	58	58	44
Q1 b	C	10	18	4	34	10	34
	IC	90	82	96	66	90	66
Q2 a	C	28	22	50	63	50	69
	IC	72	78	50	37	50	31
Q2 b	C	28	13	34	56	38	56
	IC	72	87	66	44	62	44
Q3 a	C	24	22	84	53	74	53
	IC	76	78	16	47	26	47
Q3 b	C	18	22	64	53	76	50
	IC	82	78	36	47	24	50
Q4 a	C	56	41	70	44	78	44
	IC	44	59	30	56	22	56
Q4 b	C	6	6	22	13	36	22
	IC	94	94	78	87	64	78
Q4 c	C	8	6	30	22	60	19
	IC	92	94	70	78	40	81
Q4 d	C	8	3	28	9	52	13
	IC	92	97	72	91	48	87

Table 1: Responses (percentages) to questions 1-4 'C' – correct, 'IC' – incorrect; ME (N=50); M (N=32)

original categories ‘partially correct’, ‘incorrect’ and ‘not attempted’ into category ‘IC’ for reasons of space. In Q1-a, for instance, 34% of ME and 25% of M provided partially correct answers in the pre-test and 20% of ME and 19% of M did so in the post-test.

The results show no noticeable difference in the performance of the ME and M students in the pre-test in any of the questions. The results also show, with the single exception of ME students for Q1-b, that both groups of students displayed improved performance in the post-test. The delayed post-test performance for both groups of students is broadly consistent with their post-test performance.

Without exception, M students’ performance in the post-test and the delayed post-test was better than ME students’ performance on tangent questions and ME students’ performance in the post-test and the delayed post-test was better than M students’ performance on rate of change questions. We break from the convention of ‘objectivity’ in academic reporting for a moment to say, in the manner of David Tall, *these results are bloody amazing*. These results are not ‘soft measures’, they report on what students got right and wrong. They do not provide ‘fine detail’ on students’ concept images but they do suggest a fundamental difference in students’ understanding of the derivative. We refer to this analysis of these results as the ‘emergent trend’: M students consistently performing better on tangent items and ME consistently performing better on rate of change items in the post- and delayed post-tests.

We subjected these results to an appropriate statistical test of significance, the Mann-Whitney U test. The test was applied to the sum of students’ improvement score on the rate of change and tangent items from: pre-test to post-test; from pre-test to delayed-post test. This test produced a significant difference between ME and M students on: tangent questions from pre-test to post-test ($p=0.003$) and from pre-test to delayed post-test ($p=0.002$); rate of change items from pre-test to post-test ($p=0.027$) and from pre-test to delayed post-test ($p<0.001$). Strong supporting evidence that the ‘emergent trend’ reflects a socio-cognitive phenomena.

Question 5

Q5 simply asked students ‘What is the meaning of a derivative? Define or explain as you wish’. It was designed to access students’ concept image of the derivative over time. Repeated rereading of students’ responses generated five response categories: rate of change (RC); tangent (T); rate of change and tangent (RC&T); other explanations (OE); and not attempted (NA). Before presenting the results (Table 2) we explain how we allocated students’ responses to these categories and give examples of students’ response for each category.

RC	‘rate of change’ or ‘rate of increase’ was cited, e.g. “The derivative tells us at what rate something is changing.”
T	‘tangent’ or ‘slope of the tangent’ was cited, e.g. “The derivative is the slope of tangent line at a point on a curve.”

RC&T both RC and T criteria were met, e.g. "Derivative can be defined as the slope of the tangent line and rate of change."

OE neither the RC nor the T criteria were met, e.g. "if $f(x) = ax^n$, then $f'(x) = anx^{n-1}$."

NA the student did not respond to this item.

Response	Pre-test		Post-test		Delayed post-test	
	ME (n=50)	M (n=32)	ME (n=50)	M (n=32)	ME (n=50)	M (n=32)
RC	6	13	30	16	36	25
T	28	16	26	41	26	41
RC&T	4	0	16	16	20	6
OE	30	31	18	22	10	16
NA	32	41	10	6	8	13

Table 2: Students' responses (percentages) to question 5

The pre-test shows a roughly similar pattern of response between the two groups of students though, interestingly, more M than ME students gave rate of change and more ME than M students gave tangent reasons. In both the post- and delayed post-test, however, students' responses were consistent with the 'emergent trend', M students citing tangent and ME citing rate of change.

Question 6

This question was, as mentioned, only administered once, at the time of the delayed post-test. The question presented the core ideas of both rate of change and tangent approaches to the derivative in the words of two imaginary students, Ali and Banu. Students had to choose an interpretation and give reasons for their choice. We present the results in two parts: their choices (Table 3) and reasons for their choices (Table 4).

	ME		M	
	Q6-a	Q6-b	Q6-a	Q6-b
Ali (A)	51	49	19	13
Banu (B)	27	49	63	78
Both (A &B)	22	2	16	3
Not Attempted (NA)	0	0	3	6

Table 3: Students' responses (percentages) to Q6

The results for the M students are consistent with the 'emergent trend', M students choosing Banu's tangent description. The results for the ME students are evenly divided for Q6-b though some support for the 'emergent trend', ME students choosing Ali's rate of change description, can be seen in Q6-a. We put forward a possible reason for this 'even split' with regard to Q6-b in the Discussion.

We now attend to students' reasons for their choices. Repeated reading of students' responses produced five categories: real life & application; mathematical & scientific; department; practice; and not categorised. Before presenting the results, in Table 4 we explain how we allocated students' responses to these categories and give examples of students' response for each category.

'Real life & application' was allocated to a student's response when their response made reference to real life or application, e.g.

> ME 1: Ali's definition, because his definition is related to real life. It shows the area of application of derivative. I think that maths concepts are attractive in as much as they are applicable.
>
> ME 2: I would support Ali. I am thinking with an engineer mentality. This makes me tend to be closer to the practicality and the concreteness.

'Scientific & mathematical' was allocated to a student's response when their response referred to mathematical or scientific ways of thinking, e.g.

> M 1: Banu gives the definition while Ali gives the explanation. I would support B because Banu explains it in a scientific way.
>
> M 2: Banu's understanding is closer to mine because she explains it in a mathematical way.

'Department' was allocated to a student's response when their response referred to a department or being an engineer or a mathematician, e.g.

> ME 3: Calculating rates of change seems to me more real. On the other hand what Banu says is not far away… But since I am going to be an engineer, Ali's idea would be just different. Because I would be the one who makes mathematics concrete.
>
> M 3: Banu interprets the derivative from a mathematician's perspective, and Ali interprets it from a physicist standpoint. At the end of the day, since I too am from mathematics department, I find Banu's explanation closer to myself.

'Practice' was allocated to a student's response when their response referred to their

Response	ME				M			
	Q6-a		Q6-b		Q6-a		Q6-b	
	A (51)	B (27)	A (49)	B (49)	A (19)	B (63)	A (13)	B (78)
Real life& application	29	0	29	0	9	0	6	0
Scientific/mathematical	0	13	0	29	0	25	0	28
Department	11	0	11	0	0	19	0	25
Practice	9	4	7	0	0	19	0	9
Not categorised	9	9	9	20	9	6	6	22

Table 4: Students' written responses (percentages) to Q6

calculus courses, e.g.

ME 4: We are using it in that way and learning it that way.

M 4: It is similar to what we are learning now and easier to answer.

'Not categorised' was allocated to a student's response when their response did not provide 'appropriate reasoning' to allocate it to another category, e.g.

ME 5: Ali's explanation is closer to me but Banu's explanation is correct.

M 5: I would support Banu because Banu's is closer to my understanding.

Note that responses fall under more than one category. Other than the diagonal pattern of zeroes in the first two rows of Table 4, which is consistent with the 'emergent trend', there appears to be no obvious pattern to these responses. We return to these responses, however, in the discussion section.

CHARACTERISTICS OF THE CALCULUS COURSES

The final results we present concern the ME and the M calculus courses. We focus on rate of change and tangent aspects, on 'theorems and proofs' and 'definitions' and note types of examinations questions set.

The ME calculus module, three 45 minute lessons per week, devoted 15 hours (20 lessons) to the derivative. The M calculus module, six 40 minute lessons per week, devoted 24 hours (36 lessons) to the derivative. Both calculus courses were observed and compared with students' notes to gain insights into which aspects of the derivative were 'privileged'. The analysis of observations and students' notes are shown in Table 5. ME students were taught more rate of change aspects of the derivative compared to M students who were taught more tangent-oriented aspects.

	Rate of change		Tangent	
	ME	M	ME	M
Duration (examples)	≈133 minutes (9 examples)	≈11 minutes (no examples)	≈10 minutes (no examples)	≈85 minutes (7 examples)

Table 5 Analysis of calculus courses with regard to rate of change and tangent

In introducing the derivative the ME course lecturer spent about 10 minutes on 'tangent', 'slope of the tangent line' and 'equation of the tangent line to the curve in a particular point'. He did not solve any tangent examples. He spent about 133 minutes on rate of change. He introduced the idea of rate of change through velocity, distance and acceleration and immediately following this solved five rate of change questions.

The M course lecturer also used tangent ideas to introduce the derivative, attending to the 'slope of the line', 'equation of line' and 'tangent line and secant line'. He spent very little time on rate of change ideas. He mentioned rate of change when he talked about the physical meaning of the derivative and later mentioned rate of change when

he attended to acceleration with regard to the second derivative. He did not solve any examples on rate of change.

With regard to theorems and proofs both lecturers presented 20 theorems: the ME course lecturer proved 10 of these whilst the M course lecturer proved 17. The ME course lecturer introduced 10 and the M course lecturer introduced 14 definitions.

An analysis of the ME and M departments' mid-term and final calculus course examinations indicates that lecturers reinforced the different emphases in their teaching in examinations questions: ME examinations included rate of change and not tangent questions whilst the opposite was the case for M examination; M examinations included proofs but ME examinations did not include proofs.

DISCUSSION

The literature on CI & CD and results from our study raise many issues about students' developing concept images of the derivative. We focus our discussion of these issues around the following three themes:

- Students' developing concept images of the derivative;
- The relationship between teaching and students' developing concept images;
- Students' developing concept images and their departmental affiliation.

We pay particular regard to evidence for claims we make because we feel that the interpretative nature of these claims leaves us open to the danger of making claims about students' concept images which are not supported by data or clear reasoning.

Students' developing concept images

We speak of students in general terms in this paper, noting trends in patterns of responses over time but are aware that some individuals do not fit with this trend and that the developmental paths of individuals providing similar responses will not be identical. Research designs and tools, like all tools, have constraints and affordances. We can say nothing on microgenetic development but we can provide broad and detailed descriptions of groups of students over time. Data from all six questions supports the thesis that there was a development in students' concept images of the derivative over the first year of their studies. Responses to Q1-4, the emergent trend, provide evidence that ME and M students developed different understandings of the derivative that affected their performance on different kinds of questions. Q5 says nothing about performance but gives us an insight into concept images of the derivative at three temporal points in students' studies. Q6 provides evidence about student preferences towards two representations of the derivative.

Evidence from Q1-4 is the emergent trend, that overall, ME students did better than M students on all forms of rate of change questions whilst M students did better than ME students on all forms of tangent questions. This was consistent in both post- and delayed post- tests and the base-line pre-test showed no significant difference

between ME and M students' performance. It is, as mentioned, rather striking evidence but it provides little insight into students' concept images.

Q5, an open question which asked students to define or explain what the derivative means, provides insight into students' concept images. It is quite consistent with the emergent trend. Looking at the RC and T responses we actually see that more M than ME students in the pre-test presented rate of change responses whilst more ME than M students presented tangent responses. In the post- and delayed post-tests, however, this response pattern was reversed and reflected the performance indicated in the emergent trend.

Q6 was only administered at the time that delayed-post test was administered but the response pattern in Table 3 is consistent with the general trend, though less marked than in Q5 in the ME students. (We note in passing that something interesting is happening within the ME students, more than might be expected chose Banu's explanation, but we cannot say what it is. We believe it may concern viewing pure mathematics as 'proper mathematics' whilst retaining rate of change conceptions.) With regard to Q6-a half of the ME students found Ali's rate of changed-oriented interpretation closer to their own understanding but 27% found Banu's tangent-oriented interpretation closer to their own view (see Table 4). The opposite, only stronger, was the case with M students. In Q6-b the preferences of the M students remain similar but the ME students are equally divided. Overall, however, the results show that ME students showed a stronger preference towards rate of change orientations and M students showed a stronger preference towards the tangent orientations.

We feel confident in concluding that the results to all six questions show that students' concept images of the derivative changed as they progressed from entry to the end of the first year: ME students' concept images of the derivative developed in the direction of rate of change orientations and M students' concept images developed in the direction of tangent orientations. The question of interest then, is 'what brought about these changes?' Of many possible things that may have brought about these changes we address two, which are grounded in our data and appear important, in the following subsections: teaching and departmental affiliation. We report on these separately but these two aspects are, almost certainly, intertwined.

The relationship between teaching and students' developing concept images

Vinner (1992, p.200) anticipated our work:

> The concept image ... is shaped by common experience, typical examples, class prototypes, etc. with a given textbook and a given teaching, one can predict the outcoming concept images and can predict also the results of cognitive tasks posed to the students.

We closely monitored the classroom calculus practices that each group of students was exposed to; it was here where the students encountered 'typical examples' and experienced 'given teaching'. We focus on the analysis of ME and M calculus

courses and discuss their potential contribution to the emergence of different concept images between ME and M students.

Table 5 shows that the ME calculus course emphasised rate of change aspects of the derivative (including examples) and that the M calculus course emphasised tangent aspects. In the words of Kendal & Stacey (2001), the ME calculus lecturer 'privileged' rate of change aspects whilst the M calculus lecturer 'privileged' tangent aspects of the derivative. Such teaching, as Vinner notes, 'shaped' their developing concept images of the derivative. Although we emphasise this rate of change/tangent difference we wish to note that this is not the only difference. Side by side with the tangent orientation of the M calculus course was a greater emphasis on definition and proof: 17/20 theorems proved in the M course compared to 10/20 in the ME course and 14 definitions in the M course compared to 10 in the ME course. Our interpretation of the import of this is that the tangent (rate of change) emphasis of the M (ME) course was an aspect of 'different mathematics' presented to the two groups of students: theoretical mathematics to the M students and practical mathematics to the ME students.

So are we saying 'you get what you teach'? Well, yes, to a certain extent we are but we think the lived-in-experience of being an undergraduate mathematician or engineer is more complex than conditioned response to teaching stimuli. Vinner (1992) speaks of the 'common experience' of students, of which, the calculus classroom is only a part. ME and M students will share a number of common experiences as undergraduates but the experience that is unique to each group is their experience of their departments, to which we now turn.

Students' developing concept images and their departmental affiliation

Data which we consider provides us with insight on both concept images of the derivative and departmental affiliation are students' open responses to Q6. Note that this question does not state which department the students are from and this, to us, is a positive feature of the question as it require greater 'sense making' from the students than a similar question with "Ali is an engineer …" in it.

Table 4 summarises our coding of students' explanations for their choices. Some categories represent about a quarter of each student group and some categories have voids but there is no majority category and no category with all voids. The most representative category for the ME students who chose Ali's interpretation (respectively, M students who chose Banu's interpretation) was 'real life and application' (respectively, 'scientific and mathematical').

Although we established an 'objective' method for assigning responses to categories it was, at times, difficult 'at a gut level' to judge the category of the response of some ME students between 'real life and application'/'department' and of some M students between 'scientific and mathematical'/'department'. For instance, ME 2 was assigned to both 'real life and application' and 'department' categories as he mentioned both "real life" and "an engineer mentality" and ME 3 was assigned to both 'department'

and 'real life and application' categories as he mentioned "an engineer", and "real" and "concrete". These two co-joined categories account for the majority of ME students choosing Ali and the majority of M students choosing Banu and are, we believe, intertwined in students' thoughts and experiences. 'Practice' (teaching), by comparison, accounts for much fewer responses. We do not claim that teaching is less important than departmental affiliation in the emergence of students' developing concept images but simply that students' perceptions of their department impact upon how they position themselves and their developing preferences and concept images.

CONCLUSIONS

Mechanical engineering students' concept images developed in the direction of rate of change aspects of the derivative whilst mathematics students' concept images developed in the direction of tangent aspects. Further to this, it appears that students' developing concept images and the way they build relationships with its particular forms are closely related to teaching practices and departmental perspectives.

Considering studies with regard to individuals' concept images in the literature, it appears that almost all studies of students' understanding of the derivative have focused on cognitive aspects and the individual mind. We do not dismiss cognitive studies, nor do we ignore the individual, but we feel that they must be seen in *context*, individual cognitive functioning is influenced by others, by the setting and by the way individuals position themselves in settings. From this perspective differing concept images of the derivative are not really surprising, they are simply interesting phenomenon to explore. We are aware that our investigation leaves much unexplained (how student 'positioning' develops, as well as accounting for students who do not appropriate departmental stances, e.g., ME students who do not appropriate rate of change interpretations of the derivative). Future studies of undergraduate students' concept images, of the derivative and of other concepts, should not ignore students' departmental affiliations.

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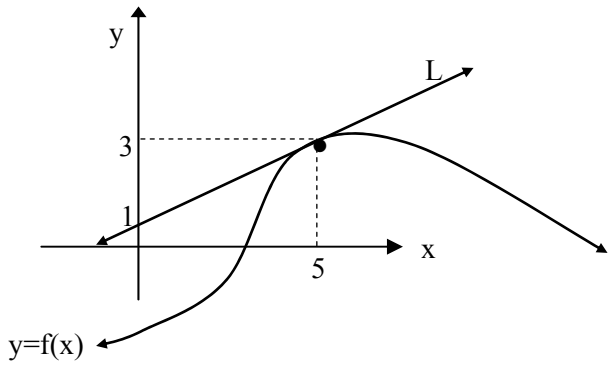
APPENDIX

Question 1

Line L is a tangent to the graph of $y=f(x)$ at point $(5,3)$ as depicted in the graph below.



$f(5)=?$

$f'(5)=?$

What is the value of the function $f(x)$ at $x=5.08$? (be as accurate as possible) (Amit & Vinner,1990)

Question 2

a) Find the equation of the tangent to the curve $y=x^3-x^2+1$ at (1,1)

b) Find the equation of Line L by making use of the graph given below

NB This question presented the graph of $f(x)=-2x^3+x^2+x+1$ and a tangent line L through the point (1,1).

Question 3 (Q3)

Find the rate of change with respect to the given variable of the following functions at the values indicated.

a) $f(x) = x^2 - 7x$, when $x = 3$

b) $g(x) = (x^2 - 1).(x + 1)$, when $x = 1/3$

Question 4 (Q4)

For a certain period the population, *Y*, of a town after *x* years is given by the formula $Y=1000(50+2x-x^2/6)$. Find:

a) The initial population,

b) Its initial rate of increase,

c) The time at which the rate of increase is 1000 people per year,

d) The time at which the population stops growing and its value at this time.

Question 5 (Q5)

What is the meaning of a derivative? Define or explain as you wish.

Question 6 (Q6)

Two university students from different departments are discussing the meaning of the derivative. They are trying to make sense of the concept in accordance with their departmental studies.

Ali says: Derivative tells us how quickly and at what rate something is changing since it is related to moving object. For example, it can be drawn on to explain the relationship between the acceleration and velocity of a moving object.

Banu, however, says: I think the derivative is a mathematical concept and it can be described as the slope of the tangent line of a graph of y against x.

a) Which one is closer to the way of your own derivative definition? Please explain!

b) If you had to support just one student, which one would you support and why?

MAKING MEANING FOR ALGEBRAIC SYMBOLS: PROCEPTS AND REFERENTIAL RELATIONSHIPS

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In this study, we analyse and discuss how a middle school student developed his meaning for symbols in the context of changing situations with two variables. We discuss the difficulties he encountered in trying to understand a symbolic equation as an abstract representation of a changing situation, with which he could flexibly engage a process and think about a concept regarding the changing situation. The results suggest that this student could operate with symbols proceptually (Grey and Tall, 1991) when generating an equation from a narrative or tabular representation of a situation but was constrained to a process when starting from an abstract equation. Kaput's (1991) model of referential relationships helps explain this apparent anomaly.

THEORETICAL FRAMEWORK

Understanding symbolism is one of the major difficulties for students when learning algebra. Hiebert and his colleagues (1997) attributed the difficulty to the fact that "meaning is not inherent" in symbols (p. 55). They insisted that without meaning symbols could not be used effectively. The National Council of Teachers of Mathematics (2000) encouraged using symbols as a tool to represent and analyse mathematical situations and structures in its Algebra Standards. In particular, the Standards emphasized using problem contexts to help students develop meaning for symbols and appreciate quantitative relationships.

The literature on symbols seems to agree on two complementary roles for symbols: symbols as signifying process and symbols as signifying concept. The literature suggests that a learner is introduced initially to symbols as a vehicle to signify process, becomes familiar with the process, and later conceives the symbols as an object carrying a concept. Thus, researchers have tried to explain the cognitive shift from symbol as process to symbol as concept.

Without denying the two roles of symbols, Grey and Tall (1991) considered symbols as a "pivot" that can allow symbol users to switch their focus between process and concept. They referred to the cognitive construct as a procept. Thus, the construct procept enables a learner to conceive symbols not only as signifying a process (e.g. compute or manipulate) but also as a concept, and to switch between them flexibly. Thus, "being able to think about the symbolism as an entity allows it to be manipulated itself, to think about mathematics in a compressed and manipulable way, moving easily between process and concept" (Tall et al., 2001, p. 8). This perspective

on symbolism is very useful in that it provides a tool to analyse the ways students use symbols in relation to their concept development and to explain the development of symbolism from elementary mathematics to formal mathematics. Before reaching the proceptual level, students can do a specific computation by knowing a specific procedure and then they become more flexible and efficient forming a process out of multiple procedures (see Figure 1).

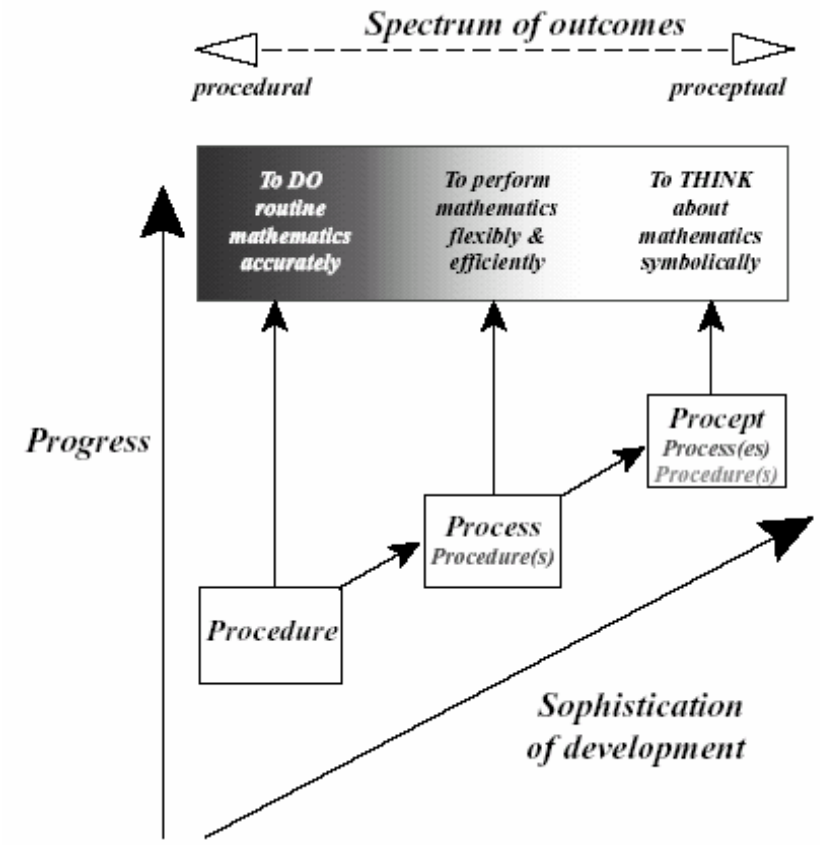

Figure 1: A spectrum of performances in carrying out mathematical processes (from Tall et al., 2001, p. 88).

In adopting the notion of procept, we investigated how a middle school student, who had recently been introduced to algebraic symbols, made sense of them in this study. We analysed to what extent the student could conceive symbols as signifying both process and concept, and switch between them flexibly.

This study, however, does not focus on algebraic symbols alone. It is an investigation of students' symbolism in relation to other forms of representation. Kaput's (1991) representation model was found to be helpful in explaining these "referential relationships" when considered together with the procept model. Kaput (1991) explained the referential relationships between 'notation A' as a representation and 'referent B'. The model (see Figure 2) had two rectangles describing the referential relationships between A and B. The bottom rectangle represented what we considered as consensual (e.g. the symbolic expression $y = 3x + 2$ [A] represents the graphical line [B] with slope 3 and y-intercept 2). Thus, through their instructions,

teachers may expect their students to make the referential relationship that they believed was consensual. However, individual students may not necessarily make the referential relationship that their teachers expected. Hence, the upper rectangle represented referential relationships that individual learners would make.

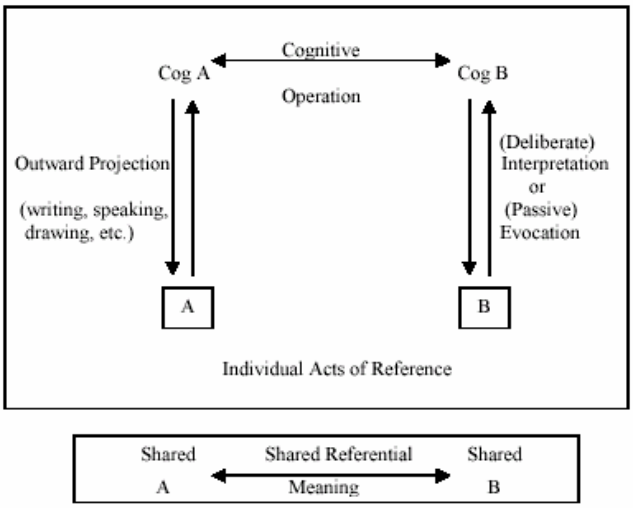

Figure 2. Kaput's referential relationship (1991, p. 60).

SETTING, MATHEMATICAL CONTENT, AND PARTICIPANTS

The students who participated in this study were seventh graders in Ms. Moseley's[1] class in a rural middle school in the southeast of the U.S. During the conduct of this study, Ms. Moseley was participating in Project CoSTAR[2] (Coordinating Students' and Teachers' Algebraic Reasoning), which was investigating how teachers and students understood their shared classroom interactions and the ways that they worked together on the teaching and learning of middle school algebra (Izsak et al., 2002). As the data for the project, Ms. Moseley's classroom activities for selected units were video recorded and observed, and she and her students were separately interviewed based on the classroom activities.

Ms. Moseley's class worked on the unit *Variables and Patterns* from the Connected Mathematics Project[3] (CMP) 7th-grade curriculum. The unit was designed with a single context (a bicycle tour) to represent a changing situation with two variables. It emphasized that a changing situation could be represented in various ways: narrative, tabular, graphical, and eventually, with algebraic symbols. In order to focus on how students develop their meaning for algebraic symbols, we designed interview tasks

[1] All names are pseudonyms.

[2] CoSTAR is supported by a grant from the National Science Foundation, grant # REC 0231879. The opinions expressed in this paper are those of the authors and do not necessarily reflect the views of NSF.

[3] CMP is a reform curriculum developed with support from NSF and now published by Prentice Hall.

that required students to write an equation to represent a changing situation presented in another form of representation, and vice versa.

For this report, we focus on the interviews with one of the four students in the study, Jeffery. He was paired with Angela for the interviews. In fact, pairing Jeffery with Angela was complementary and helpful for both to make their thinking explicit. While Angela was always cheerful and willing to express her thinking, Jeffery was very quiet unless he was confident of his idea. Mathematically, Jeffery had a good understanding of numerical relationships and was fluent in computational skills, while Angela was less confident in her numerical computation.

RESULTS

In order to address the questions posed in this study, we chose selected interview episodes with Jeffery. First, we will describe how Jeffery engaged the interview tasks below. With two sections of description, we will show to what extent he represented a changing situation with symbols and on which aspects of a changing situation he focused. Later, we will interpret how he made sense of an equation for a changing situation and suggest what might hinder him from considering an equation as a pivot between process and concept.

When writing an equation for a changing situation

In general, Jeffery could write an equation to represent a changing situation given in a narrative, a table, and a graph without much effort. In a given situation, Jeffery easily identified the changing quantities, recognized the operational patterns between them, and generated an equation. When a situation was narrated, he paid attention to the word 'per' (e.g., $3 per hour), and interpreted it as a multiplicative pattern. Likewise, he could see the operational patterns in both a table and a graph. It was interesting to note, however, that similarly to Angela, he interpreted an operational pattern additively at first and then converted it into a multiplicative one (e.g. "you add $30 for each customer" was converted into "$30 times the number of customers.")

In one interview, we provided Table 1 below, which the students were to use to write an equation for the total cost of the bike tour. This task was different from the others in that it had more than two changing quantities and complicated operational patterns. However, the students were supposedly not intimidated by the task because they were familiar with the bike tour situation in their class. We asked them to calculate the total cost for 1 customer and 5 customers before asking them to write an equation for any number of customers. Both Jeffery and Angela added the sub-costs in a row of the table for each number of customers. When writing an equation for any number of customers, Jeffery wrote an equation $NC \times Br + Fc + Vr = C$ and changed it into $Br + Fc + Vr \times NC = TC$, indicating that both were the same. We asked him to use the equation to calculate for 5 customers, and he said, " I did 30 plus 125 plus 700, and got 855. Then times 5… I don't think it will be right."

Number of customers	Bike rental	Food and camp costs	Van rental
1	$30	125	700
2	60	250	700
3	90	375	700
4	120	500	700
5	150	625	700
6	180	750	700
7	210	875	700
8	240	1000	700
9	270	1125	700
10	300	1250	700

Table 1: Bicycle Tour Costs (adapted from CMP unit *Variable and Patterns*)

While explaining how he used the equation, he figured out his equation was not correct. By being asked to compare the numbers of vans in two sums calculated from the table and his equation, Jeffery realized that his equation multiplied the number of vans also. Then he explained, "I could do 30 plus 125 and then when I got the answer, I times it by 5 and then add 700." He accordingly revised his equation into $155 \times NC + 700 = TC$.

After Jeffery wrote a correct equation, we asked both him and Angela to write a new equation for a situation when the bike tour organizers had a free van. Both students simply took out 700 from the previous equation and quickly wrote $155 \times NC = TC$. They explained that they did not need to add 700 because they had a free van.

Although Jeffery's first equation used letters to represent constant values in the situation and omitted the parentheses, he was still able to write an equation for the situation. In addition, he eventually modified his equation according to a new situation without revisiting the data in the table.

When interpreting an equation for a changing situation

In contrast with his performance in the previous task, Jeffery had difficulties interpreting an equation in terms of a changing situation. Without his partner, he struggled with making up a story for the equation $d = 8t$.

Jeffery could not even begin to think of a story and it turned out that he was not sure whether putting a number and a letter together, like 8*t*, meant addition or multiplication. After several questions aimed at helping him recall the way Ms. Moseley wrote shortened forms for the multiplication of a number and a letter symbol in her class lesson, he was asked again to make up a story. The following excerpt from the interview reveals the difficulties Jeffery had in trying to make up a story for the equation.

Jeffery: It could be dance equal 8 times *t*. It could mean dance or something.

Interviewer: Dance? What do you mean by it?

Jeffery: Like the cost of, to get in a dance could be…

Interviewer: Oh, ok. So what does *d* stand for then?

Jeffery: Dance. It would be like what you were going to or something.

Interviewer: It's more like a cost to go to the dance, right?

Jeffery: Uh-huh.

Interviewer: How about *t* then there (pointing to the *t* in the equation)?

Jeffery: It will be like the number that you are multiplying 8 by.

Interviewer: So what kind of number could that be?

Jeffery: Time or…

Interviewer: Time?

Jeffery: I think it could be ***any number*** (emphasis added).

Jeffery seemed to think of going to a dance and paying some money to get in, but he was not able to finish his story relating to the equation. He had no sense of quantitative units for the values of *d* and *t*. In an effort to understand him, the interviewer asked Jeffery to pick some numbers for *d* and *t* and tell a story using the numbers. Jeffery was able to select a pair of numbers for *d* and *t* and completed the equation as $16 = 8 \times 2$, but he was still unable to tell a story with the equation.

Focusing on his comment that *t* could be any number, the interviewer asked him whether *d* could be any number also. He answered;

> You could use like for *d*, you could put another number, like whatever you want and then 8 would be the same and then the *t* would just be what this number that you would do it by 8.

With a specific number chosen for *d*, Jeffery could tell what *t* had to be. That is, Jeffery could choose numbers satisfying the given equation due to his understanding of multiplicative relations, but he could not relate his numerical understanding into a changing situation with variables. In fact, he could successfully generate a table and a graph with the given equation but was still unable to generate a narrative situation that embodied the multiplicative relation between *d* and *t*.

Discussion of Results

Jeffery's very contrasting and uneven performance in the two tasks compelled us to re-examine his performance carefully. We focused on his word, "*any number*" and analysed his conception of variables. Jeffery could generate specific data pairs according to the given equation. Thus, the letters could be "any numbers" for him as long as the equation held true with the chosen numbers. That is, he could apply the operational relationship between two chosen numbers for the letters, but he could not see the general relationship among pairs. For Jeffery, pairs of chosen numbers for the

letters were simply the collection of pairs that happened to share the same operational relationship.

In fact, Jeffery's interpretation of the given equation did not conflict with his performance in the tasks of writing an equation for a changing situation. He could write an equation by generalizing the operations between two quantities in a given situation. For him, the letters representing the variables in a changing situation are simply substitutes of any numbers. Thus, Jeffery's conception of variable can best be described in terms of Küchemann's (1978) classification of "letters as generalized numbers." Moreover, Jeffery could generate a table and a graph according to a given equation by reversing his process of generalizing the operational relationship between two numbers.

Consequently, we implied that for Jeffery an equation represented mainly the operational relationship (i.e. the *process*) rather than the changing situation as a whole (i.e. a relation between two varying quantities). Moreover, his conceptions of variables led him to interpret a table or a graph representing a changing situation locally rather than globally. That is, he seemed to understand the relation between specific data pairs or a simple collection of specific data pairs procedurally, and did not conceive the whole set of data pairs as a trend or as the embodiment of a quantitative relation between two variables.

CONCLUSIONS

In terms of the Procept model (Gray and Tall, 1991) illustrated in Figure 1, Jeffery made sense of algebraic equations procedurally and conceived of them (at best) as standing for a numerical process. He had not yet progressed to the level of sophistication wherein he could conceive of the equation representing a quantitative relation between two variables (a concept) and, thus, was not able to switch from process to concept in order to generate a quantitative situation from a given equation. However, when faced with a narrative situation (or tabular data) that involved relations among two or more varying quantities (e.g. the bike tour data) he was able to represent the situation using letters to stand for the different quantities, and, eventually, to discriminate among the different relations among the varying quantities to produce an equation for a new quantity derived from the situation (e.g. the total cost of the tour). The ease with which he then modified this equation to represent a change in the given situation (a free van) suggests that he was able to make the switch from process to concept when working in the direction of narrative to symbolic representation. This result suggests that we need to pay attention to the direction and nature of the referential relationships involved when students are attempting to make sense of algebraic notations.

Kaput's (1991) representational model provides a framework for understanding these referential relationships. By overlaying the Procept model on Kaput's diagram for the referential relationships (see Figure 2), we gain an explanation for Jeffery's apparent difficulties in one direction and his apparent sophistication in the other.

When going from the narrative representation of a changing situation (B) to an algebraic equation (A), the student is engaged in acts of deliberate interpretation and can make use of all of his available cognitive operations (and thus make the switch from process to concept), whereas when going from an abstract symbolic equation to a narrative referent the student has to attempt to make an outward projection of the operational relations embodied in the symbolic form. This requires an act of invention rather than interpretation. Jeffery's inventions were limited to his available numerical processes (in which he was fluent) and, thus, he was not able to switch from process to concept and generate a reasonable quantitative situation that this equation could model.

We can now see why it is a challenge for young learners to understand that an equation with letters can represent a changing situation. The NCTM (2000) recommended providing students with "extensive experiences in interpreting relationships among quantities in a variety of problem contexts before they can work meaningfully with variables and symbolic expressions" (p. 225). Jeffery's case supports this recommendation in that the referential relationship involved in interpreting relations among quantities in a variety of problem contexts can help students progress toward a proceptual level of working with algebraic symbols, whereas starting with the symbols restricts them to (at best) a process level and may hinder their further development.

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MATHEMATICAL PROOF AS FORMAL PROCEPT IN ADVANCED MATHEMATICAL THINKING

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National Changhua University of Education

In this paper the notion of "procept" (in the sense of Gray & Tall, 1994) is extended to advanced mathematics by considering mathematical proof as "formal procept". The statement of a theorem as a symbol may theoretically evoke the proof deduction as a process that may contain sequential procedures and require the synthesis of distinct cognitive units or the general notion of the theorem as an object like a manipulable entity to be used as inputs to other theorems. Therefore, a theorem could act as a pivot between a process (method of proof) and the concept (general notion of the theorem). The author hypothesises that mature theorem-based understanding (in the sense of Chin & Tall, 2000) should possess the ability to consider a theorem as a "formal procept", and it takes time to develop this ability. Some empirical evidence reveals that only a minority of the first year students, studying in one of the top five ranked mathematics department in the UK, could recognise a relevant theorem as a "concept" (having a brief notion of a theorem) and did not have the theorem with the notion of its proof as a "formal procept". A year later some more successful students showed their conception of the theorem as a "formal procept" and the capability of manipulating the theorem flexibly.

INTRODUCTION

Mathematical proof is one of the most important aspects of formal mathematics. From most mathematics textbooks we can simply see the process of a mathematical proof as the development of a sequence of statements using only definitions and preceding results, such as deductions, axioms, or theorems. Theoretically the *process* of a mathematical proof occurs when the proof is built up and looked at subsequently as a process of deducing the statement of the theorem from definitions and the specified assumptions. A proof becomes a *concept* when it can be used as an established result in future theorems without the need to unpack it down to its individual steps. The author chooses to focus on this sequence of proof as a process of deduction becoming encapsulated as a concept of proof in a manner that would seem natural to most mathematicians. It is noted that there are alternative theories, for example, Dubinsky and his colleagues (Dubinsky, Elterman & Gong, 1988) focus on the use of quantified statements as processes becoming turned into mental objects by applying the quantifiers. Pinto and Tall (2002), in contrast, show how some students are capable of building formal proofs by reconstruction of prototypical imagery used in thought experiments.

ORIGINAL NOTION OF PROCEPT

Gray and Tall (1994) suggest the notion of "*procept*", which is taken to be characteristic of symbolism in arithmetic, algebra and calculus, defined in the following terms:

> An *elementary procept* is the amalgam of three components: a *process* which produces a mathematical *object*, and a *symbol* which is used to represent either process or object.
>
> A *procept* consists of a collection of elementary procepts which have the same object.
>
> (Gray & Tall, 1994, p.120)

The original definition was made in the context where the authors were aware of a wide range of examples and the definition was framed to situate the examples within the definition. In this primary consideration it is a "descriptive definition", in the sense of a definition in a dictionary, rather than a "prescriptive definition", in the sense of an axiomatic theory. However, if we consider the definition of "procept" in a prescriptive view, it seems applicable to extend the original notion of "procept" to the notion of formal proof, which can be called "*formal procept*", by adding the following analysis.

EXTENDED NOTION OF FORMAL PROCEPT

It should be noticed that there are three components of an elementary procept: *process*, *object*, and *symbol*. Now we can put the frame of Gray & Tall's "procept", particularly in the form of an "elementary procept", on the notion of formal mathematical proof. The *symbol* is the statement of what is going to be proved (which can be a theorem). The *process* is the deduction of the whole proof. And the *object* is the concept of the general notion of proof. Therefore, a theorem, for example, which is considered as a formal procept could act as a pivot between a process (method of proof) and the concept (general notion of the theorem). It should be stressed that the individual is not considered to conceive the real meaning of a theorem until the theorem has become a formal procept. With the above interpretation we could see the role of a symbol as being pivotal not only in elementary mathematical thinking but also in advanced mathematical thinking to allow us to change the channel between using a symbol as a concept to reflect on and link to other concepts, and as a process to offer the detailed steps to deduce a proof. However, an immediate argument arises. It seems that the above corollary does not always follow because even mathematicians sometimes use certain theorems without fully understanding their proofs. However, the author finds this viewpoint an *advantage* to our analysis, for it simply shows that such individuals are *not* using theorems as formal procepts, they only have *part* of the structure, usually the statement of the theorem which they then use as an ingredient in another proof without fully understanding the *totality* of the structure. The author considers the whole notion of a theorem to be grasped when the notion of proof of the theorem is also assimilated in the individual's understanding.

MEAN VALUE THEOREM AS AN EXAMPLE OF "FORMAL PROCEPT"

The notion of formal procept is applicable to trigonometry and calculus. Many trigonometric formulae and theorems in calculus, such as Mean Value Theorem [MVT] and Intermediate Value Theorem [IVT], can be seen as formal procepts. We shall take MVT as an example for further interpretation. Basically proving MVT has to apply Rolle's Theorem as a means. Besides, to prove Rolle's Theorem, we have to refer to Maximum-Minimum Theorem and the theorem: "c is a critical number if f if $f'(c) = 0$ where c is an interior point of an interval I". Furthermore, to prove Maximum- Minimum Theorem, we have to apply Least Upper Bound Axiom and Heine-Borel Theorem. If the students could understand these sequential theorems, which should be considered as individual "formal procepts", when proving MVT, it seems very clear that the students have developed the whole notion of MVT as a "formal procept" since they could integrate the relevant ideas together. However, if the students only recite the products (the statements of the theorems) without understanding the ideas of the proofs, they could not be able to apply these theorem to solve practical problems flexibly. Besides, when more and more formulae and theorems are learned, the less able students will become trapped in reciting all these products which increase the burden upon an already stressed cognitive structure.

HIERARCHY OF THE DEVELOPMENT OF SYSTEMATIC PROOF

Chin and Tall (Chin & Tall, 2000) postulate a hierarchy running through the development of systematic proof, in stages consisting of *concept image-based*, *definition-based*, *theorem-based*, and *compressed concept-based*. These stages show successive compressions of knowledge in the sense suggested by Thurston (1990). The first stage, which is concept image-based sees the student having a concept image of a particular concept built from experience, but very much at an intuitive stage of development. The transition to definition-based involves the first compression. From amongst the many properties of the concept-image, a number of generative ideas are selected and refined down to give the concept-definition. During the definition-based stage, the definitions are used to make deductions, all of which are intended to be based explicitly on the definitions. Many students, however, remain in the concept-image based stage, basing their arguments not on definitions and deductions, but on thought experiments using concept images (Tall & Vinner, 1981; Vinner, 1991). Bills and Tall (1998) introduce the term 'formally operable' definition (or theorem), proposing that:

> A (mathematical) definition or theorem is said to be *formally operable* for a given individual if that individual is able to use it in creating or (meaningfully) reproducing a formal argument.
>
> (Bills & Tall, 1998, p.104)

Tracing the development of five individuals over two terms in an analysis course, focusing on the definition of "least upper bound", they found that many students never have operable definitions, relying only on earlier experiences and inoperable

concept images. Furthermore, it was also possible for a student to use a concept without an operable definition in a proof using imagery that happens to give the necessary information required. Thus, we already know that the development from the concept-image based stage to the compressed notion of operable definition is a difficult one for many students. Even so, they are then expected to move on to the next, theorem-based stage, when theorems that have been proved by the process of proof are now regarded as being compressed into *concepts* of proof, to be used as entities in the process of proving new theorems. For this to be fully successful, I hypothesise that students who have developed mature theorem-based understanding should possess the ability to consider a theorem as a "formal procept". The author further hypothesises that individuals with this capacity to use theorems flexibly as processes or concepts are developing a compressed concept level of mathematical thinking that enables them to think with great flexibility and conceptual power.

Some evidence here shows that only a few students understand the notion of proof as a formal procept, but the empirical research also shows that, over time, more students grasp the subtlety of the idea.

EMPIRICAL STUDY

The plan of the empirical study, which included a cross-sectional probe and a longitudinal probe, was to obtain a global perspective of the first year mathematics students' general understanding of some relevant theorems, then to investigate whether and how the students' understanding improves. In the cross-sectional probe, 277 first year mathematics students, following a course in one of the top five ranked mathematics departments in the UK, responded to a questionnaire on equivalence relations & partitions" when just having learned the topic for several weeks. Thirty-six out of these 277 students were interviewed. In the longitudinal probe, fifteen selected students answered the same questionnaire and were interviewed during the first term in their second academic year. Their marks for the first year study are widely distributed — three are over 80, four between 70 to 79, four between 60 to 69, one between 50 to 59, and three between 40 to 49. This presentation is focused on two questions in the questionnaire which are generally designed to examine how the students manage to apply a relevant theorem to make their deductions.

"Equivalence relation" at the theorem-based level

The following question was designed to examine if the students improve their understanding from the definition-based level to theorem-based level:

> A relation on a set of sets is obtained by saying that a set X is related to a set Y if there is a bijection f: $X \to Y$. Is this relation an ***equivalence relation***?

It is necessary to note specially that the following theorem, which can be directly applicable to this question, had been taught before the topic of "relations" was introduced in the lecture:

(1) The *identity* map is a *bijection*.

(2) The *composition* of *bijections* is a *bijection*.

(3) The *inverse* of a *bijection* is a *bijection*.

This involves compression of the proofs of (1), (2), (3) (as processes) into useable concepts (theorems).

In the cross-sectional probe, only a small percentage of the students (13%, 36 out of 277) tried to apply the above theorem to make their deductions. Nearly a half of the students (132 out of 277) still went back to examine the definitions step by step to answer this question (they were categorised as "definition-based"). Less than a half of these thirty-six students (14 out of 36) only briefly referred to the theorem without giving more detailed interpretation. That is these fourteen students could only state the theorem but seemed not able to unpack its meaning. For these fourteen students, the notion of proof cannot be considered as a formal procept yet because they did not seem to know the process (method of proof) but only the brief concept (statement of theorem). In addition, it should be noticed that, within the thirty-six interviewees (six out of these thirty-six interviewees were categorised as “theorem-based”), thirty-three expressed that they had impression of the relevant theorem. It seems to suggest that most of the students should know or, at least, have some kind of impression of this relevant theorem, even though the majority did not manage to apply the theorem to the practical question.

In the longitudinal probe, twelve out of fifteen were able to apply the theorem in the second year, whilst only three were categorised as “theorem-based” in the first year. As was found in the cross-sectional probe, the students’ concept images of this topic were not solid at that time. Although most of the students seemed to know the relevant theorem, they did not really have a clear idea how to apply it to this practical problem. JULSON (68% for his first year study) was an example offering a definition-based response (as follows) but he vividly expressed in the first year interview — “*I remember I learned it [the theorem] in the lecture a couple weeks ago, but I’m sorry I haven’t put it in my head yet*.”

Always bij. $f: X \to X$, $\therefore$ set $X \sim$ set X

If bij $f: X \to Y$, then bij $f^{-1}: Y \to X$

If bij $f: X \to Y$ and bij $f': Y \to Z$, then bij $f'': X \to Z$

(JULSON (68%), 1st year)

Compared with their former responses, the quality of these fifteen students’ deductions seems to indicate that the notion of the theorem had become more workable in their concept images. JULSON’s response for the second year (classified as “theorem-based”) could offer us some evidence.

- $id = bij : X \to X$
- if $f = bij : X \to Y$ then $f^{-1} = bij : Y \to X$
- if $f = bij : X \to Y$ and $g = bij : Y \to Z$
  then $gf = bij : X \to Z$

(JULSON (68%), 2nd year)

In the second year, JULSON not only stated the theorem but also explained how the theorem can be proved (in the interview). Thus he clearly showed that the notion of proof of this theorem had become a “formal procept” in his concept image as he knew both the statement of theorem (as general concept) and the method of proof (as process).

The following quoted conversations recorded in the interview with DIAHUM might offer us some more delicate insight into how the successive moves — from informal to definition-based, then on to theorem-based conceptions — happened with the individual.

DIAHUM (48% for his first year study) gave the following response (classified as "informal definition-based") in the first year:

$a \to a$
If $a \to b$ $b \to a$.
If $a \to b$ $b \to c$ have same no. of elements
$a \to c$.

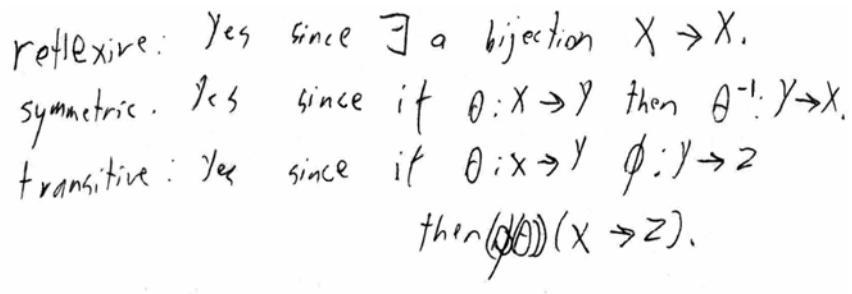
(DIAHUM (48%), 1st year)

He cleared up what he meant in his response as follows:

> I was trying to apply the definition of equivalence relation to make the answer more formal. But I don’t think my answer was formal enough because I didn’t really know how to apply the definition even though I can remember it. And another problem is I can’t recall the definition of bijection. What I can remember is a bijection is one-to-one and onto. That means the two sets have the same number of elements (he explained later that this idea was from what he learned at A-level).

He also expressed that he knew the theorem which is directly relevant to this question in the interviews. But the theorem seemed to be something only in his understanding in a theoretical manner rather than in his intuition which can be freely referred to at any time.

In the second year, he responded in terms of the relevant theorems as follows:


reflexive: Yes since ∃ a bijection $X \to X$.
symmetric. Yes since if $\theta: X \to Y$ then $\theta^{-1}: Y \to X$.
transitive: Yes since if $\theta: X \to Y$ $\phi: Y \to Z$
then $(\phi\theta)(X \to Z)$.

(DIAHUM (48%), 2nd year)

Although he did not use the term "identity" to mention the bijection mapping from the set X to itself, he could precisely write down the composition of two bijections whilst some others mentioned it in the wrong order. In addition, he could explain the idea to prove the theorem in the interview. When being asked why he answered in this way in the second year, he gave the following interpretation:

> Well, I think it's fairly natural for me to make the deduction like this. When I faced the question, the theorems burst upon my head and I just wrote down the proof.

DIAHUM's case seems to suggest that he cannot freely apply a formal conception until it is assimilated in his concept image as an embodiment. When DIAHUM could only recite the formal definition of equivalence relation but was still struggling with the implication of it, it is natural for him to consult the relevant ideas he learned at school to make his first deduction because they were more embodied and secure in his concept image. Having a year of time to digest all these notions, the theorem, which he only knew about before, had been assimilated into his concept image as a formal procept that he could recall intuitively in the second test.

In the students' (written or oral) responses, we can see that most students seemed to apply the relevant theorem directly to this practical question in the second year whilst most of them only gave a definition-based response in the previous year. This kind of result is consistent with the successive move from definition-based conceptions to theorem-based conceptions over time during which the ideas are being used formally (Chin & Tall, 2000). From the improved quality of the students' deductions, the author considers, at least for some students, the notion of proof of the theorem seemed to have become a "formal procept" in their concept images. They only appeared to know the general concept (statement of the theorem) but not the process (method of proof) of the notion of proof of the theorem before. But, a year later, some students seemed to be able to unpack the notion of the theorem to the proof process and to apply the theorem to the question more flexibly.

Linkage between "equivalence relations" and "partitions" (at the compressed concept-based level)

Theoretically the notion of "equivalence relations" is linked to the notion of "partitions" as there is the following theorem stating that "an equivalence relation can produce a partition of a set and vice versa" which is always formulated as the conclusion of the topic.

> Let ~ be an equivalence relation on a set X and $E_x=\{y \in X / x \sim y\}$ be the equivalence class (with respect to ~) of $x \in X$. Then $\{ E_x / x \in X \}$ is a partition of X. The relation "belong to the same piece as" is the same as ~.
>
> Conversely if X is a partition of X, let ~ be defined by $x \sim y$ if and only if x and y lie in the same piece. Then ~ is an equivalence relation, and the corresponding partition into equivalence classes is the same as X. (Stewart & Tall, 1977, p.74)

The following question is asked in order to examine whether the students appreciate the idea practically.

> Write down <u>two</u> different ***partitions*** of the set with four elements, $X=\{a,b,c,d\}$. For the first of these, please write down the ***equivalence relation*** that it determines.

In the cross-sectional probe, the students' reponses to this rather easy question with only four elements in the set reveal that only few students (sixty-one out of 277) show there is a workable linkage between the two notions in their concept images. The others gave two correct partitions with incorrect or without corresponding equivalence relations, or incorrect partitions with incorrect or without corresponding equivalence relations, or totally wrong answers. However, all the thirty-six interviewees said that they knew there is a theorem linking the two notions "equivalence relations" and "partitions" together, whether they appreciated it or not. It seems fairly clear that being aware of the statement of a theorem does not mean that the theorem is operable in one's concept image. I consider that the notion of proof has not become a "formal procept" yet, since the students could only remember the statement of the theorem as general concept but did not have the access to proof as process, the method of proof. Thus they could still not apply the theorem to this practical problem in the first year.

In the longitudinal probe, there were only five out of the fifteen subjects being able to apply the idea of the relevant theorem by successfully giving two correct partitions with a correct corresponding equivalence relation in the first year, and the number increased to eleven in the second test. As to the other four students, three gave two correct partitions without corresponding equivalence relation and one even failed to offer two correct partitions without giving any corresponding equivalence relation. Please note that all the fifteen expressed that they remembered they had seen, in the lecture, the theorem which links the two notions together.

HELTON, getting 61% for his first year study, can be a representative of those who failed to offer a correct response before but solved the question successfully in the second test. HELTON seemed to try to define an exact relation as follows in the first year:

$\{a\}, \{b, c, d\}$ – $x \sim y$ iff x and y are in the same partition.
$\{a, c\}, \{b, d\}$

(HELTON(61%), 1st year)

In the interviews, he confessed that he could just remember the theorem without really understanding the meaning of it. Besides, he did not recognise the mistake he made in the above response saying "in the same *partition*". In the second year, HELTON simply gave the following response but carelessly missed the two pairs (*b*,*d*), (*d*,*b*) in his equivalence relation:

$X = \{a, b, c, d\} = \{a\} \cup \{b, c, d\} = \{a, b\} \cup \{b, c, d\}$
$R = \{(a,a), (b,b), (c,c), (b,c), (c,b), (c,d), (d,c), (d,d)\}$

(HELTON(61%), 2nd year)

When asked why he was able to offer an almost correct answer in the second time, HELTON indicated directly that when preparing the examination, he studied how the theorem is proved and then grasped the idea of the theorem; thus he could simply solve the problem in the second year.

MAUHAM (71% for her first year study) is someone who failed to offer a correct answer to this question twice. She was able to give two correct partitions followed by a big question mark in the first test.

$\{a\} \{b, c, d\}$ $\{ab\} \{c, d\}$

?

(MAUHAM(71%), 1st year)

She gave the following response in the second test:

$[a], [b]$
$[a] = \{x \in X \mid x \sim a\}$

(MAUHAM(71%), 2nd year)

In the interviews, MAUHAM successfully mentioned the content of the relevant theorem, but she confessed that he only recited the statement of the theorem and had no idea how the theorem can be proved.

The result of this question appears to parallel the former question in many instances. Most of the students sensed the relevant theorem linking the two notions together but

only a few could practically apply the theorem to the question in the first year. A year later, some students' understanding had progressed to reach a more mature theorem-based level. The theorem was no longer a "general concept" only but also a "process" which suggests the method of proof to make the whole notion of proof of the theorem as a "formal procept" in their concept images. However, only trying to recite the statement of a theorem without understanding the notion of proof of the theorem is not helpful for improving the student's understanding.

DISCUSSIONS AND CONCLUSIONS

The proceptual encapsulation in advanced mathematics seems to be slightly different from that in simple arithmetic (Gray & Tall, 1994), in which pupils appear to build up the notion of proceptual structure from encapsulating various processes, to obtaining the concept, then on to forming the procept of a symbol. The empirical data of this presentation reveal that most students, at the university level, seem to have the product (the statement of a theorem) first, then to develop the notion of proof if possible. There is evidence that being stuck in processes of calculation seemingly prevents pupils from obtaining the concept (e.g. Blackett, 1990, Gray & Pitta, 1997). However, the use of the computer to carry out the process, and so enable the learner to concentrate on the product, significantly improves the learning experience (Gray & Pitta, op. cit.; Gray & Tall, op. cit.). This kind of evidence suggests that concentrating on the product first, then to develop the notion of procept is possible and also helpful for improving student's learning.

The empirical evidence presented in this paper gives us confidence to make a conclusion that the notion of procept of Gray & Tall can be extended to advanced mathematics. At the beginning, most students just have the product (the brief notion of the theorem) in their concept images only. But they cannot grasp the essence of the theorem and have more flexible thinking until they perceive the notion of proof of the theorem. Therefore, the ambiguity of process and product represented by the notion of formal procept also provides a more natural cognitive development at the university level which gives the students enormous power to develop more flexible mathematical thinking.

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TWO STUDENTS: WHY DOES ONE SUCCEED AND THE OTHER FAIL?

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This article considers two students who both pass a preliminary course in college algebra at the same level; one proceeds satisfactorily in the next course whilst the other finds it impossibly difficult. We analyse the responses of the students in semi-structured interviews on topics from the first course to seek reasons for this difference in later performance. We propose a theory that explains the difference in terms of the nature of the students' mental connections and the richness of their mental concepts as revealed in their interviews. We suggest that this theory can be used to analyse students' thinking in terms of the cognitive units they have to use and reflect upon in their mental structures which prove to be rich and well-connected in those who succeed, but limited and poorly connected in those who eventually fail.

INTRODUCTION

Attainment is increasingly being measured in terms of criterion based tests, to decide whether students have the required pre-requisites to succeed in a given course. Invariably end-of-course tests focus on what students have learned in order to pass an exam, rather than whether they are likely to succeed in a subsequent course. In this article we consider two students, both with a grade "B" in their intermediate algebra course, yet who fare very differently in the following college algebra (pre-calculus) course. We analyse interviews on material from their preliminary course to see if there is any evidence of higher level thinking that will explain reasons for their different performances on the subsequent course. We weave the empirical evidence with a theory of cognitive development relating to existing theories that give insight into the different performance of individuals in a variety of different contexts. This theory is founded on Barnard and Tall's (1997) idea of 'cognitive units' (the elements of thought that an individual manipulates in his or her conscious focus of attention), the richness of interior structure of these units (in the sense of Skemp, 1969) and a range of other theories from the analysis of Krutetskii (1976) through the notions of procedural and conceptual knowledge (Hiebert and Lefevre, 1986, Hiebert and Carpenter, 1992), and the nature of cognitive structure and conceptual linkages.

We hypothesize that the student who is ready to proceed to the next course has fundamental differences in his or her knowledge structure from the student who is not ready, and that these fundamental differences are not necessarily evident from performance on exams in the prerequisite course.

The college intermediate algebra course in this study is for all the students enrolled in it a preliminary course for entry to a college algebra course (pre-calculus). These community college students are adults who may have failed at an earlier stage, have

often been traumatised by their previous experiences, and are committed to attempt to overcome their previous difficulties. Nevertheless, we hypothesise that our general theory of cognitive structure is of value in a far wider domain of mathematical and more general cognitive development. It concerns the difference between those individuals who reflect on their knowledge in ways that create rich, mentally manipulable concepts and productive linkages that are powerful, and those who merely acquire knowledge in bits and pieces without organizing them into a connected structure. In this sense, we claim that the theory has a wider general applicability than simply the performance of students in remedial algebra courses.

BACKGROUND

> "I'm telling you, I'm not good at math…I don't really know. I hate math. I have a bad attitude with math."
>
> "Math is my hardest subject, all through high school and everything…"
>
> "I'm not too good in math, I'll tell you that…I've just never been real good. I've never gotten hold of basics in high school too good"

All of these comments were made by community college students during interviews covering intermediate algebra topics. Each of the students struggled in intermediate algebra, was ultimately successful (some with grades as high as "B"), but then had serious difficulty in the next course, the one required for university general studies. Colleagues of the author frequently describe the students as "stupid", "lazy" and "unmotivated", with "no carry-over from one course to the next".

Leaving aside the pejorative nature of these comments, we should ask a deeper question: why it is that some students who have done "well" in intermediate algebra have extreme difficulty in pre-calculus, while others—who have done equally "well"—find it routine?

Our search for an explanation has drawn us to an investigation of student cognitive structures—what do students actually *do* when they are working problems, and *how* do they do it? What is it about their cognitive structure, as evidenced by their problem-solving processes, that allows one to succeed while the other does not? Does the successful student merely have more available procedures, or is there a fundamental difference in his or her cognitive activities?

We use semi-structured interviews to gain some insight into why some are ready to progress, while others are not. A major focus in such an interview would be to look at not only *what* the student does, but *how* and *why*. However, before we address the specifics of such a quest, it is necessary to review a number of major ideas already well-established in the literature.

RELEVANT RESEARCH LITERATURE

Hiebert and Lefevre (1986) distinguish between procedural knowledge following step-by-step instructions and conceptual knowledge that is "a connected web of

knowledge, or network". Hiebert and Carpenter (1992) further discuss these ideas in the following terms:

> We believe it is useful to think about the networks in terms of two metaphors … structured like vertical hierarchies or … like webs. When networks are structured like hierarchies, some representations subsume other representations, representations fit as details underneath or within more general representations. Generalisations are examples of overarching or umbrella representations, whereas special cases are examples of details. In the second metaphor a network may be structured like a spider's web. The junctures, or nodes, can be thought of as the pieces or represented information, and the threads between them as the connections or relationships. Hiebert & Carpenter (1992, p. 67)

While these ideas have been part of mathematics education theory for many years, they have been used primarily as general philosophy rather than specifics about cognitive structure. We plan to use the distinction between procedural and conceptual knowledge and extend the ideas to an analysis of student cognitive structure in college algebra courses.

In our analysis a particular idea proves to be helpful. Barnard and Tall (1997) introduced the idea of a "cognitive unit" as a "piece of cognitive structure that can be held in the focus of attention all at one time", which may be considered as nodes in the web metaphor of Hiebert and Carpenter. However, additional insight is possible. While these nodes may be viewed individually as cognitive units, a single node may be unpacked to reveal an internal structure which is again a web of connected cognitive units. This shift from an individual node to a web of nodes, and back again, was described by Skemp (1979) as a "varifocal learning theory", in which the nodes of webs are subtly connected conceptual schemas. It is important to be able to "zoom in" and "zoom out", to compress a collection of related ideas, each of which is a cognitive unit, into a single cognitive unit. According to Barnard (1999), the entire entity can—if necessary—be conceived of as a unit "small enough to be held in the focus of attention all at one time". This describes a form of mental compression, in which you "can file…away, recall…quickly and completely…, and use…as one step in some other mental process" a concept which can be utilised as a single entity or unpacked as a whole schema of ideas (Thurston, 1990). This is discussed further in educational terms by Gray & Tall, (1994) as what enables the student to use a whole complex of ideas from one context as a foundational unit in a subsequent context.

Problem 1

For example, consider the equation $y = mx + b$. One can consider it a concept as a network of internal ideas: m is the slope, b is the y-intercept; any linear equation can be represented by substituting appropriate numbers in for m and b; one can draw the graph if there are two points available, or one point and the slope, etc. A student may thus see $y = mx + b$ as a *single entity* rich with properties and links easily brought into the focus of attention.

However, consider the following:

The equation $y = 2x + 3$,
The equation $y - 5 = 2(x - 1)$,
The graph of $y = 2x + 3$,
The line through $(0,3)$ with slope 2,
The line through the points $(1,5)$ and $(3,9)$,

each of which may be considered individually as a cognitive unit which can be linked with any or all of the rest as representing the same underlying concept—the same straight line or equivalent linear relationship between x and y. We could express this diagrammatically as five separate nodes with appropriate connections as in Figure 1.

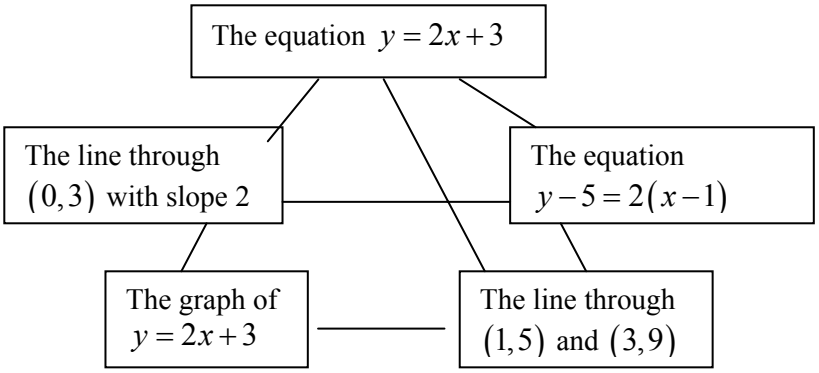


Figure 1. Relationships

To seek insight we will explore the cognitive structure demonstrated in interviews with two students working problems involving graphs of linear functions from the Intermediate Algebra course. They took place when the students were enrolled in the pre-calculus course, so would already show longer-term understanding rather than the knowledge the students may have learned at the time of an exam.

INTERVIEWS

We asked the students to answer the question, "What is the slope of the line on the graph here?" (shown in figure 2)

Natasha had a flexible solution.

Natasha: What is the slope on the graph here? It goes down…do you want me to solve it using the points?

Researcher: I don't care how you solve it.

Natasha: Because you can just really look at it and tell…it's a negative 2 slope.

Researcher: So you did it by counting squares.

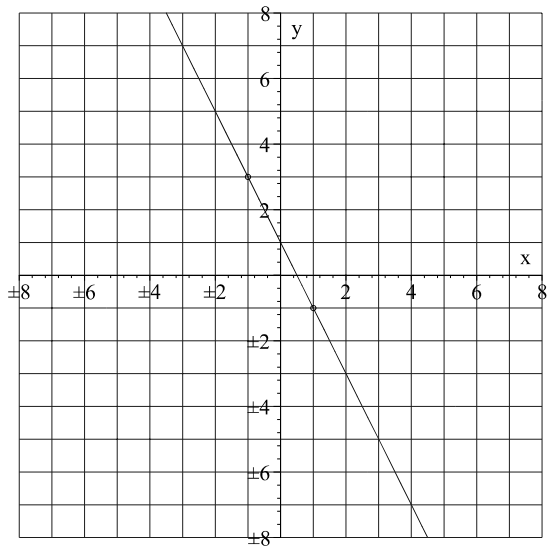


Figure 2. The Graph

Natasha: Yeah, just by looking at it. I mean, if the squares weren't there, I could do it by taking the two points and finding the slope like I've done…

Kathy's solution process (in figure 3) was less straightforward:

Kathy: One, negative one, and negative one, three. …shoot.

Researcher: *(pointing at the left and right sides of the equation.)* So how did you get from here to here?

Kathy: By subtracting…you've got me all bamboozled. My handy-dandy calculator, I rely on that. That would be four.

Researcher: Four over two? Does that look right?

Kathy: I think it does, but I could be wrong.

Researcher: Well, you made a mistake, not a very big one…

Kathy: One I can't catch, though. It'll make me feel really stupid later.

Researcher: *(Encouraging.)* I'm just trying to protect you from making a dumb mistake on your exam! … The denominator, what's minus one minus one?

Kathy: Zero.

Researcher: No, it's not.

Kathy: Two, is that it?

Researcher: It's negative two, isn't it?

Kathy: Yeah.

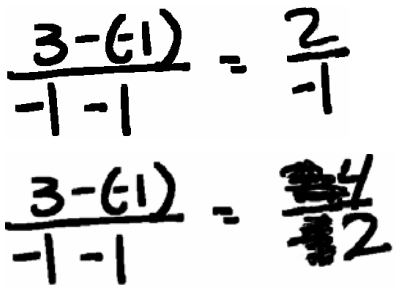


Figure 3. Kathy's solution

Kathy had great difficulty with negatives and fractions; she froze when asked to manipulate them without a calculator. When she had a calculator, she used it to do her arithmetic and only then was able to cope with simple calculations. Her solution process is illustrated in the figure 4.

Analysis

Natasha had flexibility; she could either read the slope off the graph, if that was easy, or she could find the slope by identifying two points and using the slope formula. This flexibility saved her a lot of effort on this particular problem. Kathy only evoked a formula for the slope, which required two points. She was unable to link the slope to the change in y over the change in x from the graph. This problem was, for her, consequently much more complicated, ultimately involving computations with negatives, which caused her much difficulty. She was also insecure when she finished the problem; she always checked with an authority figure—the instructor, the

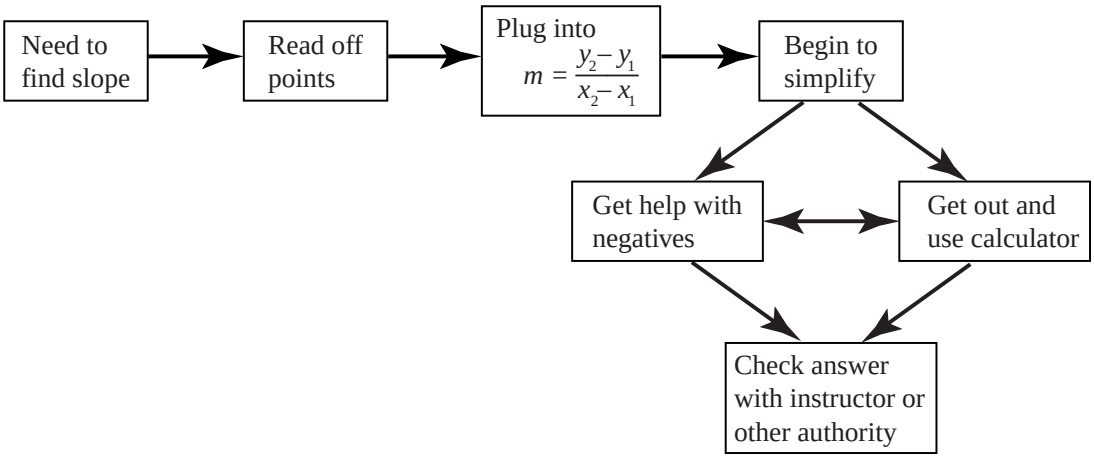
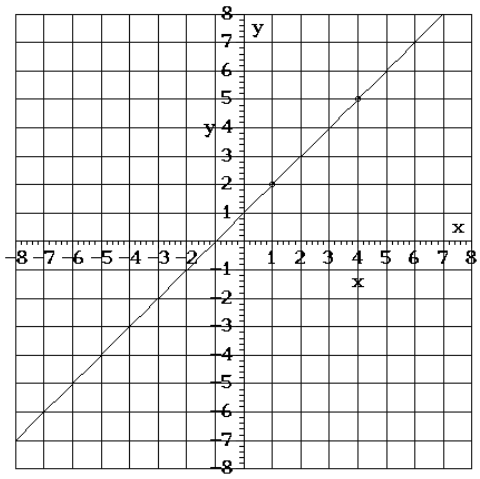


Figure 4. Kathy's Solution Process

interviewer, or the answers in the back of the book—for assurance that she had found the correct answer.

Problem 2

In a second problem to write the equation of a straight-line graph (figure 5), Natasha's solution was not the most efficient. She was able to read off the slope, but then used the point-slope formula to write the equation, whereas it would have been more efficient to simply use the *y*-intercept from the graph. Nevertheless, she demonstrated flexibility in finding the slope, and again in checking to ensure that the point she was using in the formula was correct. An outline of her strategy is given in figure 6.



Write the equation of the line.

Figure 5. Problem 2

Meanwhile, Kathy approached the problem by reading off two points (1,2), (4,5), using the formula for the slope to find it is 1, and then the formula for a line through (1,2) with slope 1. She too made an error, but made no effort to correct it until she was prompted by the researcher.

Problem 3

A third problem to find the equation of a straight-line graph had the same format of problem 2, but with the negative slope from problem 1. Natasha again showed flexibility; she read the slope m=–1 by inspection and then read off the *y*-intercept *b* numerically to give the equation in the form $y = -1x + 1$.

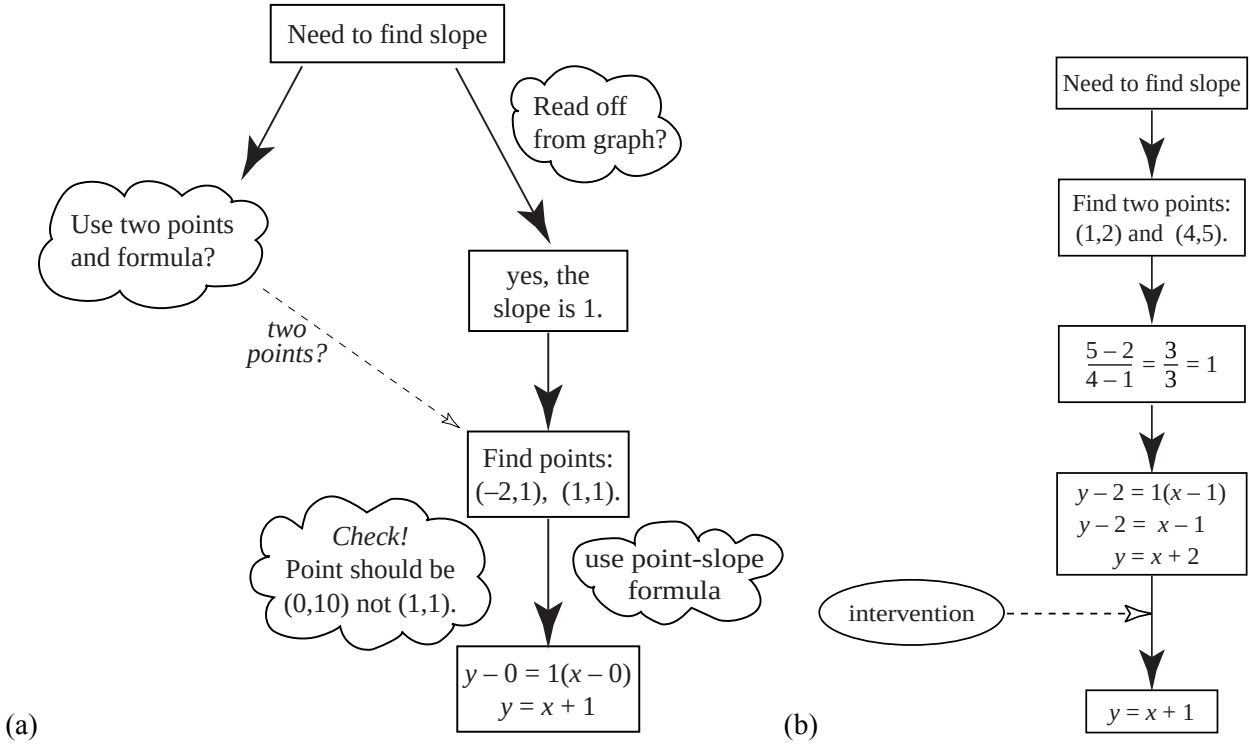


Figure 6. (a) Natasha's and (b) Kathy's Solution Processes

Kathy did not exhibit the same flexibility or checking mechanisms:

Kathy: Three, negative two…I need to find another point…

Researcher: That's a good plan.

Kathy: Four, negative three…

Researcher: What are you going to do now?

Kathy: Find the slope first. Negative three…

She then uses the formula incorrectly (figure 7) …

Researcher: Now wait a minute. Why did you…this is right. Why did you tell me that $-3-2$ is -1?

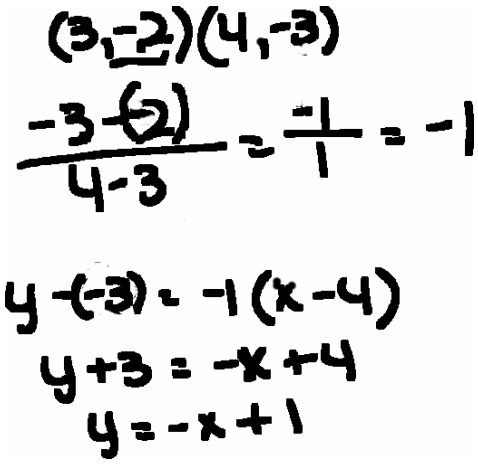


Figure 7. Kathy's writing

Kathy: Wait, is it negative 5?

Researcher: Yes, however, you made a mistake earlier. Where you had $-3-2$, it's $-3-(-2)$, isn't it?

Kathy: Oh, yes.

Researcher: So then what do you want up here?

Kathy: That would be positive 5?

Researcher: Isn't minus a minus a plus?

Kathy: Yes.

Researcher: Minus three plus two is minus one.

Kathy: Which is what I got, I just didn't have the negative sign.

(note: she had made canceling errors)

Researcher: So you've got the slope, now what are we going to?

Kathy: Use the…

Researcher: What's the *y*-intercept on that line?

Kathy: The *y*-intercept?

Researcher: Where does it cross the *y*-axis?

Kathy: One, er, zero, one.

Researcher: So you should get something *x* plus one, shouldn't you?

Kathy: Yeah.

In each problem, Natasha demonstrated flexibility in choosing a route to a solution, thus showing evidence of links between graphs, formulas, and other aspects. She also routinely checked her work using alternative methods, another indication of useful links. She found her own error in the second problem. Kathy, on the other hand, had at most a single procedure in each case and was prone to make mistakes.

SUMMARY

In all the questions considered there is a broad common thread. Natasha demonstrated links between graphical and symbolic representations, as well as links to and between procedures. Although she made mistakes, she had methods of checking and self-correcting. She did not always make the necessary connections and had some fears about negative numbers, but was broadly successful. Kathy obtained the same grade on her examination but merely learned a set of procedures and had

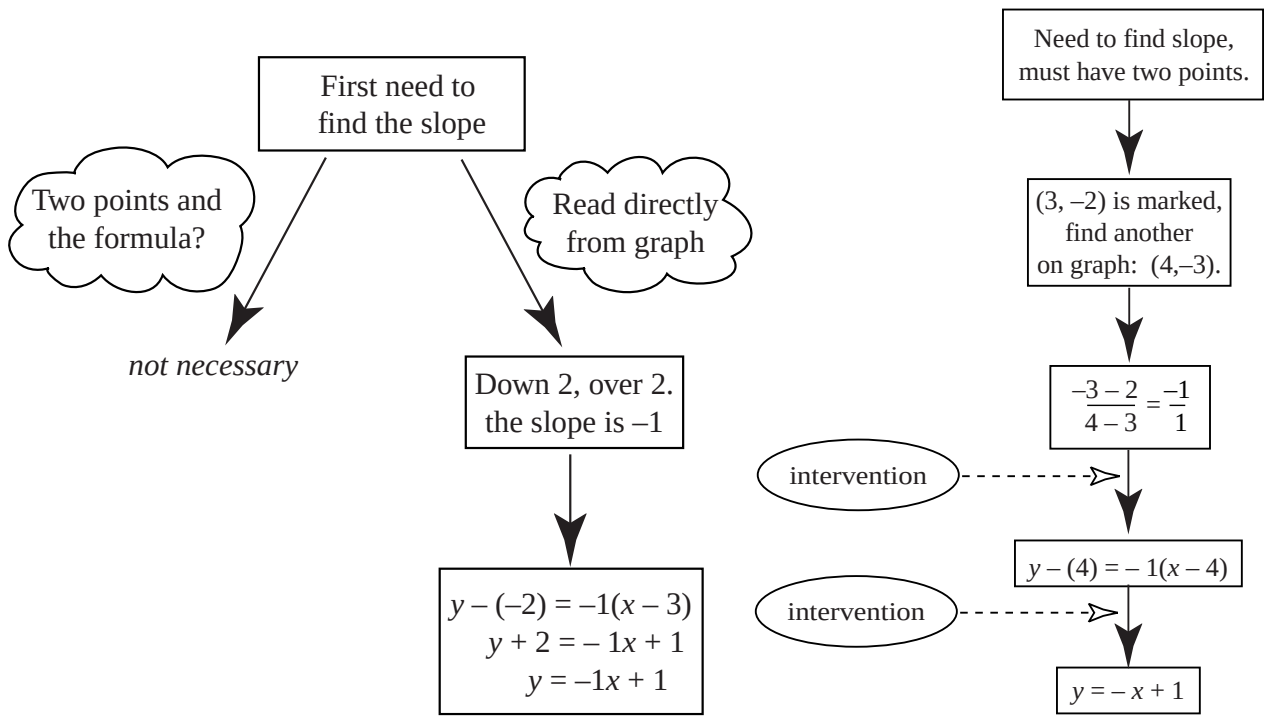


Figure 8. Natasha and Kathy's Solution Processes

difficulties with negative numbers and fractions that she coped with in routine questions by using her calculator. The procedures she has learned have allowed some success, but she must work very hard, and the procedures are not organized in a useful way that would allow her to build on them in the subsequent pre-calculus course.

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ESTIMATING ON A NUMBER LINE: AN ALTERNATIVE VIEW

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This paper considers children's accuracy in estimating the position of numbers on a 0 to 100 number line. Children with median ages ranging from 6.5 to 10.5 were invited to position paired numbers on the number line. Difference between their estimates and the actual numbers indicates that they exaggerate magnitudes in the lower half of the line but underestimate them in upper half. The findings call into question explanations of behaviour orientated towards linear or logarithmic models and suggest the alternative view that to some extend accuracy is dependant upon perceptual and imagined "correctors".

INTRODUCTION

Within England and Wales the process of improving standards in mathematics are specified within two documents. The first, the National Curriculum for Mathematics (DfEE, 1999a), identifies the contents of the mathematics curriculum that should be taught during Key Stage 1 (KS1, age 5 to 7) and Key Stage 2 (KS2, age 8 to 11). The second, the National Numeracy Strategy (NNS) (DfEE, 1999b), identifies, almost through a process of genetic decomposition, how and what should be taught to instill a sense of being numerate. Programmes of Study outlined within the National Curriculum are fully aligned with the NNS, the latter providing a detailed basis for implementing the statutory requirements. Its developmental approach for the acquisition of numerical skills and concepts sees the transition from the use of perceptual items to "more sophisticated mental counting strategies" finally sublimated by the acquisition of basic number facts and the development of knowledge of the number system. To achieve these ends the NNS suggests that a daily mathematics lesson is appropriate for almost all pupils with a high proportion of the time devoted to whole class teaching through which mental and oral work should feature strongly.

The question arises as to whether or not the initiatives within the UK have provided opportunities for children, particularly for those considered to be at the lower end of the spectrum of achievement, to develop a sophisticated means of thinking. This paper gives a response to this question by considering one small aspect of the initiative, the accuracy with which children with a median age range of 6.5 to 10.5 estimate the position of numbers on a number line.

RESEARCH FRAMEWORK

In their review of mathematical representations, identified as mediators of mathematical concepts and processes between teachers and pupils, Lesh, Post & Behr (1987) categorise the number line as a manipulable model that has built in relationships and operations to fit many learning situations. The association between

number (real number) and line has been evident since Babylonian times. The Greeks intuitively conceived of real numbers as corresponding to linear magnitudes, implying that one may think of "numbers as measured off on a line" (Bourbaki, 1984, p. 121). The number line is, therefore, an abstraction of a representation strongly associated with the notion of a measure instrument and since continuity underscores it all real numbers can be mapped onto it. Herbst (1997) indicates that the number line is a metaphor of the number system and he defines it as the consecutive translation of a specified segment U as a unit, from zero. U itself can be partitioned in an infinite number of ways (i.e. fractions of U). In its completeness the number line is a very sophisticated metaphor but it is conjectured that recognition of this sophistication can grow through experience and use.

Though it is clear that the number line possesses certain underlying features there have been a range of suggested modifications and uses of it as a representation to support the development of mathematical skill and understanding. It is considered as a diagnostic tool for both the pupil and the teacher (Rousham, 1997) and it can be a helping tool for teaching whole number operations (see for example Behr & Post, 1992). The fact that pupils can physically act on it allows them to be cognitively involved in their own actions and make decisions about where to make a mark and write down the numbers on the line (Beishuizen, 1997; Klein et al, 1998).[1] Thus the "dense" nature of the number line permits it to be seen as a representation of the number system and also be used as a "solving and justifying" tool (Herbst (1997). It may help in obtaining the solution to a problem on the line, as well as provide justification of the way of thinking and to the way an answer is obtained.

Identified as a key resource to help children master numeracy skills effectively (DfEE, 1998), it is the solving and justify view that is associated with its strong development and use within the NNS. However the NNS adopts both of the above themes. On the one hand it associates the number line with an overview of development of the number system whilst on the other it provides examples of its use as a tool supporting the development of addition and subtraction procedures with particular reference to notions of partitioning and implicit reference to associativity and commutativity. This paper focuses on one aspect of the former, that of being able to associate inherent knowledge of the number system from 0 to 100 with magnitude to estimate the position of numbers on a 0 to 100 number line.

Throughout the NNS reference to using the number line to support the development of an understanding of the number system is continual. It is associated with the development of counting skill to order numbers to 10 (Reception, median age (MA) 4.5), to 20 (Year 1, MA 5.5), to 100 (Y2. MA 6.5) and it is used to develop

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[1] The reader should note that it is not the purpose of this paper to consider modifications to the notion of number line that include the 'empty number line' (see Klein et al., 1998), nor to discuss the merits or otherwise of associated models such as the number track, a notion that may cause considerable confusion for those who are not aware of the conceptual differences between the two (see Skemp, 1989).

understanding of fractions (Y2 onwards), decimals and negative numbers (Y4 onwards). Additionally we may see it used from Y2 to develop skill in addition and subtraction. Throughout the early stages of the document there are recurring references that children within years 2 to 4 (median ages 6.5 to 8.5) should be able to have experiences that would enable them to interpolate the position of a particular number on a partially numbered or an empty number line, record their estimates and find the difference between the estimate and the actual number. For Year 2 children the presented examples include number lines with the ends marked 0 and 10 and for Years 3 and 4 number lines are marked from 0 to 100. It is implicit that children within the years 5 and 6 possess the skill and understanding to do this.

Several studies have investigated number line representation of individuals with particular reference to estimating a particular number. Dehaene (1997) considered the internal representation of numbers by adults and suggested that they express a logarithmic pattern, smaller numbers are to the left and larger numbers, tending to be compressed into a smaller space, are to the right. He argued that this is an inherent and intuitive mental idea of the number line that we all possess and that this idea:

> … is automatically activated whenever we see a number and which specifies that 82 is smaller than 100 without requiring any conscious effort. This "number sense" is embodied in a mental number line oriented from left to right.
>
> (Dehaene, 1997, p.151)

Siegler & Opfer (2003) concluded that children's (2nd, 4th and 6th grades) estimations of numbers in the range 0 to 100 expressed linear mapping between the numbers and their magnitudes that improves in consistency with age. Children's estimation of numbers from 0 to 1000 expressed a logarithmic pattern which:

> … exaggerates the distance between the magnitudes of numbers at the low end of the range and minimizes the distance between magnitudes of numbers in the middle and upper ends of the range. (Siegler & Booth, 2004; p. 429)

Siegler & Booth (2004) concluded that kindergarteners' estimations expressed a consistently logarithmic pattern, 1st graders an amalgam of logarithmic and linear patterns and 2nd graders purely linear.

Siegler (2005) suggested that estimations in general are far from accurate because of limited real knowledge and conceptions of numbers. He also suggested that linear representations are a result of age and experience, are associated with high achievement and are the result of an approach to estimation were the individual divides the line at specific points and uses these points as references to locate other numbers results in the linear pattern. An accumulative approach of estimating the position of numbers results in the logarithmic pattern. Given the recommendations of the NNS which suggest that that younger children should be given opportunities to estimate positions on a 0 to 100 line it is our intention was to consider the estimations of children from Y2 and above within one English primary school on a similar line.

METHOD

The results presented in this paper are part of a wider study on children's use and understanding of the number line representation. The sample data, drawn from children from a school within the English Midlands had, by accident rather than design, remarkable similarities to that of Seigler — the school was within a relatively deprived area with children from diverse cultural backgrounds. External assessment of mathematics teaching within the school was identified as “good” whilst in general the children’s attainment was “below average” (OFSTED 2004). The school has extensively used the NNS since 1999.

The paper reports on quantitative outcomes derived from the analysis of a questionnaire distributed to 91 children (18 from Y2, 18 from Y3, 19 from Y4, 17 from Y5 and 19 from Y6, (that is children with median ages that ranged from 6.5 to 10.5). Qualitative data obtained from interviews is alluded to but not a focused upon. The questionnaire was designed so that the children were presented separate sheets with a series of numbers and required to place each number on a 0 to 100 horizontal number line on each sheet. The questionnaire was designed so that the children were presented on separate sheets with a series of numbers and required to place them on separate 0 to 100 number lines. The individual items were compatible with items identified from within the NNS (See p8, 9, 17) but were also selected to consider differences in the children’s ability to estimate numbers that were equidistant from the end points 0 and 100. The numbers, in the order that the children responded to them were 93, 45, 12, 5, 75, 3, 7, 25, 97, 95, 88 and 55. The analysis presented here is essentially descriptive.

RESULTS

Figure 1 provides an illustrative example of one pair, 88 and 12, of each child’s estimate and the distribution of these estimates within each year group.

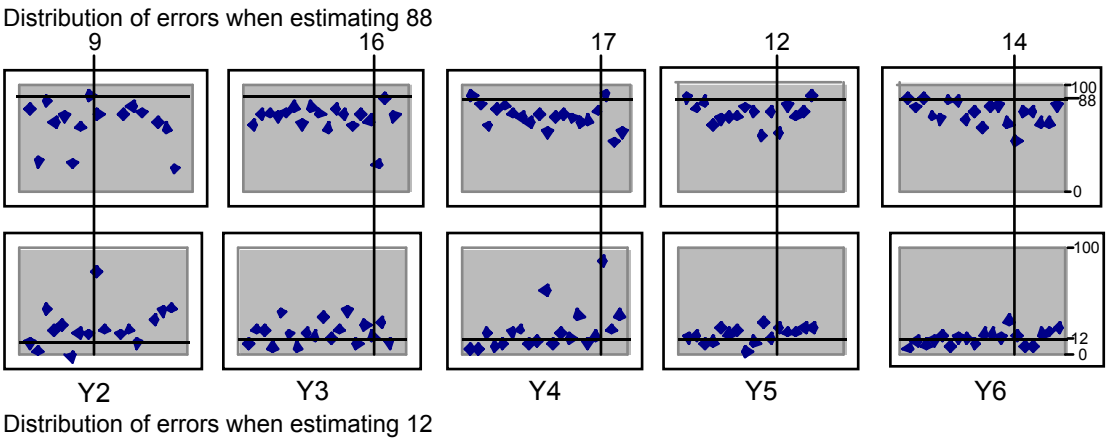

Figure 1: Full sample estimations of 12 and 88 identified by year group

Each child’s estimate of the numbers 88 and 12 (marked on the figure) is identified on a vertical scale marked 0 to 100. The distributions are arranged so that children within each year group are given in the same order to reflect their overall

achievement within mathematics as measured by inter school tests and teacher predictions of achievement within Standard Attainment Tasks. Low achievers are to the left and high achievers are to the right.

Even though the numbers to be estimated are equidistant from the end points of the line several features emerge from Figure 1:

1. A general trend amongst the majority of children to underestimate the position of 88 and over-estimate the position of 12.
2. The overall accuracy of estimations tends to be associated with age. The estimations of the older children, particularly their estimation of the position of 12, are generally far more accurate than those of any of the younger children but though we note a gradual convergence in the children's estimates of 88, even amongst the older children there is a tendency to extensively underestimate.
3. A child's ability to provide a reasonable estimate of one number does not imply that the child will also provide a reasonable estimate of its paired counterpart. Note particularly the relatively good estimates of 12 given by children 16 (Y3), 12 (Y5) and 14 (Y6) (see vertical lines in Figure 1) and yet each child gives one of the worst estimates of 88 in their particular class. Conversely, consider the good estimates of 88 by children 9(Y1) and 17(Y4) but their relatively poor estimates of 12.
4. Since each diagram is formed to reflect an order of achievement there is little evidence to suggest that mathematically higher achieving children are better at estimating that their lower achieving peers.

The general trends identified within points 1 and 2 (above) may be seen from a clearer perspective within the illustrative example given in Figure 2. It illustrates the size and direction of the difference between the average of the absolute difference between each child's estimation of the position of a number allocated, corrected to establish whether each average is an underestimate or an over-estimate, and the actual point. The curves are diagrammatic to illustrate the differences between Y2 (continuous line) and Y5 (dashed line). For example, the mean of the absolute differences between Year 2 children's estimates of the position of 12 on the number line and its actual position was 17 whilst that for

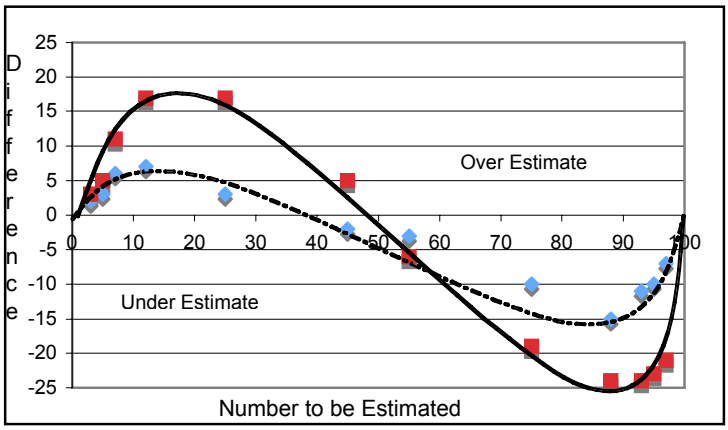

Figure 2: Difference between actual numbers on the number line and the mean of the absolute differences between that number and the estimates given by Years 2 and 5

Y5 children was 7. The general observations noted from within Figure 1 become clearer within Figure 2:

- Apart from one instance, estimating the position of 55, the magnitude of the differences identified from the results of the older children are always less than those of the younger children (Y2).
- Whist Figure one indicates that the magnitude of 12 is over-estimated and that of 88 is underestimated, Figure 2 suggest the tendency to over estimate magnitudes below 50 and underestimate magnitudes above 50.
- 45 is the only number for which the absolute differences are in opposite direction. Y2 over-estimate and Y5 underestimate.

These general features within the figure are characterised by the curve. This is not intended to have any mathematical significance beyond the descriptive. It is simply a mechanism to illustrate features that emerge from the distributions of the estimations within the two groups. Figure 3 draws upon these features to present a comparative illustration of the trends in the distributions of the mean differences of all of the year groups across the range of estimated numbers. To achieve simplicity it does not include pinpointed differences.

The indications from Figure 3 are that:

- amongst all year groups there is a general tendency to over-estimate the positions of numbers in the first half of the number line segment 0 to 100 and

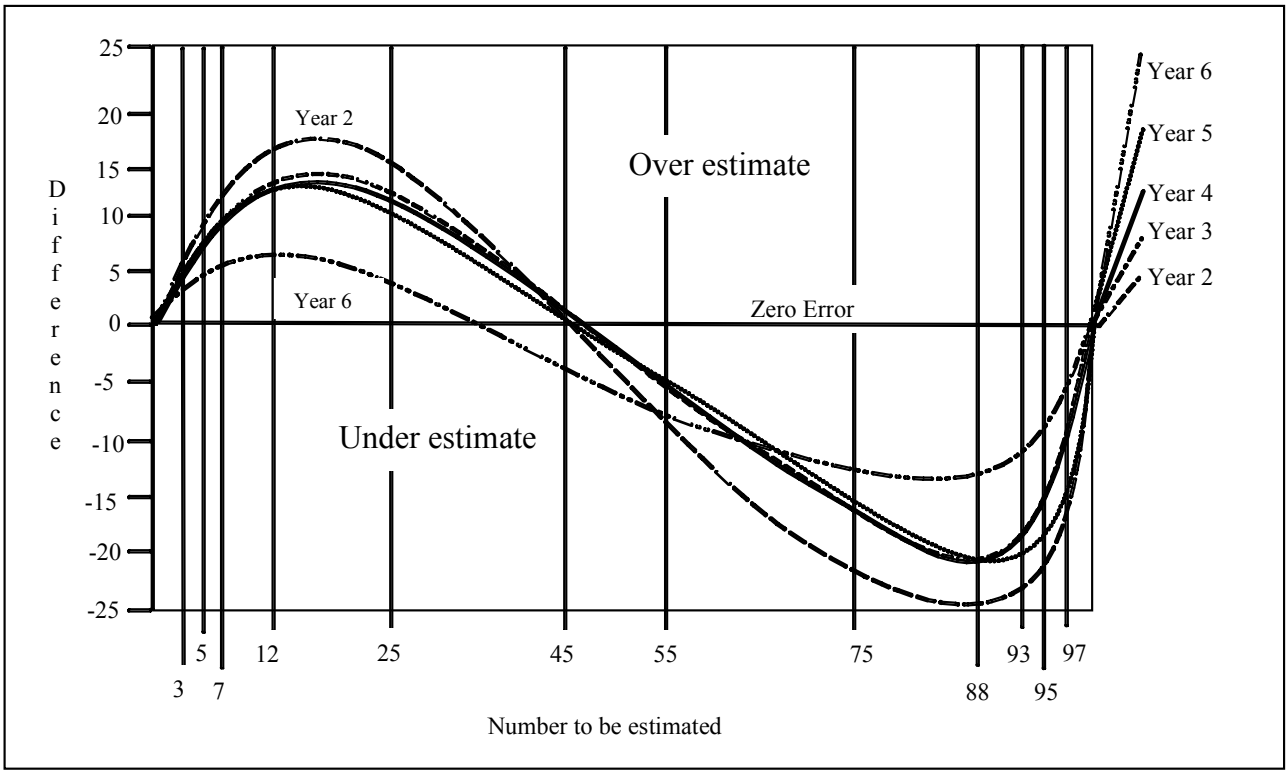

Figure 3: Summary of general trends in children's efforts to estimate position of numbers on a number line

underestimate the position of numbers in the second half.

- the maximum extent of underestimation can be greater than that of overestimation.
- the trends noted amongst children within years 3,4 and 5 are similar. Years 2 and 6 demonstrate departures from these in that differences identified within Year 2 are greater that those of all other years whilst those identified for Year 6 are less.
- the remarkable accuracy with which all year groups, apart from year 6, identify the position of 45. It is equally remarkable that they do not then reflect similar accuracy with 55 (since 45 and 55 are equidistant from the middle of the line segment.)

DISCUSSION

Given that two points marked on the number line segment were 0 and 100 it is perhaps unsurprising that as children are required to pinpoint numbers close to the extremes the errors that arise are generally small and unidirectional. Accuracy is increased because as numbers approach the end points there is a decreasing opportunity to estimate to an alternative side of the target — to make a negative error when pinpointing near zero, and to make a positive error when pinpointing near 100.

However, when the differences between the mean of the estimates of numbers within the extreme decades are considered views on end point accuracy seems less tenable. The mean deviation of the magnitude for 3, 5 and 7 is an overestimate of 5 while that for the numbers 93, 95 and 97 is an underestimate almost three times as large, -14. Although the children cannot place a number beyond 100 a sense that they display accuracy because the number is near 100 is questionable. Though they appear to illustrate number sense associated with a left to right orientation of a number line, thus enabling them to specify that 93 is less than 100 or that 7 is greater than 1 (Deheane, 1997), the relative equality of the magnitudes of these numbers in the context of their defining frames seems to be missing despite their experiences following the NNS.

What is striking from Figure 3 is not so much that children of different ages appear to be more accurate in estimating magnitudes on the 0 to 100 number line segment line but that the distribution of the magnitudes should have such a remarkable similarity. The general evidence would seem to support the conclusions of Seigler and Booth (2004) in that the children exaggerate magnitudes in the lower half of the line but underestimate them in the upper half. However, the nature of the distribution of the differences does not support either a linear or a logarithmic theory. Nor does it allow us to subscribe to the view that with the range of children we considered there is a change in the pattern of the distribution and although there is some evidence of growing accuracy this appeared to stagnate between years 3 and 5

Over 45% of the children below Y6 within this study chose to estimate the position of a number by counting and marking ones starting from zero. A further 45% used

counting without marking. Only 20% of year 6 counted from zero marking counted ones. Although there was very limited evidence of children applying a strategic approach to their estimations (halving and quartering of the number line), the degree of accuracy associated with 55 and 45, particularly the latter, seems to suggest an additional explanation. The visible points 0 and 100 provide cues as to the position of numbers relatively close to them, they serve as "correctors" where adjustments could be made. However, we also suggest that a middle number, possibly 50, may also serve the same purpose but its effect is limited (as also appears to be the case of the visible correctors). Certainly the bias towards underestimation and overestimation within the frame presented to these children is worthy of further study. It may be that a clearer view of the children's behaviour can be related to Fischer's (1994) suggestion that the bias is associated with perception, both real and imagined and that in some instances perception overcomes procedural approaches. Whatever the real reasons the results would suggest that experience advocated by the NNS do not appear to translate into strategic approaches that would include the application of general number knowledge to a sophisticated representation.

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LINKING THEORIES IN MATHEMATICS EDUCATION

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Mathematics education as a scientific discipline is very different today from how it was when David Tall, Eddie Gray and other colleagues entered the field. At the time, explicit theoretical considerations were the exception and were important to only a few dedicated individuals; today every respectable contribution to a conference or journal is expected to have an explicit theoretical framework.

I entered mathematics education as a researcher in theoretical physicist with some experience in high school mathematics and physics teaching. The field in which I had specialized, mathematical physics, is theory par excellence, with empirical physics playing only a minor role. But even experimental physics researchers work within strict theoretical boundaries, and often the purpose of their empirical investigations is the construction of theoretical models. In the vast majority of cases, they work for a long time within a single theoretical framework, and contribute to "normal science" (Kuhn, 1962). In this way, they amass large amounts of data. Except in the very rare cases, in which data clearly refute a theory, theories are basically stable; they may be elaborated or connected, but they have a tendency to grow slowly and steadily.

Our physical world is governed by four different types of interaction, acting at very different scales: the gravitational, the electromagnetic, the strong and the weak forces. Most of us are best acquainted with the electromagnetic interaction because it is manifest at our own scale. The gravitational interaction is relevant only at much larger scales, and the other two at much smaller scales. Physicists attempt to find what they call a Grand Unified Theory (2006), unifying three of these interactions, or even a Theory of Everything, intended to provide a single framework for describing all four interactions. The idea that such a theory exists is based on the belief that all four interactions are, in fact, manifestations of the same underlying basic principle. I will come back to these remarks later.

During and after my process of transition from mathematical physics to mathematics education, three people had an overwhelming influence on my professional formation and life in mathematics education, one in Israel, one in the US, and one in Europe. The one in Europe is David Tall. I learned from colleagues that David has had a similar influence on many others. There is also purely numerical evidence for this. For example, a look at the bibliography of the "PME-30 book" (Gutierrez & Boero, 2006) shows that with 55 citations, David Tall is by far the most cited researcher in the book. Eddie Gray, with 30 citations, occupies the very respectable 15th position out of about 300 cited researchers.

Since I began my research in mathematics education, wherever I turned, I found that David had already been there and left his mark. Some of the topics I was involved with and where I found David's prior work helpful are the use of computers, mainly

their dynamic features, for learning; advanced mathematical thinking, including the learning of the concepts of calculus; abstraction, in particular the notion of procept, which Eddie and David proposed; and more recently, the integration of theories in mathematics education.

In the present paper, I want to propose some thoughts on this last issue, and this is also the reason why I mentioned theoretical physics. In contrast to researchers in physics, we in mathematics education tend to invent theories, or at least theoretical ideas, at a pace faster than we produce data to possibly refute our theories. As a consequence, there is a considerable supply of theories, not all of them supported by a suitable amount of empirical evidence. Not infrequently, researchers are proposing a new theory, or modifying an existing one to fit, support or explain a collection of data they happen to have, and then go on to the next, quite independent research. Such a theory has never been challenged. A challenge could be that other researchers collect suitable data in a context different from the previous ones, and analyse these data in light of the theory.

Moreover, many theories in mathematics education are local in the sense that their domain of validity is limited. In view of the young age of our domain, this situation may be unavoidable, and even beneficial. Nevertheless, we should not forget that

- A theory will only provide insight if it supports and explains empirical data, and if it does this for a suitably wide range of data. For this purpose, we need to collect lots of data and analyse them in the light of the theory; we need to carry out normal science in Kuhn's sense;
- As a community of researchers, we should collaborate in order to reduce the number of theories and to increase their consistency.

I will here relate only to the second of these issues. As mentioned above, today every respectable contribution to a conference or journal has an explicit theoretical framework. *Educational Studies in Mathematics*, for example, requires that submissions "provide a well founded and cogently argued analysis on the basis of an explicit theoretical and methodological framework" (ESM, 2006). Theory-building is a crucial necessity for mathematics education to grow into a scientific discipline. David Tall and Eddie Gray have done their share to promote this idea, and so have many others. For example, the community of French researchers in mathematics education has shown how theoretical ideas like didactic engineering and instrumentation, and theories like the theory of didactic situations and the anthropological theory of didactics, have grown over many years in intensive interaction within a scientific community.

The awareness among researchers of the necessity of theory, has led to a multitude of theories, many of which serve only a limited group of researchers, for a specific kind of research. There are several reasons for this multitude of theories. One is the diversity of educational cultures and the relative isolation of many researchers in their own cultural environment: Mathematics education is different from physics in that it

grows within, or at least with the local or national education system. Frequently research sites are classrooms within that system, and research is linked to that system. This may make it difficult to adopt theories from one system to another.

Another reason is the complexity of our objects of research. In addition to the mathematical contents and the view of these contents adopted by the education system and by classroom teachers, and in addition to the cognitive aspects of learning processes concerning these contents, researchers may want or need to take into account the declared or hidden aims of the system and of the teacher, students' personal learning histories and their beliefs about mathematics, physical conditions such as the availability of computers and the frequency and kind of use that students are making of them, social conditions of the learning environment, affective and motivational issues, and other factors. Since any specific research study can take only some of these factors into account, different studies have different theoretical needs; and theories may be more or less well designed to deal with the different aspects of the learning process. As a result, many researchers may find it more convenient and more efficient, to produce their own theory rather than to read, digest, and adopt theories produced by others.

In order not to offend others, I take our own research group as an example. Rina Hershkowitz, Baruch Schwarz and I (2001) have, over the past ten years, built a theory of abstraction in context. This theory is designed to serve as an analytical tool for the description of processes of emergence of students' new knowledge constructs, taking into account the social, physical, and historical context in which these processes occur. At the same time, our theory fails to deal with other aspects that may be relevant for these processes such as gesture and movement, or students' affect. Nor have we, heretofore, tried to examine whether the theory can be used as a predictive tool for the processes of emergence of knowledge constructs. Other topics, such as institutionalization or issues concerned with entire curricula maybe orthogonal to the theory of abstraction in context.

Our theory-building effort has certainly been influenced by our personal experience, without us being aware of this influence. We may be unaware of hidden assumptions we have made. Our theory may therefore fail to be useful in some educational systems. We are interested in putting it to the test, but this is not easy since only positive results tend to come to our attention. Indeed, a number of other researchers have found it useful for their own purposes and have adopted (and in some cases adapted) it. Some such instances are mentioned in Hershkowitz, Hadas, & Dreyfus (2006). Nevertheless, one of the reviewers of the latter paper recommended rejecting it because "this does not really extend a large body of research, it merely adds to the framework of a small group of researchers".

Groups using a particular theory may be smaller (such as in the case of abstraction in context) or larger (such as in the case of the theory of didactic situations), but this is secondary. Essential questions concern the distinctiveness of each theory, and the

overlap between different theories; the measure to which they can, or cannot be combined or integrated into a larger framework and the reasons for this, the extent to which we can connect theories and thus reduce their number. Some efforts in this direction have been made during the past few years, and once again, David Tall and Eddie Gray have been deeply involved.

In a Research Forum at PME 28, the idea leading the forum organizers (Eddie Gray and myself) was to deepen the community's understanding of processes of abstraction by looking in parallel at three theories, one of them represented by David Tall (Boero et al., 2002). The Forum showed that while the three presented theories have common aims such as the detailed description of processes of learning about abstract mathematical notions, they differ in many respects, for example in how they take into account the context in which the learning processes occur. Two of the conclusions were that

- Applying different theories to the same data could help researchers see how the theories are complementary and/or contradictory, and what their limitations are;
- Mathematics education at present needs models to fit specific research questions rather than big, unified theories.

Notwithstanding these conclusions, David Tall (2004) came forth soon thereafter with his encompassing "three worlds" view of mathematics (education), in which he proposes a unified way to see the development of mathematical thought from early childhood to research level mathematics, a unified view that allows for diversity in students' development.

In 2005, the program committee of CERME-4 called into being a working group on different theoretical perspectives and approaches. Once again, Eddie Gray played a central role in the leading team, and David Tall was an active participant (Dreyfus et al., 2005). The central term that emerged from the working group was *networking*. The overall conclusion was that there was no expectation that theories would be integrated into a "grand unified theory" in the near future. If we can develop and maintain a certain degree of networking between some of the advocates of the different theoretical stances that are currently evident within mathematics education, this will constitute an important step on the path towards establishing mathematics education as a scientific discipline. Collaborative work between teams using different theories with different underlying assumptions is necessary for substantial progress towards networking theories. Such collaboration could take the form of separately analysing the same data and then meeting to consider and reflect upon each other's analysis.

One example for such collaboration, albeit in a rather limited range of theories, all dealing with affective issue, is presented in a Special Issue of *Educational Studies in Mathematics* (Zan et al., 2006). This issue presents a welcome example of looking at one data set through different theoretical lenses. On the other hand, it does hardly

attempt to network the various theories. Collaborative research between research teams using different theories should also attempt to become aware of the underlying assumptions of each theory, to describe what each theory can explain, as well as what it can not explain, and thus how it could be supplemented by other theories. In other words, such collaboration is expected to exhibit what other theories can contribute to the understanding of phenomena produced by a specific theory, and thus eventually, what a group of theories can achieve in common.

David Tall and Eddie Gray have extensive research experience, a strong theoretical inclination and a suitably general view of mathematics education to have an important role to play in this endeavour. We should still expect a lot from both of them in the future.

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THE IMPORTANCE OF COMPRESSION IN CHILDREN'S LEARNING OF MATHEMATICS AND TEACHERS' LEARNING TO TEACH MATHEMATICS

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This paper examines the pragmatic and theoretical basis of the Connecting Mathematics for Elementary Teachers (CMET) Project. In particular we attempt to answer the following question: "Why connect research on how children learn mathematics with preservice teachers' learning of mathematics?" We answered this question easily at a pragmatic level but found that the notion of compression (i.e., for complex understanding to occur one must compress previous ideas into compact, more precise mathematical objects) was invaluable in justifying our theoretical basis. Using compression as a lens provided preservice teachers with insights into how children learn mathematics and also helped them with their own learning of mathematics. Focusing on compression also enhanced preservice teachers' motivation to learn mathematics. Our analysis indicated that the CMET approach, explaining how mathematical knowledge is compressed, was essential because some knowledge is not 'decompressable' without examining research on children's learning of mathematics.

The Connecting Mathematics for Elementary Teachers (CMET) Project attempts to help preservice elementary teachers connect the mathematics they are learning in content courses with how children learn and think about mathematics to tie research on children's learning of mathematics with practice. To this end we have developed a supplement that parallels the typical mathematics content course topics. Our intent in helping preservice teachers make these types of connections is that they will both improve their own understanding of mathematics and eventually improve their future teaching of mathematics to children.

The CMET materials primarily consist of descriptions, written for prospective elementary teachers, on how children think about, misunderstand, and come to understand mathematics. These descriptions are based on current research. Some of the connections related to the work of David Tall include: how children come to know number, addition as a counting activity (Gray & Tall, 1994), how manipulatives may 'embody' (Tall, 2004) mathematical activity, and how concept image (Tall & Vinner, 1981) relates to understanding in geometry. For example, we discuss how linking cubes may embody the concept ten in understanding place value and at a more sophisticated level of mathematical thinking Base Ten Blocks (Dienes Blocks) may be a better embodiment of the standard algorithms for addition and subtraction.

In addition to these descriptions the CMET materials contain:

- problems and data from the Third International Mathematics and Science Study (TIMSS) and the National Assessment of Educational Progress (NAEP)
- our own data from problems given to elementary school children,
- And questions for discussion by preservice teachers.

It is important to note that the CMET supplement is not a methods textbook. It is designed as a supplement for traditional mathematical content courses for elementary teachers; hence it is not a mathematics textbook either.

WHY CONNECT LEARNING

In analyzing both the theoretical and practical basis for our project, a significant question became: "Why should connections to research on how children learn mathematics be useful and necessary in preservice teachers' learning of mathematics?" This question can be addressed on several different levels. At the most basic and obvious level, making connections with how children learn mathematics helps preservice teachers see the usefulness of the mathematics they are learning and how they will apply and use this mathematics in their future teaching. As Bransford, Brown, and Cocking (1999) suggest, learners are more motivated to learn and develop richer understandings when they see the contexts of applicability. This explanation is one that both teacher educators and students might give for the benefits of such an approach. In fact, in our preliminary evaluations, students did provide similar reasons for why they thought our materials were useful. Many indicated that they are learning how children think and are reflecting on their future teaching. A few even mentioned that they had used the ideas presented in the supplement in opportunities they had to work with children. Hence, from a pragmatic and motivational perspective this approach was beneficial.

Another goal of our project was for preservice elementary teachers to improve their mathematical understanding. We maintain that by understanding how children understand mathematics, prospective teachers will enhance their own understanding of mathematics. Preservice teachers often have an instrumental (Skemp, 1987) view of mathematics because the mathematics prospective elementary teachers learn in content courses is disconnected from what they will be teaching. Typically they have only their own, often negative experiences learning mathematics, to relate to the mathematics they are learning. As a consequence, mathematics learning becomes a disassociated group of facts and procedures without meaning; mathematics is not learned as a sense-making activity. In post surveys some students indicated that the CMET materials helped them understand mathematics better. Making these connections then also helps preservice teacher understand mathematics relationally (Skemp, 1987). This understanding should likewise help them in their future teaching of mathematics to children.

Yet these practical justifications do not fully explain why such an approach is necessary. These justifications suggest that this approach will improve preservice teachers' future teaching and perhaps their motivation to learn mathematics. But we

further maintain that these connections will help prospective teachers in their development of mathematical understanding and, more particularly, help them achieve the mathematical knowledge that is necessary for teaching that goes beyond basic mathematical content knowledge (Hill & Ball, 2004).

COMPRESSION

A deeper analysis is necessary to address our overarching question as to why this type of approach is viable. A theoretical justification for the CMET project, and the primary focus of this paper, is the notion of *compression* (Tall, 2004; Thurston, 1990). Compression is the idea that in order for complex thinking to occur one must compress previous ideas into compact, more precise mathematical objects. The hugely complex organ, the brain, can only function by focusing on a few essential aspects of any given situation or problem at one time to make decisions. Consequently, other aspects and information must be suppressed. In order for the brain to convert information into manageable pieces, it must *compress* that information into a useable form. In order to achieve this feat, it is helpful for the mind to compress mathematical processes so that they can be thought of as mental mathematical objects. The mind can then operate with and on the object without recreating the process each time.

In order to understand the concept of compression, it is useful to consider David Tall's work on how children learn to count and to do arithmetic. Tall writes:

> [W]e need to look at how we human beings develop from a single cell and by successive cell-division which depends on our genetic structure, the embryo develops into a complex individual. As the cells divide, they specialize according to the genetic code where this cell is part of a finger-nail and that cell is a neuron in the brain. The child is born with a huge number of connections between the various parts of the brain linked to the rest of our body. There are myriads of connections between the brain and the eye, the brain and the ear, the brain and the hand, which require organising by experience. This making of connections in the brain is how the child learns. As the child moves its arm around, its hand may touch an object it sees and over time this action is remembered by strengthening the connections in the brain and the child begins to build up a sequence of actions known as 'see'-'grasp'-'suck' whereby it coordinates sight and touch to grasp an object to bring it to its mouth to taste.
>
> Arithmetic develops in exactly the same way. The parent talks to the child and sings nursery rhymes such as 'One, two, three, four five, once I caught a fish alive' and other activities which involve the first words of the number sequence; these lead to the learning of a sequence of numbers and the act of pointing successively to objects in a collection, one at a time, saying the numbers as we go, to build up the sequential organisation of the counting scheme. The child may count several toys 'one, two, three' and, no matter which order they are counted in (provided each one is counted once and once only), the count always ends with 'three'. The realisation that the number we reach when counting a set is always the same is called 'conservation of number'. It is technical language to say

that the number found from counting a particular set is always the same. The focus of attention changes from the action of changing to the concept of number.

The learning of the counting sequence and the conservation of number take many months, perhaps years to learn. It brings the first example of what Thurston (1990) calls 'compression'. In this case the counting schema, including all the pointing and saying of the sequence of numbers, is compressed into a single concept, the concept of *number*. While counting occurs in time and we have to *do* it, a number can be represented by a single number word and we can *think about* it. This allows the child to begin to use numbers themselves as thinkable things to operate with and to do arithmetic. However, before going on to do arithmetic, the child has a range of increasingly subtle ways of counting to learn to be able to do the operations more efficiently and more powerfully.

This notion of compression can also be applied at a more advanced mathematical level by considering the natural progression in the ways that children solve 4 + 5. At first children will need actual objects to count such as pencils.

The child initially counts each pencil in the first group, *touching* (and later in the child's development, pointing at) each pencil as he or she counts, then the child counts the second group in a similar manner, and then to the amazement of most adults combines the two groups and counts the total, again touching each pencil as the child counts. At a later stage the child may recognize that there are 4 pencils in the first group and simply 'count on' from four saying, "5, 6, 7, 8, 9". Here the child has *compressed* the concept of number rather than having to go through the procedure to count four objects in order to know that there are four; the child has compressed the counting procedure into a single number concept. The child no longer needs to count each group to know how many are in it, he or she can focus on the aspect of the number without having to make it.

By building on experience, the child develops more sophisticated and more compressed methods of doing arithmetic – the more compressed, the more powerful the technique. Later in the child's development when asked what is 4 + 5, he or she may respond, "4 + 4 = 8 and since 5 is one more than 4, 4 + 5 = 9." Similarly, using the knowledge that three and two makes five may lead the child to the idea that twenty-three and two makes twenty-five. Finally, the child, like the preservice teacher, instantaneously knows that 4 + 5 is 9. When asked why, he or she says, "I just know it" or "I have it memorized." When a learner has this kind of "instantaneous" mathematical knowledge, we call that knowledge "derived facts." What is interesting about derived facts is that they often represent multiple levels of compression.

In the last two examples, the child has compressed the process of counting to find a sum to a "mathematical fact." The child will eventually compress counting and known facts into standard algorithms. As a final example of compression, consider the ways that a child learns multiplication. To solve 3 x 6, a child will initially think of multiplication as repeated addition and that 3 x 6 means 6 + 6 + 6. When asked, the child might respond, "6 + 6 = 12" and then may count on: "13, 14, 15, 16, 17, 18." In the example the child uses his compressed knowledge (e.g., number, derived facts) to make sense of multiplication. The average child will eventually compress the notion of multiplication and continue the process of using compressed knowledge with more sophisticated mathematical thinking such as division. Layer upon layer of compressed knowledge is constructed.

Compression is a lens by which to view learning. In our case, the CMET Project, we are looking at the learning of mathematics after the fact. We are not trying to help children learn mathematics but rather describe how they learned mathematics. We are attempting to help preservice teachers learn mathematics by looking at how children, and they themselves, have compressed mathematical knowledge.

DECOMPRESSING MATHEMATICAL KNOWLEDGE

Our intent as mathematics educators is that preservice teachers gain a rich mastery and understanding of mathematics in an efficient, compressed format. This level of knowledge gives the teacher a powerful way for *them* to think about mathematics. It is often what is termed basic content knowledge. While content knowledge is necessary for teaching, it is not sufficient. To encourage children to learn mathematics in a powerful way it is necessary to have insight into how children go through the process of compressing knowledge from carrying out mathematical actions to compressing them into fluent processes and even further into mental objects that they can manipulate flexibly in the mind.

Our project materials describe the natural processes that children may use in doing mathematics and how children develop more sophisticated mathematical thinking through compression. From our perspective teachers should be able to decompress, or as noted by Gray and Tall, "decompose and recompose at will" (1994) their mathematical knowledge in order to teach children mathematics.

Elaborating on prior characterizations of compression, we make the following distinction: there are two types of compressed knowledge important for teachers. The first type is compressed knowledge that can be decompressed through reflective thought or by focusing on the learning process. For example, most adults, without too much thought, can explain that multiplication is repeated addition e.g., 3 x 6 is 6 + 6 + 6. However, the second type of compressed knowledge is so far buried in our learning that our conscious mind cannot recreate it. The only way to develop this knowledge is to study how children learn mathematics.

For instance, a closer analysis of how children might actually add to solve this problem reveals that how children might add is not how most adults add. A child who does not have his multiplication facts committed to memory might solve this problem this way, "I know that 6 + 6 = 12, and 12, 13, 14, 15, 16, 17 18." Those who work with children on a regular basis will recognize this solution as a reasonable response. Significantly, this solution strategy is not readily apparent or decompressable to most adults. An adult that does not work with children on a regular basis when asked how a child might solve 3 x 6 may indicate that the child would add or even count, but he or she probably would not arrive at the detailed solution strategy previously described. Perhaps somewhat surprising is the fact that as children we most likely solved multiplication problems in the same manner as the describe child, but most of us do not remember this process. We have compressed knowledge that we have forgotten how to decompress – we cannot decompress it. As a final illustration of the ways that adults forget how to decompress knowledge, consider how many adults actually remember having to *touch* objects in order to count them.

Our supplement is helping preservice teachers understand how mathematical knowledge is compressed. In one sense the supplement asks these adults to relearn the mathematics they have already compressed and constructed. We find that the teaching of content mathematics to undergraduates is conducive to introducing them to research on how children learn and think about mathematics. As they are studying undergraduate mathematical content, they are also learning how this knowledge is compressed by children and how they can decompress their own knowledge. Work like ours whether with preservice or practicing teachers is essential to participants' learning and development as a teacher. Certain mathematical knowledge can be easily decompressed through activities that focus now how children learn or reflective thought. In many ways teachers likely do this intuitively in their teaching. Likely asking themselves, "How did I learn this mathematical concept?" or "How do I understand this concept?" However, there is mathematical knowledge that is not easily decompressed or there are subtle aspects of decompressable knowledge that are not readily apparent. Research on children's mathematical thinking provides an avenue for helping teachers understand how children learn mathematical knowledge that is not readily decompressable.

CONCLUSIONS

The concept of the compression of mathematical knowledge is one reason that connections to how children learn mathematics are *useful* in preservice teachers' learning of mathematics. At the pragmatic level, understanding how children think mathematically is useful because it is more motivating for preservice teachers in their own learning mathematics. They may see how what they are learning is applicable to their future teaching. The notion of compression may also provide motivation for preservice teachers because they may gain a better understanding of mathematics as they come to understand how mathematical knowledge is compressed. They may see

how mathematical knowledge is compressed by children and themselves. They are able to examine mathematics from children's perspectives, not just from a mathematical perspective, which is sometimes alien to them. Understanding how children think is essential to enhancing teaching. However, one cannot understand how children think without the knowledge of how children compress mathematical knowledge.

It is important that prospective teachers gain both types compressed knowledge – knowledge that can be decompressed through reflection and knowledge that cannot be decompressed without examining research on how children learn. The CMET curriculum provides tools to help preservice teachers reflect on their own learning and consider how child learn mathematics. CMET is essential in helping these preservice teachers gain both types of compressed knowledge as the project emphasizes self-reflection and presents research on children's learning of mathematics.

The process of unpacking compressed knowledge is an integral part of preservice teachers' development as teachers. Understanding what mathematics children compress and how they compress that mathematics provides powerful insights to the teaching and learning of mathematics. Not only do preservice teachers learn how children think but they also develop deeper mathematical understandings by looking at children's mathematical thinking and at compression in particular. This knowledge is different from mathematics content knowledge. Likewise it is different from pedagogical content knowledge – knowledge of how to teach mathematics to children. Yet, this knowledge is reflexively related to both content and pedagogical knowledge and has a profound influence on both types of knowledge. In a broader sense we are attempting to incorporate a theoretical framework which uses research on children's learning of mathematics applied to teaching practices.

Compression is only one of the ideas that David Tall has contributed to the CMET project. As previously noted, his insight on 'embodiment' and 'concept image' was particularly significant contributions to our project. One of the daunting tasks in undertaking this project has been the breadth of mathematics covered in the elementary school curriculum. There were many areas which required extensive research on our part. To our amazement, and David's delight, he was quite versed and knowledgeable in each of these areas of mathematic research for the CMET project. Thank you David for all your help and support.

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CONCEPT IMAGES, COGNITIVE ROOTS AND CONFLICTS: BULDING AN ALTERNATIVE APPROACH TO CALCULUS

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In this paper, we describe an alternative approach for the concepts of limit and derivative in early calculus. This approach is grounded on the notion of local straightness as a cognitive root and on the theoretical framework designed in my doctoral work (Giraldo, 2004). We propose a learning sequence based on progressive cognitive expansion from the primitive perception of the global variations of a graph towards the formal definition of gradient. This sequence is supported by a specifically designed computational learning environment.

INTRODUCTION – FORMAL MATHEMATICS AND TEACHING MODELS

> A first fact must astonish us, or rather should astonish us, if we were not accustomed with it. How is it possible that there are people who do not understand mathematics? If mathematics evokes nothing but the rules of logic, those which are accepted by all the well developed spirits, if its evidence is founded upon principles that are common to all the men and that no one could not deny without being insane, how is it possible that there are so many people for whom it is entirely opaque?
>
> Henri Poincaré, 1908 (our translation)

One of the most challenging aspects of activity of mathematics teaching is the fact that it demands radically contrasting branches of knowledge: subjective mechanisms of human thinking and rational, sound rules of logic. Traditional mathematics teaching practices are often built upon tacit assumptions such as: *if the matter is explained clearly enough, then the students will understand it*; or *if they know the meaning of every word in a definition, then they will understand the mathematical meaning the defined concept*; or yet *if they understand the mathematical meaning of the concept, then they will be able to operate with it*. As a consequence, pedagogical organization and presentation often attempts to imitate the neat order of the subject's formal theoretical structure: axioms → definitions → propositions and theorems. The eventual failure of this approach leads mathematicians and mathematics teachers to face the puzzling question: *How is it possible that there are people who do not understand mathematics?*

The most extreme version of this teaching model is probably the so-called *New Math Movement*, which gained popularity in the 1960s. However, development of research in mathematics education opened ways to a number of alternative models.

In their (today classic) paper, Tall & Vinner (1981) used the term ***concept image*** to describe "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes". According to the authors, it is built up over the years through experiences of all kinds, changing

as the individual meets new stimuli and matures. In the same work, they define ***concept definition*** to be a form of words used to specify that concept. The authors claim that we may learn to recognise concepts from experience, and then appropriately use them in different contexts without knowing a precise definition for them at all. In other words, the absence of a concept definition does not impede the development of a rich and flexible concept. On the other hand, a concept definition may be meaningfully learnt or learnt in a rote manner, and it may or may not be coherent with the formal definition (that is, a concept definition which is accepted by the mathematical community). Moreover, the concept definition may or may not be consistently related with the whole concept image. An individual can successfully memorize a concept definition and know it by heart without having a rich concept image associated. In this case however, it is likely to be meaningless. Therefore, a broad concept image does not necessarily include an accurate concept definition (or any definition at all) and, conversely, the ability to state a concept definition is not necessarily associated with a rich concept image. This theory suggests that effective teaching approaches must not only focus the formal mathematical structure of the concepts, but also aim to enrich students whole set of ideas related to them.

Many authors have pointed out that formal definitions may embody obstacles to students in early stages of learning (e.g. Sierpinska, 1992). One first obstacle arises from the language itself: definitions are built upon words that have meanings in current language, but, in order to understand definitions, one must abstract from those meanings. The word *limit*, for instance, is often understood as a barrier that cannot be reached. Another – and perhaps stronger – obstacle is that most definitions are thoroughly unfamiliar to students. From a historical standpoint, formal definitions are products of development processes that, in many cases, took centuries to mature within particular social and theoretical contexts. Such aspects are merely bypassed if the pedagogical sequence that imitates the logical structure. When definitions are used as first introductions to concepts, they refer to alien objects that are completely unknown. As a consequence, students are likely to stick with the current language (if at all), instead of focusing on the intended mathematical meanings. Hence, teaching approaches based on the logical order of concepts – axioms → definitions → propositions and theorems – are actually anti-pedagogical inversions.

COGNITIVE ROOTS AND LOCAL STRAIGHTNESS

The other side of the coin is the teaching strategy according to which teachers attempt to informally motivate concepts using intuitive references from learners' previous experiences. In the hope to "simplify" the matter, teachers present ideas in a restricted context, in which they hold properties that are not valid in general. Incautious use of such a strategy may also bring about obstacles and consequently narrow students' concept images. For instance, calculus teachers often appeal to students' experiences with tangent straight lines to circles and other elementary (convex) curves. These images are likely to lead to misconceptions such as: *if a*

straight line is tangent to a curve at a point, it must not intercept the curve a second time; or *if a straight line is tangent to a curve at a point, then the whole curve must remain on one same side of the straight line*.

The cognizance that both logical-formal and intuitive-informal teaching strategies may result in narrowed concept images naturally leads to a question: *which features must hold an appropriate introductory idea to a mathematical concept*? A possible answer is given by Tall (1989). The author defines a ***cognitive root*** to be "an anchoring concept which the learner finds easy to comprehend, yet forms a basis on which a theory may be built". In Tall (2000), this notion is re-formulated as a cognitive unit holding two fundamental conditions: (i) to be (at least potentially) meaningful to the student at the time; and (ii) to contain the seeds of cognitive expansion to formal definitions and further theoretical development.

In calculus teaching, introduction to the concept derivative is traditionally supported by the well known picture of a tangent straight line being graphically approximated by secant straight lines, which accompanies the definition by means of limit of the ratio: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$. The notion of *secant* is likely to appear in learners' concept images as *a straight line that crosses the curve more than once*. Therefore, the presentation of *tangent* as an antithesis of *secant* stress the idea of tangent as *a line that touches the curve at a single point* – which certainly does not match with the concept of tangent in the context of infinitesimal calculus. Besides, the formal notion of limit is not an appropriate cognitive root for derivatives, because, as we have argued, condition (i) above does not hold, even though condition (ii) certainly does.

Tall (1989) claims that the notion of ***local straightness*** is a suitable cognitive root for the concept of derivative. The idea of local straightness is based on the fact that a (differentiable) curve looks straight when highly magnified. According to the author, it is therefore a profound representation of the notion of gradient of a graph, which appeals directly to the sense of vision and the mental images of curves and allows the gradient function to be seen as the changing gradient of the graph itself. It can also be used to give insights into highly subtle ideas. In an approach based on the notion of local straightness, the gradient of a differentiable function at a point is introduced as the slope of the straight line which the graph mingles with when highly magnified on the neighborhood of the point. Tall (2000) affirms that "local straightness' is a primitive human perception of the visual aspects of a graph. It has global implications as the individual looks *along* the graph and sees the changes in gradient, so that the gradient of the whole graph is seen *as a global entity*".

In Tall (1986), a ***generic organizer*** is defined to be "an environment (or microworld) which enables the learner to manipulate *examples* and (if possible) *non-examples* of a specific mathematical concept or a related system of concepts". Tall (2000) claims that generic organizers allow the development of new sequences in curriculum starting, not from *mathematical foundations*, but from *cognitive roots*. Therefore, the design of a generic organizer must be based on a central cognitive root.

The generic organizer associated to the notion of local straightness is a computational environment that allows users to change graphic windows and observe consequent changes in a curve's appearance. Therefore, learners may observe that a differentiable curve looks straight when magnified, whilst a non-differentiable one does not; for instance it may have corners, or look progressively more wrinkled.

DESCRIPTIONS AND CONFLICTS

Concept images, concept definitions and cognitive roots theories suggest that formal definitions should figure not as starting points, but as goals of mathematics teaching. A concept definition can only be meaningful if it is comprised within a broad and flexible concept image. Therefore, teaching must focus on continuing expansion of learners' concept images. On that account, effective approaches should include a wide range of representations, pictures and references to each concept. Each reference or representation to a mathematical concept keeps particular limitations, as it they place emphasis on certain aspects of the concept but overlook others. Hence, no reference or representation can provide a faithful picture of the concept. In the case of real functions, for instance, three main representations are traditionally employed, namely algebraic (formulae), graphical (graphs) and numeric (tables). Formulae make explicit the precise relation and exact values but do not give visual insight. Graphs provide visual insight into properties, such as growing and maxima, within given limits but leave precise values and details unclear. Tables provide discrete examples and insight into variations but give no information about the behaviour on continuous subsets of the domain.

In my doctoral work (Giraldo, 2004), I defined ***concept description*** to be any reference to a mathematical concept, stated with a pedagogical context, which does not exhaust the concept, that is, which comprises limitations, in the sense that it stress some facets, but overshadows others. Thus, descriptions may be oral or written references, symbols, mathematical notation, sketches, diagrams, pictures and so forth. In the case of the concept of function, formulae, graphs and tables used in classroom are instances of descriptions. Besides, computer environments provide a collection of dynamic descriptions for many concepts. When we mention 'the function x^2' in the classroom, for example, we usually mean the function $f : R \to R$ that maps each real number x into its square x^2 (although it is not clearly stated in the sentence). That is, we are describing the function f.

Such limitations may lead students to situations of confusion. For instance, depending on constraints such as scale and horizontal and vertical ranges, a graph may give restricted visual information about the function displayed. If a student sticks with this information, he or she may be led to wrong conclusions. I considered (Giraldo, 2004) a ***conflict situation*** to be the perception of a (at least apparent) contradiction brought out by the limitations associated with one or more description (see also Giraldo et al 2003, 2004). Mathematics education literature has reported positive and negative effects of conflict situations.

In Brazil, Abrahão (1998) observed the behavior of a group of teachers dealing with computer outputs displaying potentially contradictory information. The author reported that, in most of the cases, the participants accepted the visual information from the machine, even when clearly conflicting with their previous knowledge. For instance, in one of the interviews, a teacher was shown the graph of a third degree polynomial function that looked like a parabola due to graphic windows limitations. He then affirmed that it *was* a parabola, although he had claimed, a few minutes earlier, that the graph of a third degree polynomial could not have that shape. Abrahão conclude that, in some cases, computational limitation seemed to have *atrophied* participants' concept images.

In Giraldo (2004), I defined the ***narrowing effect*** to be a process as the one instanced by Abrahão's experiment: the limitations associated with descriptions lead to limitations on the concept images consequently developed by learners.

On the other hand, Hadas et al (2000) presented a set of activities on a dynamic geometry environment designed to cause uncertainty from situations in which the construction seemed to contradict students' intuition. The number of deductive explanations increased considerably and the authors conclude that proofs were brought into students' actual arguments. Similarly, in my own work (Giraldo, 2004), I observed undergraduate students in Brazil dealing with conflict situations triggered by the confrontation of computational descriptions for derivatives and limits with other kinds of descriptions. There has been evidence found that, for five out of six participants, the experience with conflicts acted to *expand* concept images. Results have been presented in Giraldo et al (2003a), Giraldo et al (2003b). This research suggested that, within a pedagogical approach suitably designed, conflict situations may act not to *narrow*, but to *expand* concept images. Therefore, narrowing effects usually are not associated with the *occurrence* of conflicts, but with their *absence*.

If a single description is used as the predominant reference in a pedagogical approach, we should expect the limitations associated with that reference to narrow learners' concept images. On the other hand, if descriptions are placed within a rich context, in which the associated limitations are emphasized, rather than avoided, they can be very positive. No single description can exhaust a concept, but a rich set of descriptions can act to expand concept images.

AN APPROACH TO THE CONCEPT OF DERIVATIVE

In this work, I outline the proposal of a pedagogical sequence for the introduction of the concept of derivative, based on the framework described in the previous section. It is organized in five steps, starting from the cognitive root – local straightness – and developing towards the formalization. In each step a particular kind of description is employed. The limitations associated with each of those were used to introduce the following steps. Before the beginning of the sequence itself, students were given tasks that involved graph sketching on the computer, changing graphic windows, observing and analyzing consequent changes in the curve aspect. All the

computational activities were performed with software Maple V. This proposal was tested in a year-long undergraduate calculus course. Results are currently being analyzed.

Step 1: Numerical Approximation

Activities involving local magnification were first presented. Students were asked to sketch the graph of a function f on the computer and gradually approximate the window near a fixed point $(x_0, f(x_0))$. They were instructed to stop the process when the curve acquired the aspect of a straight line and then approximate the slope by substitution, employing the usual formula: $\frac{\Delta y}{\Delta x}$. After each evaluation, students were encouraged to compare their results with classmates and observe that the values would be different, but close to each other. This discussion was used to stress that the local magnification is an approximation process: curve looks like a straight line, but does not really become a straight line. The conflict situations engendered by the fact that different students obtained different values naturally prompted the questions: *Is there a real number that represents the slope of the straight line which looks like the curve? Is it possible to obtain this value analytically?*

Step 2: Algebraic Approximation

Using a magnified curve as reference, students were given the task of finding the slope a of the straight line the looks like the curve near the point $(x_0, f(x_0))$, using the approximation formula: $f(x_0 + h) \approx ah + f(x_0)$. To aid evaluation, polynomial and simple rational functions were used. The real number a was then presented as *the local gradient of the curve at the point*. They could then start to perceive the gradient as a magnitude depending on the point, that is, as a function. When performing the algebraic approximation, student naturally eliminated the terms with order greater than one. This led to the questions: *Why does this procedure work? In what sense does this straight line approximate the curve?*

Step 3: Geometrical Differentiation

After evaluating the slope a, students were asked to sketch the straight line and the curve in a magnified window (where they mingled), then gradually enlarge it and observe the straight line detaching from the curve. In the preceding steps the existence of straight line which "best approximates" (or the tangent) is suggest by means of numeric and algebraic approximations. Now, this line is geometrically actualized. In the traditional approach, the gradient appears as an attribute of a line, which is external to curve and "happens to have" a geometrical relationship with it (tangency). Differently, in this approach, the gradient appears as an attribute of the curve, which is deeply related to its local behavior – and the tangent line is a geometrical actualization for the gradient. In the former case, the gradient is an external property that has a vague relation with the curve, whilst in the latter, it is arises as an intrinsic property of the curve, an expression of its local picture.

Step 4: Conceptualization

This step was supported by a specially designed generic organizer, *BestLine* (Giraldo, 2001). For a detailed description, see Giraldo & Carvalho (2003). The aim is to make clear the mathematically precise meaning of the local approximations. The first inputs of the generic organizer are the function f, a point x_0, and a value for the slope a (which need not be the derivative of f at x_0). Then, it displays the graph and the graph of f and the straight line $r(x) = ax + f(x_0)$. It allows the user to perform the local magnification, by choosing $h = \Delta x$ (the horizontal length of the graphic window), and to observe the numeric variation of two magnitudes: the difference between the straight line and the curve, $\rho(h) = |f(x_0 + h) - r(h)|$; and the ratio between this difference h, $\frac{\rho(h)}{h}$. By comparing the behavior of the difference and the ratio, for $a = f'(x_0)$ and $a \neq f'(x_0)$, learners are led to conclude that the difference $\rho(h)$ vanishes, whichever is the value of a (provided that the function is continuous), but the ratio $\frac{\rho(h)}{h}$ tends to zero if, and only if, $a = f'(x_0)$. Thus, one might say that any straight line crossing the curve "approximates" it, in the sense that $\rho(h)$ tends to zero. However, only the tangent line makes $\frac{\rho(h)}{h}$ tend to zero. In this sense, it is *the best local approximation to the curve*. The generic organizer provides numeric and graphic descriptions for this fact. As $\frac{\rho(h)}{h}$ vanishes, it might be said that $\rho(h)$ tends to zero "faster" than h. Since h sets the horizontal length of the window, it acts as a scaling unit for the picture. If $a = f'(x_0)$, the curve and the straight line quickly become undistinguishable as the user zooms in. If $a \neq f'(x_0)$, the user is able to tell them apart throughout the whole process. Therefore, the experience with the generic organizer may be used to suggest the following interpretation: *The tangent line becomes closer to the curve faster than the user can zoom in*.

Step 5: Formalization

In this final step, the definition of derivative as limit was introduced as natural re-writing of the expression for the approximation established in the previous step.

RESULTS AND REMARKS

In this paper, I have briefly reviewed the concept images and cognitive roots theories and describe their influence on the theoretical framework developed on my doctoral work. The pedagogical sequence reported above was designed as an application of this framework. The research addresses the influenced of this learning sequence on students' concept images for derivatives and limits. On that account, the research design is based on a qualitative study with a sample of five students. The methodology is based on semi-structured and task-based interviews. Global results are currently being analyzed. Partial results are presented on Giraldo & Carvalho

(2006). Some excerpts of the interviews with two participants, Eduardo and Flávio (pseudonyms), are transcribed bellow.

Eduardo (step 1) Roughly speaking, we could compare it [the local magnification process] to the view of the Earth planet we'd have if we observed it from the space, something like a sphere. But, if we'd come back and observe it from the surface again, it has details, which were undistinguishable before.

Flávio (step 4) So, this guy [points $\rho(h)$ on the screen] will have to decrease faster than h. Because if h went faster, than this guy would ever remain, and this length here [points the vertical segment on the screen] would remain too. […] But in the case of the derivative it doesn't happen. We see it's a very special characteristic. […] There exist infinitely many straight lines that are really close to the curve at the point. At the point, they all coincide anyway. […] But the derivative is that one which glues to the curve. Actually, I think it's even more than that.

Flávio (step 4) We actually have found a way of rewriting the definition. But it's more […] things aren't so hidden, you know? It's on one's face. You can feel what's going on. Actually we did write the definition, but this way is much stronger than the way we all know, with the limit and so on. Here it really shows the relation of those two magnitudes, if we divide one by the other, we compare them, and realize what goes on when it tends to zero.

As the excerpts illustrate, results confirm Tall's argument that local straightness is a suitable cognitive root for derivatives. In fact, Eduardo's statement confirms that it is a familiar notion, which belongs to students' actual experiences and knowledge. Flávio's statements show that the experience acted to expand his concept image and also to deepen his understanding of the definition of derivative. The main aim of this research is to investigate the design an alternative to the "axioms → definitions → theorems" approach – a pedagogical sequence that allows learners to naturally expand concept images and, consequently, to build up solid formal understanding.

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VISUALISATION AND PROOF: A BRIEF SURVEY

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The contribution of visualisation to mathematics and to mathematics education raises a number of questions of an epistemological nature. This paper is a brief survey of the ways in which visualisation is discussed in the literature on the philosophy of mathematics. The survey is not exhaustive, but pays special attention to the ways in which visualisation is thought to be useful to some aspects of mathematical proof, in particular the ones connected with explanation and justification.

FOREWORD

Let us start by wishing Happy Retirement to David, the relentless teaser of brains, who inspired so many students and colleagues to pursue diverse and often rather wild ideas in mathematics education, and showed amazing tolerance for the many ways of carrying out research in that field. In his well-deserved retirement we also wish David good health, happiness and a continuing enjoyment of the world's wonders. We recommend taking the scenic route, always. Since one of David's many interests is the use of visualisation, we hope the following brief survey will be of interest to him.

INTRODUCTION

A number of mathematicians and logicians have investigated the use of visual representations, in particular their potential contribution to mathematical proofs (Brown, 1999; Davis, 1993; Giaquinto, 1992, 2005; Mancosu, 2005). Over the past twenty years or so these investigations have gained in scope and status, in part because computers have increased the possibilities of visualisation so greatly. Such studies have been pursued at many places, such as the Visual Inference Laboratory at Indiana University and the Centre for Experimental and Constructive Mathematics (CECM) at Simon Fraser University in British Columbia. At most of these institutions, the departments of philosophy, mathematics, computer science and cognitive science cooperate in research projects devoted to developing computational and visual tools to facilitate reasoning.

A key question raised by the intensified study of visualisation is whether, or to what extent, visual representations can be used, not only as evidence or inspiration for a mathematical statement, but also in its justification. Diagrams and other visual representations have long been welcomed as heuristic accompaniments to proof, where they not only facilitate the understanding of a theorem and its proof, but can often inspire the theorem to be proved and point out approaches to the construction of the proof itself. And of course every mathematics educator knows that they are essential tools in the mathematics curriculum.

It is only in the last two decades or so, however, that visual representations have begun to be considered seriously as substitutes for traditional proof. Today there is

still much controversy on the role of visual representation in proof, and a number of researchers are actively pursuing the topic. In their positions on this issue these researchers span a broad range. At one extreme are those who say that visual representations can never be more than useful adjuncts to proof, as part of their traditional role as facilitators of mathematical understanding in general. At the other extreme are those who claim that some visual representations can constitute proofs in and of themselves, rendering any further traditional proof unnecessary.

Between these two extremes one finds a variety of positions that are more nuanced or perhaps simply less clear. Some authors, for example, do not envisage a visual representation constituting an entire proof, but would maintain that an appropriate visual representation is acceptable as an integral component on which the proof as a whole would stand or fall. Other authors seem to be hesitant or inconsistent in their positions, and to this extent the division of the rest of this paper into three seemingly well-defined sections is of necessity somewhat forced.

VISUAL REPRESENTATIONS AS ADJUNCTS TO PROOFS

Francis (1996), for example, maintains that the increased use of computer graphics in mathematical research does not obviate the need for rigour in verifying knowledge acquired through visualisation. He does recognize that "the computer-dominated information revolution will ultimately move mathematics away from the sterile formalism characteristic of the Bourbaki decades, and which still dominates academic mathematics." But he adds that it would be absurd to expect computer experimentation to "replace the rigour that mathematics has achieved for its methodology over the past two centuries". For Francis, then, visual reasoning is clearly not on a par with sentential reasoning.

Other researchers have come to similar conclusions. Palais (1999), for example, is a mathematician at Brandeis University who worked on a mathematical visualisation program called 3D-Filmstrip for several years. Reporting on his use of computers to model mathematical objects and processes, he observes that visualisation through computer graphics makes it possible not only to transform data, alter images and manipulate objects, but also to examine features of mathematical objects that were otherwise inaccessible. Palais concludes that visualisation can directly show the way to a rigorous proof, but stops well short of saying that visual representations can be accepted as legitimate proofs in themselves.

VISUAL REPRESENTATIONS AS AN INTEGRAL PART OF PROOF

Very few assert that proofs can consist of visual representations alone, but a number of researchers do claim that figures and other visual representations can play an essential, though restricted, role in proofs. Casselman (2000), for example, having explored the use and misuse of pictures in mathematical exposition, concludes that a picture can indeed form an essential component of a proof.

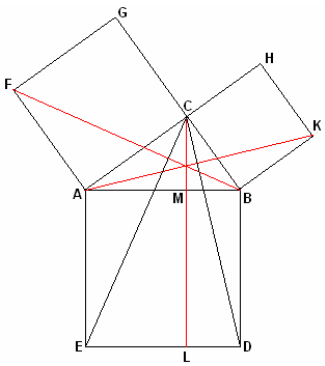

Figure 1: Pythagoras

Taking his cues from Tufte (1983, 1990 & 1997), Casselman first formulates eight suggestions for creating illustrations that foster mathematical understanding. Two of these are that "… the figures themselves should tell a story" and that one should "… ask constantly whether the figures really convey the point they are meant to" (Casselman, 2000, p. 1259). He goes on to give several examples of good mathematical illustrations, most of which adhere to the principles of visual explanation set out by Tufte (1997). The most striking is his comparison of the traditional picture for proving the Pythagoras theorem (see Figure 1) with 16 pictures taken from a computer-animated series based upon the area-preserving property of shears. (This series is now familiar to many students.) He considers the animation to be measurably better than the traditional figure, which lacks explanatory power.

Casselman (2000) not only points out the importance of pictures for understanding, however, but goes on to state that "In spite of disclaimers and for better or worse, pictures—even if only internalized ones—often play a crucial role in logical demonstration" (p. 1257) and can "convey information, sometimes a whole proof " (p. 1260).

The term visualisation is most often applied to public acts of communication: using a diagram or other representation as a vehicle to convey a mathematical idea, to explain or convince. For Giaquinto (1992, 1993), however, visualisation is an individual experience that takes place in an internal mental space. Giaquinto, a philosopher of mathematics, is concerned with the epistemic aspects of this inner experience.

One of Giaquinto's prime interests is the use of visual imagination to *discover* mathematical truths. "One *discovers* a truth by coming to believe it in an epistemically acceptable way." (Giaquinto, 1992, p. 382) This conception of *discovery* allows Giaquinto to make two important claims. The first is that discovery requires independence: One must come to believe on one's own terms; one cannot blindly accept another's assertion. The second is that epistemic acceptability hinges upon a larger congruency: A discovery is not valid if it conflicts with other independently acquired beliefs.

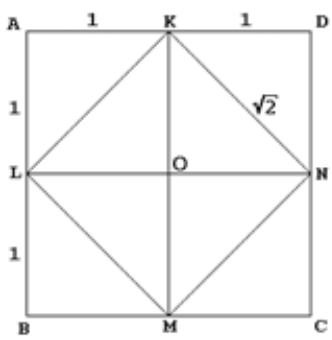

Figure 2: Doubling the square

Giaquinto differentiates between discovery and demonstration. One can believe in a discovery without having a valid justification (Giaquinto, 1992, p. 383). Alternatively, one might read a justification of a claim without being able to discover its truth. Thus he is interested in studying the role of visualisation in discovery without making any claim that it has a role in the construction of proof. He cites a few examples, among them the famous construction of the doubling of the square in the slave boy

episode of Plato's *Meno* (see Figure 2), to show that "the visual way of reaching the theorem illustrates the possibility of discovery without proof" (Giaquinto, 2005, p. 77).

Borwein and Jörgenson of CECM examined the role of visualisation in reasoning in general and in mathematics in particular, posing to themselves the two questions: "Can it contribute directly to the body of mathematical knowledge?" and "Can an image act as a form of 'visual proof'?" They answer both these questions in the positive, though they would insist that a visual representation can be accepted as a proof only if it meets certain criteria.

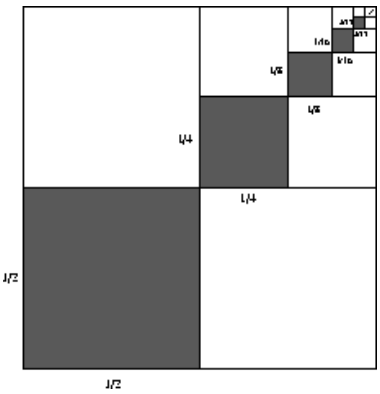

Figure 3

In arguing their position, Borwein and Jörgenson (1997) cite the many differences between the visual and the logical modes of presentation. Whereas a mathematical proof has traditionally been presented as a sequence of valid inferences, a visual representation purporting to constitute a "visual proof" would be presented as a static picture. They point out that such a picture may well contain the same information as the traditional sequential presentation, but would not show an explicit path through that information and thus would leave "the viewer to establish what is important (and what is not) and in what order the dependencies should be assessed." For this reason these researchers believe that successful visual proofs are rare, and tend to be limited in their scope and generalizability. They nevertheless concede that a number of compelling visual proofs do exist, such as those published in the book *Proofs without words* (Nelsen, 1993). As one example, they present the heuristic diagram which aims to prove that the sum of the infinite series 1/4 + 1/16 + 1/64 + ... = 1/3 (See Figure 3).

Borwein and Jörgenson suggest three necessary (but not sufficient) conditions for an acceptable visual proof:

- Reliability: That the underlying means of arriving at the proof are reliable and that the result is unvarying with each inspection
- Consistency: That the means and end of the proof are consistent with other known facts, beliefs and proofs
- Repeatability: That the proof may be confirmed by or demonstrated to others

One might wonder whether these criteria would not apply to proofs in general, not only to visual ones. One might also object that the first criterion in particular, lacking as it does a definition of "reliable," does not provide enough guidance in separating acceptable from unacceptable visual proofs. Indeed, Borwein and Jörgenson make no claim to have answered this question definitively. Nevertheless, they would not only assign to visual reasoning a greater role in mathematics in general, but would also maintain that some visual representations can constitute proofs.

Davis (1993) advocates a conception of theorem that includes images, figures and computer graphics. He offers a "wide definition" of visual theorems that includes (a) all the results of elementary geometry that appear to be intuitively obvious, (b) all the theorems of calculus that have an intuitive geometric or visual basis, (c) all graphical displays from which pure or applied mathematical conclusions can be derived almost by inspection and (d) graphical results of computer programs which the brain organizes coherently in a certain way (Davis, 1993, p. 336).

By "theorem" Davis seems to mean the statement of a mathematical result, not of a justifying argument. Nevertheless, his discussion indicates that he often sees a figure as being explanatory enough to make another proof redundant. For example, he claims that the theorem that circles cannot be plane-filling inside a circle is made trivial by a figure of a circle containing three smaller free-floating circles (Davis 1993, p. 337-8). In the case of fractals, Davis states that the "visual theorem" gives us information about the mathematical objects that may be difficult or impossible to put into words. He says that the figure is "the passage from the mathematical iteration to the perceived figure grasped and intuited in all its stateable and unstateable visual complexities" (Davis, 1993, p. 339).

Although Davis does not dwell on this point, in both these cases he would appear to believe that the figure must be accompanied by a verbal or formulaic presentation. A circle containing three circles, or a fractal image, no matter how visually evocative, would not constitute a piece of mathematics. Davis (1993) is perhaps best read as asserting that a figure, because of its explanatory value but in addition to it, could be an integral part of proof.

VISUAL REPRESENTATIONS AS PROOFS

Other researchers go further when challenging the idea that visual representations are no more than heuristic tools. Barwise and Etchemendy (1991, 1996) sought ways to formalize diagrammatic reasoning and make it no less precise than deductive reasoning. They acknowledge that the notion of proof as a derivation, consisting of a sequence of steps leading from premises to conclusion by way of valid reasoning, and in particular the elaboration of this notion in mathematical logic, have contributed enormously to progress in mathematics. They claim, however, that the focus on this notion has led to the neglect of other forms of mathematical thinking, such as diagrams, charts, nets, maps, and pictures, that do not fit the traditional inferential model. They also argue that it is possible to build logically sound and even rigorous arguments upon such visual representations.

These two researchers proceeded from what they call an informational perspective, building upon the insight that inference is "the task of extracting information implicit in some explicitly presented information" (Barwise and Etchemendy, 1996, p. 180). This view leads them to a criterion for the validity of a proof in the most general sense: "As long as the purported proof really does clearly demonstrate that the

information represented by the conclusion is implicit in the information represented by the premises, the purported proof is valid" (p.180).

The authors go on to say that whenever "there is structure, there is information", and that a visual representation, which may be highly structured, can carry a wealth of information very efficiently. Because information may be presented in both linguistic and non-linguistic ways, they conclude that strict adherence to inference through sentential logic is too restrictive, inasmuch as sentential logic is only a linguistic representation.

The question is how to extract the information implicit in a visual representation in such a manner as to yield a valid proof. Barwise and Etchemendy show examples of informal derivations, such as the use of Venn diagrams, and suggest that perfectly valid visual proofs can be built in a similar fashion upon the direct manipulation of visual objects. Unfortunately, as they point out, the focus on sentential derivation in modern mathematics has meant that little work has been done on the development of protocols for derivation using visual objects, so that there is much catching up to do if visual proof is to realize its considerable potential.

Though the view of these researchers is that proof does not require sentential reasoning, they do not believe that visual and sentential reasoning are mutually exclusive. On the contrary, much of their work has been aimed at elaborating the concept of "heterogeneous proof." Indeed, Barwise and Etchemendy (1991) have developed *Hyperproof,* an interactive program which facilitates reasoning with visual objects. It is designed to direct the attention of students to the content of a proof, rather than to the syntactic structure of sentences, and teaches logical reasoning and proof construction by manipulating both visual and sentential information in an integrated manner. With this program, proof goes well beyond simple inspection of a diagram. A proof proceeds on the basis of explicit rules of derivation that, taken as a whole, apply to both sentential and visual information.

Few philosophers of mathematics make the explicit claim that diagrams or other visual representations can constitute an independent method of justification. One of the strongest advocates of this position is Brown (1997, 1999), whose stance is closely related to his Platonist view that mathematics deals with real objects having an independent existence. For him "*Some 'pictures' are not really pictures but rather are windows to Plato's heaven"* (Brown 1999, p. 39).

In this context, Brown presents a number of theorems concerning sums and limits, and follows the statement of these theorems with "picture-proofs". Each of these consists of a single figure. He then gives a traditional proof for comparison. Brown's presentation implies that he believes these figures alone constitute proofs on the same level as the traditional arguments that follow them, consistent with his stated position (Brown 1997, p. 169-172; Brown 1999, p. 34-7).

It is not entirely clear how Brown comes to his conclusion that some visual representations constitute proofs. He seems to be using an analogy: just as proofs can

be convincing and explanatory, so too figures can be convincing and explanatory. In essence, Brown appears to believe that a proof is anything that is both convincing and explanatory – and thus that any visual representation which satisfies those two criteria is a proof.

Just by looking at them, however, one cannot even understand how most of these "picture-proofs" function as representations of mathematical objects, much less as valid mathematical arguments. Generally one finds that one has to apply to them a reasoning process, in the form of sentences, in order to understand the theorem and be convinced of its validity. Brown fails to make the case that this reasoning process, a traditional mode of mathematical thinking, is unnecessary.

Folina (1999) provides useful and succinct criticisms of Brown's account, concluding that "... not every kind of convincing evidence for a mathematical claim counts as a proof. In particular, Brown does not show that a picture, or anything 'picture-like', can be a proof. In my view, he does not really argue for this" p. 429.

CONCLUSION

This brief survey shows that we are still far from fully understanding and agreeing upon the role of visualisation in mathematics and mathematics education. While one can expect differences of opinion to continue to exist on the role of visualisation in proof, there is certainly room for more effort aimed at better ways to use visualisation in its universally accepted role as an important aid to mathematical understanding.

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DIVERSITY OF STUDENTS' SOLUTIONS OF A WORD PROBLEM AND THE TEACHER'S EDUCATIONAL STYLE

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Eddie Gray and David Tall's idea about the diversity of student's reactions to some cognitive challenges is used as a tool for the analysis of the educational style of a teacher. A given word problem was solved by nearly 500 Grade 3 pupils. Each solution was characterized by five to nine phenomena. The characterizations of all the pupils of the same class were put together to make a characterization of the teacher's educational style .

INTRODUCTION

It is a great honour for me to be a speaker at such a prestigious meeting. I would like to express my deep thanks to the organizers for the opportunity.

This paper presents the second phase of research which deals with a list of 10 phenomena. The first phase focused on the phenomena 1 to 5 and the results of this phase were published in SEMT 2005. In this paper I focus at phenomena 6 to 8. The last two phenomena have not yet been analysed.

THE BACKGROUND TO THE IDEA OF DIVERSITY

Let me start with a personal remark. I met Eddie Gray for the first time ten years ago. It happened at the one week Zbygniew Semadeni's seminar organized for PhD students in Poland. One afternoon we (Eddie and I) had a long walk during which Eddie explained to me his and David's idea about how the diversity of student's reactions to some cognitive challenge indicates his/her intellectual ability. Eddie showed me a lot of examples and described a lot of extremely interesting experiments. All he presented to me was in a perfect agreement with my own experiences. All that he told me, I knew, but I did not know that I knew this. He changed my sleeping experience into awakened knowledge. Within two years this idea has been incorporated into my lectures in didactics of mathematics for future teachers.

At the beginning of 2005 the diversity idea profoundly influenced one aspect of my research. At that time I had analysed a set of more than one hundred third grade (9-10 year old) pupils' solutions to the following word problem.

> Father had 20 wheels. He decided to make scooters and three-wheelers. How many of which can he make?

The author would like to express his thanks to all the schools and teachers who were willing to participate in the experiment.
The research has been supported by grant GAČR 406/05/2444.

The goal of the research was to map and classify pupils' different interpretation of this not uniquely formulated task. I observed that in one class, 9 different interpretations appeared and in another class all the interpretations, except for one, were the same. This fact led me to the hypothesis that the Gray-Tall diversity idea can be used possibly as a tool for the analysis of the educational style of a teacher. This made me change the previous goal of the research to the new one: to find those phenomena in pupils' solutions which can be used for the characterization of the educational style of the teacher of a given class.

METHOD

During the first half of 2005 about 50 teachers were asked to pose the given problem to their classes. 28 of the teachers who were asked (20 from CZ, 5 from Slovakia and 3 from Poland) were willing to participate in the research. The problem was presented to the pupils under following conditions:

- Before the Problem is given to the class, a teacher speaks to pupils about scooters and a three-wheelers to remind them that they have two and three wheels.
- Pupils are informed that their solution would not be evaluated. They are given enough time for their work. Pupils are asked to put all their work on the given paper.
- Possible pupils' questions are answered in a supportive way but no additional information is given. E.g. the question "Is it necessary to find several solutions?" is answered by "If you think so, you can give several solutions".

The majority of nearly 500 pupils' solutions were analyzed from the point of view of the new goal of the research. So far 10 phenomena suitable for the characterization of the educational style of a teacher have been identified. The results of the analysis of phenomena 1 to 5 have been published in SEMT 2005. Each phenomenon is discussed separately. In this paper I will focus on phenomena 6 to 8 and try to describe two different educational styles of two teachers by analysing pupils' solution in two classes denoted here by A and B.

PHENOMENA DESCRIBING THE SOLUTION TO THE GIVEN PROBLEM

The description of a pupil's written solution is decomposed into 10 phenomena.

1. Grasping. We examine how and if there is evidence about the pupil's grasping process. In general, grasping starts with reading the text and terminates with the conscious awareness "*I am going to do this and this*".

2. Interpretation. Eleven different interpretations of the problem appeared. Nine of them are coded in table 1. Two additional cases - coded (L) - concerns solvers considering that all the vehicles constructed by the father had to be of the same kind: either scooters, or three-wheelers. If they thought the father had only 20 wheels, their interpretation was coded (L1); if they thought, the father had 20 + 20 wheels, their interpretation was coded (L2).

How many wheels has father to use	How many solutions do I have to find?		
	one	more	all
all 20	I1	I2	I3
20 or 19	J1	J2	J3
not limited	K1	K2	K3

Table 1. Interpretations

3. Language. Four different languages were recognized in the solutions: letters (A)[1], ordinal numbers (C), calculations (D, E), graphs and schemes (G), and pictures (H).
4. Strategy. 17 codes are split into 7 classes and two additional signs are used: (↑) for the improvement in strategy and (↓) for the deterioration in strategy
    a. Insight (A1), marginal notes (A2), trial and error strategy (A3), magic (A4).
    b. Taking away twos and threes from 20 using numbers (B1) or marks (B2).
    c. Adding twos and threes up to 20 using numbers (C1) or marks (C2).
    d. Decomposing 20 wheels into the scooter and three-wheeler parts (D1), division 20:2 (D2), division 20:3 (D3), division 18:3 (D4).
    e. Grouping wheels in pairs (scooter + three-wheeler): 4.(2+3) = 20 (E1), handling pairs (2+3) (E2), subtracting 20 – 5 – 5 – 5 – 5 (E3).
    f. Wheels are drawn and we are not able to say how they were used by the solver.
    g. Strategic 'mess' and 'groping' without any meaningful step (G1), with at least one meaningful step (G2).
5. Answer. Let [*a,b,c*] represent the pupil's answer: *a* scooters, *b* three-wheelers and *c* wheels left. Then there are
    a. 4 results within interpretations I: [10,0,0] (A), [7,2,0] (B), [4,4,0] (C), [1,6,0] (D),
    b. 3 more results within interpretations J : [2,5,1] (E), [5,3,1] (F), [8,1,1] (G),
    c. 37 more results within interpretations K provided the case [0,0,20] is accepted (Q),

[1] Each code must be considered together with the phenomenon to which it is linked. E.g. sign (A) is used for coding three different phenomena: grasping, language and answers.

d. 3 answers within interpretation L: "*10 scooters or 6 three-wheelers*" (H1), "*10 scooters, 6 three-wheelers*" (H2), "*10 scooters and 6 three-wheelers*" (H3).

A partly correct result (Y), an incorrect result (X), a result is found but the answer is missing (Z), a mistake or incorrectness in articulating the correct result (W) (e.g. [10,9,1] instead of [5,3,1]), some commentary is added (K). Further coding: complete result (1), incomplete result (2) (e.g. [2,5,-]).

6. Communication. The range of information (words, numbers, pictures, schemas, tables, letters, icons,…) which we can find in a pupil's solution. Of particular importance are
   a. individual ideas which cannot be found in others pupils' solutions
   b. a pupil's remarks about the solving strategy used
   c. a pupil's explanation about the interpretation of the problem.
7. Approach. Three different solving approaches to the problem are identified:
   a. prescribed – a solver follows procedures taught in school, suggested by the teacher
   b. individual – a solver uses his own analysis of a given situation
   c. uncertain – we are not able to decide type of approach.
8. Activity orientation. Teachers' educational beliefs influence pupils' activity. Thus pupils' written solution allows us to make a conjecture about the teachers' educational beliefs. In some cases we discussed these conjectures with the teacher directly.
9. Climate. A hypothetical scale for 'measuring' pupil's peace of mind during his/her work. Fear is on the bottom of this scale and joy and curiosity is at the top.
10. Handling errors. A brief look at this phenomenon pointed to 4 classes. A pupil
    a. is not aware of the error or misconception in his/her solution,
    b. knows that there is an error in his/her solution and he/she gives up (resigns),
    c. is aware of the error and repeats the same solving strategy,
    d. is aware of the error and tries a different solving strategy.

CLASS A

18 pupils, 7 girls, 11 boys. Test was written on 12. 04. 05.

Communication in the class is very slight. To prove this we looked at all the written, calculative, and visual information included in the pupils' solutions.

Two boys did nothing. They returned empty sheets.

words	0	1	2	4	6	9	14
pupils	5	1	1	1	7	2	1

Average : 4.5 words/pupil

Table 2. Distribution of Words

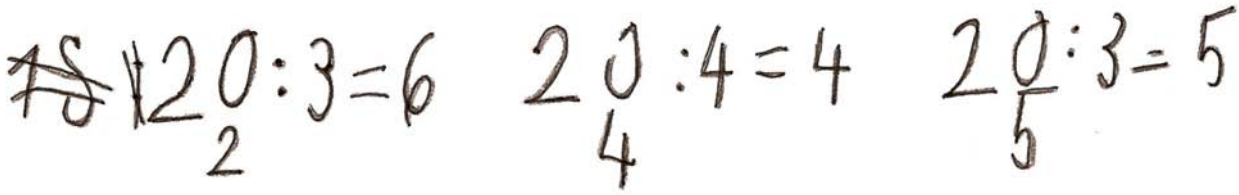

Figure 1. Richard's Work

Written information is poor. Distribution of all the 81 words written by 18 pupils is given in table 2.

Seven pupils (3 girls, 4 boys) used just 6 words: "Father can make x scooters and y three-wheelers" where 'x' and 'y' are two numbers. Two boys added to this answer 3 more words.

Only Richard wrote more than 10 words. It is worth noticing that only he found more than one solution. His solution consists of three divisions (Figure 1) and his answer consists of 6 numbers and 14 words: "Father can make 3 scooters and 6 three-wheelers or 4 scooters and 4 three-wheelers or 3 scooters and 5 three-wheelers". However we see that the solving strategy of this most communicative pupil is erroneous.

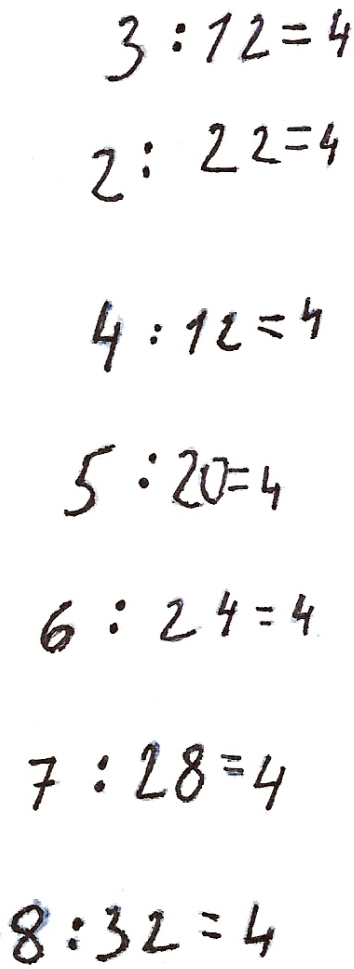

Figure 2. Eve's Work

Calculative information is scanty as well. The set of calculations (mostly division) is also poor. One girl and two boys had no calculations at all. Nobody gave more than 7 calculations. Eve with 7 calculations put a correct answer "4 scooters and 4 three-wheelers" and set of divisions written in an improper way (see Figure 2). The second and the third equation in this column are incorrect. We are not able to find the causal linkage between these calculations and the correct result.

There is no visual information given in this class. No pupils drew illustration, schema, graphs or tables.

Conclusion: the communication in this class is very low.

The pupils' approach is largely instructive. There is a lack of autonomous ideas. To prove

this we can look at pupils' solving strategies and again at words used in the pupils' solutions.

Only three pupils, Alice, Tommy, and Eric were able to grasp the problem correctly and find a reasonable solving strategy. Alice put down "20:3 = 6 (2)" and answered "Father can make 1 scooter and 6 three-wheelers". Tommy put down "18:3 = 6" and gave the same answer. Eric presented the nicest solution: "2 + 3 = 5, 20:5 = 4, Father can make 4 scooters and 4 three-wheelers".

In all other cases either the answer is missing (6 pupils) or it is incorrect (3 pupils) or the solving strategy is questionable (6 pupils) as we saw in Richard's (Fig.1.) and Eve's (Fig. 2.) solutions. The majority of pupils of this class are not prepared to analyse a problem which is not standard.

If we look at all 81words written by the 18 pupils we see that all of them come from the text of the problem. Just three words are "new": Two of them in Richard's solution: the word 'or' is used here twice. The third "new" word is in Dirk's solution: "Father can make 11 scooters and 8 three-wheelers and 0 wheels remain." The last word is the only attempt of autonomous verbal presentation in this class. Everything else is just a repetition (frequently incorrect) of instructions taught in the school.

Activity orientation is given by the teacher's belief that "demanding word problems like the given one are too difficult for primary pupils". She told me that her pupils are very good in solving word problems in which one operation namely addition or subtraction is used. She showed me a test written by her class a week ago comprising two problems:

1. Mike had 57 marbles. He lost 18 of them. How many marbles Mike has now?

2. Nina is 5 and her grandmother is 62 years old. What is the difference between grandmother's and Nina's age?

The teacher allowed me to look through the pupils' solutions. I saw about a dozen of them and in all of them the words "lost" and "difference" were underlined and in all these solutions a correct operation namely subtraction, was used. The teacher informed me that in this school there is a custom to have the 'headmaster test' on one day at the end of the school year. It is a test taken by all the classes of the school. The test is prepared by the head of the school herself. With pride, she showed me the results from the last year. Her class was first in mathematics amongst all five third-year classes.

This teacher believes that instructive teaching is the most effective educational style. Her pupils are very good in solving standard problems and very poor in solving non-standard problems.

CLASS B

31 pupils, 19 girls, 12 boys. Written 19. 10. 04.

Communication in this class is high: written, calculative, and also visual.

words	3	5	6	7	8	9	11	12	15	17	19	20	23	24	26	28	29	30	41	44
girls		2	2	3	1			1	1	1		1	1	1		1	1	1	1	1
boys	2	2	1	2	1	1	1				1				1					
all	2	4	3	5	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Table 3. Distribution of Words

Verbal communication is rich. Distribution of all 443 words used is given in table 3.

The average words/pupil is 14.3 (17.6 in girls and 9.1 in boys). The winner in this competition is Barbara with 45 words. Her solution can be decomposed into three parts. The first one is in Figure 3 and consists of the illustration (4 scooters and 3 three-wheelers, a number line with the word "reminder" ('zbitek' spelled incorrectly) and the answer with 9 words and 2 numbers: "father can make 5 scooters and 3 three-wheelers. (It) remains him 1 wheel".

The second part of Barbara's solution consists of a calculation "3+3+3+3+2+2+2+2" and answer "father can construct 4 scooters and 4 three-wheelers".

Most interesting is the third part of Barbara's solution in which the girl explains her solving strategy: "I found it since if one scooter has 2 wheels and one three-wheelers has three wheels so I calculated how many go to 10 and those others again to 10".

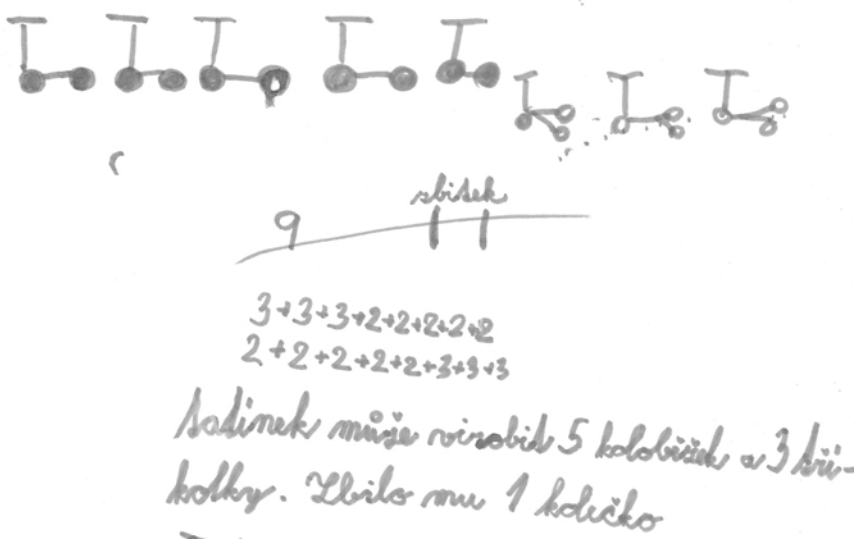

Figure 3. Barbara's Work

At the other end of the verbal scale is Arthur with only three words. He started to write a long sentence but than he erased everything and put down 3 words "father can create" and 8 lines of calculations: (original is not suitable for being copied).

3 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 = 19

6 + 2 + 2 + 2 + 2 + 2 + 2 + 2 = 20

9 + 2 + 2 + 2 + 2 + 2 + 2 + 2 = 19

12 + 2 + 2 + 2 + 2 = 20

15 + 2 + 2 = 19

18 + 2 = 20

2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 = 20

3 + 3 + 3 + 3 + 3 + 3 = 18

The architecture of these calculations shows pupils deep insight into the problem and his solving strategy. He choose the J3 interpretation (find all solutions using either 20 or 19 wheels) and in the last two rows interpretation L (all vehicles constructed by the father had to be of the same kind). One can hardly find a more economical presentation of all the results.

Calculation, compared to written or visual information is relatively poor. The one presented by 8 lines written by Arthur is the largest. On the other hand, the majority of calculations are meaningful which is very rare in other classes.

Visualization is the strongest area of communication in this class. Only 9 out of 31 pupils' solutions are without any visualization (6girls and 3 boys). The most impressive argument for pupils' ability to use illustrations and graphs in handling mathematical problems is figure 04. (Letters K and T stay for scooter = koloběžka, and three-wheeler = trojkolka). Here we see schemas or illustrations of three-wheelers used by 15 pupils. In two cases (pupils 116 and 118) there are two different visual representation of a three-wheeler, in pupil 121 there are even three.

Conclusion: the communication in this class is high.

The pupils' approach is highly autonomous. In all the observed phenomena, great diversity were identified. In the problem interpretation 9 out of 11 possibilities appeared in the pupils' solutions. 17 different solving strategies were used and even the underachievers tried to find some solution on their own. One of them did a lot of calculations and drawings on both sides of a sheet and in spite of the poor quality of her work she twice wrote "evaluated by myself for 1-" (i.e.

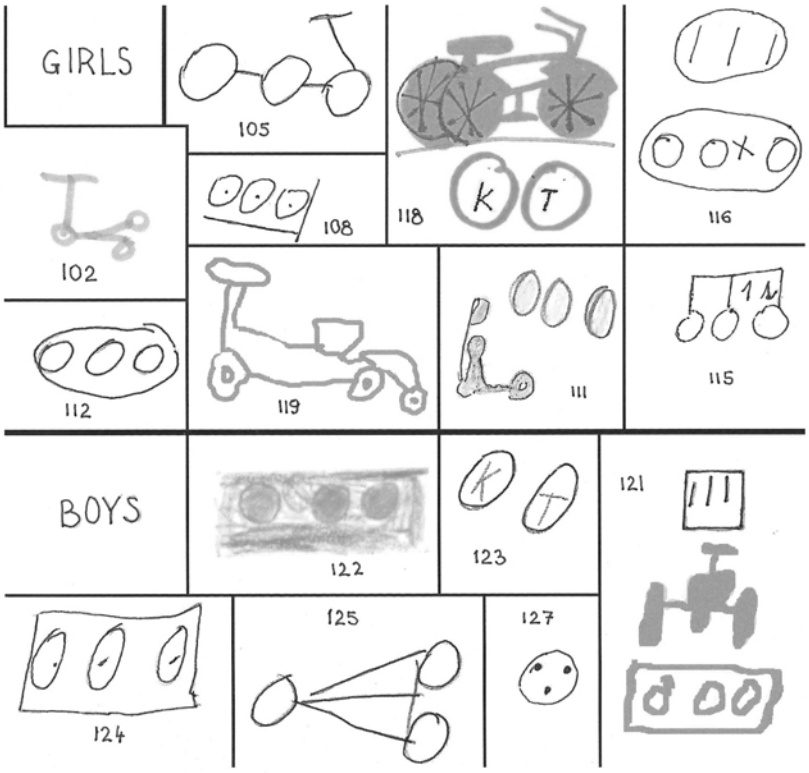

Figure 4. Diversity of Response

nearly the best). Probably the most convincing proof of the autonomous approach of the pupils is the diversity of pictures of three-wheelers on figure 4.

Activity orientation is given by teacher's belief that "some of these pupils are very clever and we learn from each other (smiling); and none of them is lazy" after a while she added "we all love to solve problems like that one".

The teacher is able to create such a climate that even a weak pupil tries to solve the given problem on his/her own. We did not find coping amongst the pupils (which was frequent in other classes). We did not observe pupils fearing to make a mistake but we found more cases in which a pupil changed his/her solving strategy several times.

CONCLUSION

The research is in progress. We hope that it will be possible to create such the guide for a teacher so that he/she will be able to analyze his/her pupils' solutions of the problem and infer valuable suggestions for the improvement of his/her work.

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RECONSIDERING THE IMAGINARY SPHERE

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In this paper I discuss Tall's (2004a) 'Three Worlds of Mathematics' theory of mathematical development. By revisiting an earlier exchange of papers (Inglis, 2003; Tall, 2004b) I discuss where one particular historical example of mathematical development fits into the theory. Whereas the theory began life as an attempt to classify the ontology of mathematical objects, I suggest it has now become an all encompassing theory of mathematical 'modes of operation'. I argue that these two aims may be mutually exclusive, and that the original aim should be abandoned. Finally, I close the paper by replying to Tall's (2004b) discussion on the language used within the theory, and suggest that using the words 'concept' and 'object' interchangeably is unhelpful.

THE THREE WORLDS OF MATHEMATICS

Gray and Tall (2001) put forward the suggestion that there are three distinct types of concept in mathematics; and the idea is appealing. During the preceding decade two highly influential theorists had proposed that learners come to understand mathematics through a sequence of considering processes on objects, encapsulating or reifying the processes into new objects, and then constructing further processes on these encapsulated objects (e.g. Dubinsky, 1991; Cottrill et al, 1996; Sfard, 1991). Such a model of mathematical development is certainly attractive, and can be readily applied to many areas of mathematics: Gray and Tall's (1994) important paper on early arithmetic perhaps providing the most persuasive example.

However, as noted by Tall (1999), there are large problems associated with assuming that all mathematical objects are encapsulated processes: what of geometrical shapes, or of axiomatic structures? It is very difficult to see what process has been encapsulated to become, for example, the discrete topology. Recognising this, Gray and Tall (2001) proposed that there are three "fundamentally different" types of object, the ontological origins of which should be considered separately:

> For several years now [...] we have been homing in on three [...] distinct types of concept in mathematics. One is the embodied object [...] Another is the symbolic procept [...] The third is an axiomatic concept in advanced mathematical thinking. (Gray & Tall, 2001, p. 70) .

Thus, by categorising mathematical objects into three different categories, Gray and Tall (2001) neatly avoided the over-application of their proceptual brand of process-object theory. However, this threefold categorisation has since been expanded into the notion of "three worlds of mathematics". Not only are there three different types of object, but they are operated with in different ways (Tall, 2002, 2004a, 2004b):

- The conceptual-embodied world "grows out of our *perceptions* of the world and consists of our thinking about things that we perceive and sense, not only in the physical world, but in our own mental world of meaning" (Tall, 2004a, p.285). Geometrical shapes and Euclidean proofs belong in this world.
- The proceptual-symbolic world consists of concepts which are encapsulated processes. Following Gray and Tall (1994), these *procepts* are distinguished by their symbols, which are used to "switch effortlessly from processes to *do* mathematics to concepts to *think* about" (Tall, 2004a, p.285).
- Finally, the so-called formal-axiomatic world consists of defined objects that are manipulated with logical deductions. These formal objects are then incorporated into new axiomatic structures to define new objects and so on. It is here that the discrete topology lives.

Far from simply being a categorisation of three different types of object, the three worlds became a characterisation of three entirely different types of cognitive development; each with a different type of object, a different mode of operation, and a different warrant for truth. This jump from a threefold categorisation of mathematical concepts to a threefold categorisation of mathematics is something of a leap. Indeed, Tall (2004b) wrote:

> The conceptual leap from three forms of concept to three worlds of mathematics may seem simple, but in practice it has proved to be both profound and daunting. It is one thing to have an insight into three different kinds of mathematical concept formed in different contexts; it is a much greater leap to claim that there are (at least) three distinct worlds of mathematics each with a different mode of development and […] each with a different kind of warrant for mathematical truth. (Tall, 2004b, p.31).

Tall is correct, the jump from a threefold categorisation of objects so as to avoid the over-application of a pre-existing theory, to a full threefold categorisation of the whole of mathematics is indeed a big conceptual leap. But is it justified?

WHAT KIND OF OBJECT IS THE IMAGINARY SPHERE?

When I first came across the three worlds of mathematics I was intrigued by the idea. What had originated in an attempt to avoid making the same mistakes as other theorists by applying their theory too widely, had become an all encompassing account of mathematical development; it would be hard to imagine a theory claiming more generality than that! Could this bold claim, that there are three distinct ways of constructing and operating with objects, be sustained?

I thought not. In a short exploratory paper (Inglis, 2003), I argued that there were examples of mathematical objects which do not fit into Gray and Tall's (2001) threefold categorisation. I briefly recapitulate those arguments here.

A particularly interesting example of 'cognitive development' comes from studying the research of the 18th century Swiss mathematician Johann Heinrich Lambert. In common with all mathematicians of his time, Lambert was interested in the problem

of Euclid's parallel postulate. Concerned that this over-complicated axiom did not seem as obviously true as Euclid's other axioms, he attempted to prove that it was indeed 'true'. Instead of trying to construct a direct proof, however, Lambert tackled the problem from reverse. He showed there were three possibilities:

1. If there is a triangle with angle sum <180°, then every triangle has angle sum <180°.
2. If, as in standard Euclidean geometry, there is a triangle with angle sum =180°, then every triangle has angle sum =180°.
3. If there is a triangle with angle sum >180°, then every triangle has angle sum >180°.

If Lambert could successfully show that cases 1 and 3 were self-contradictory he would have shown that the parallel postulate was 'true'. He quickly disposed of the first case, but the third proved more difficult. Indeed, we now know that this task is impossible: in 1868 Beltrami managed to show that the third case is as consistent as the second (Gray, 1989).

Although Lambert struggled to find a contradiction, he did manage to derive some rather interesting results. In particular, he found that, in the third case, the area of a triangle (with angles α, β and γ) was proportional to $\pi-(\alpha+\beta+\gamma)$. Recalling that the area of a triangle on a sphere of radius r is $r^2(\alpha+\beta+\gamma-\pi)$, he noticed that letting r=i gives $i^2(\alpha+\beta+\gamma-\pi)=\pi-(\alpha+\beta+\gamma)$, which is the formula Lambert derived for the angle sum under the third hypothesis. He commented:

> I want to say: if of two triangles one has a greater area than the other then the angle sum of the first triangle is smaller than that of the other […] I should almost therefore put forward the proposal that the third hypothesis holds on the surface of an imaginary sphere. (Fauvel & Gray, 1987, pp.518-520).

Crucially, Lambert is not defining his 'imaginary sphere' by the geometry that is true on its surface. He is pointing out that, given a sphere of radius i, the geometry on its surface would satisfy the third hypothesis.

But what kind of mathematical object is Lambert's imaginary sphere? In which 'world' does it belong? In my 2003 paper I argued that it does not satisfactorily fit into any of them. It is not based upon an embodiment of things we perceive or sense; it is not proceptual as it has neither process nor symbol; and, for Lambert at least, it was not axiomatic. Indeed, interestingly, he believed it *couldn't* be axiomatised as that would be contrary to the 'true' postulates of Euclidean geometry.

I suggested that Lambert's conception of an imaginary sphere is an example of a mathematical object that does not fit easily into any of the three worlds. Created by

analogy with symbolic equations, neither based on formal definition nor perception, it seems to lie somewhere between all three.[1]

Of course, this observation is hardly a great threat to the pillars of the theory, it merely points out the danger of over-ambitious claims of generality. However, when considering the issue, Tall did not agree with my analysis. In the next section I briefly review Tall's (2004b) response, which was interesting for several reasons, and raise the question of what exactly the purpose of the theory is.

IS THE IMAGINARY SPHERE EMBODIED?

Tall (2004b) explained how Lambert's imaginary sphere fitted into the three worlds framework in a slightly unexpected fashion. He wrote:

> [The imaginary sphere] is clearly an example arising out of the proceptual world of symbolic manipulation in which the algebra of spheres $x^2+y^2=r^2$ [sic] is applied to the case where $r^2=-1$. In this context the conceptually embodied meaning of spheres no longer applies. [...] The imaginary sphere is part of the natural process of extending the manipulations of symbols that have meaning in the proceptual world to a situation where the corresponding link to the embodied world no longer holds. It is parallel to the idea of using the square root of a negative number as manipulable symbol before it has a conceptual embodiment. (Tall, 2004b, p.31).

Of course Tall was correct to say that Lambert's sphere arose out of symbolic manipulations from the proceptual world, but the parallel with standard complex numbers is suspect: in the case of considering the roots of negative numbers, there existed a well known *procedure* of taking square roots. A complex number is clearly proceptual: it has procedure, symbol and object. There is no parallel at all with Lambert's sphere, which has neither procedure nor symbol. Although it may have arisen out of manipulating procepts, it itself is clearly not proceptual. However, Tall continued:

> The fact that neither Inglis nor I can 'see' an embodiment, just as Cardan and Tartaglia could not 'see' a conceptual embodiment of $\sqrt{(-1)}$ is not a denial of the distinction between the conceptual-embodied world and the proceptual-symbolic world. […] In my publications I have used the term 'embodiment' with a meaning that I believe is consistent with the colloquial notion of 'giving a body' to an abstract idea. This includes all cases of conceptions in visuo-spatial terms, not only those that arise from perception of actual objects. As a result of Inglis's intervention, I have found that many individuals interpret my writing to refer only to perceptual embodiment. I have therefore moved to alert the reader to this fact and now use the named 'conceptual embodiment', at least in the initial stages, to alert the reader to my intended meaning. (Tall, 2004b, pp.30-31).

Tall was correct to argue that there is a clear distinction between the conceptual-embodied and proceptual-symbolic worlds, and he seems to be hinting that the

[1] In the 2003 paper I gave the similar example of a four dimensional cube constructed through analogy with lower dimensional cases (Inglis, 2003).

imaginary sphere should be considered to be a conceptually-embodied object which belongs in the embodied world. Even though it cannot be perceived, it is nevertheless embodied.

Whilst this slight clarification of what the conceptual-embodied world is welcome, I will argue that it, unfortunately, introduces several additional problems into the theory, and raises important questions regarding the theory's purpose.

A THEORY OF OBJECTS OR OF 'MODES OF OPERATION'?

One of the fundamental distinctions between the three worlds is, their so-called *modes of operation*: how individuals operate in them. A key part of this idea is that each world has its own types of "warrant for truth":

> Initially something is 'true' in the embodied world because it is *seen* to be true. […] Increasing sophistication in geometry leads to Euclidean proof, which is supported by a visual instance and proved by agreed conventions. […] In arithmetic, something is 'true' because it can be calculated. […] In the formal world, something is 'true' because it is either assumed as an axiom or definition, or because it can be *proved* from them by formal proof. (Tall, 2004a, p.287).

So, in the embodied world something is true either through visual observation or through a proof supported by a visual instance. This claim seems very hard to reconcile with Tall's (2004a) explanation that the embodied world need not consist of only perceptual objects. How does one *observe* something to be true, or construct a proof supported by a *visual instance*, about an object which cannot be visually seen?

Returning specifically to the case of the imaginary sphere, it is clear that the warrants for truth associated with this object are not the perceptual or visually-supported proofs seen with other common conceptually-embodied objects. As we have seen, they are proceptually based symbol manipulations. Here we see an object from one world (the embodied) being dealt with using the modes of operation and warrants for truth from another (the proceptual). Thus by expanding the definition of the embodied world to include curious "artful specialisations" such as the imaginary sphere, Tall (2004b) has sacrificed one of the main planks of his theory: that of distinct modes of operation for each of the worlds.

What then, is the three of worlds of mathematics a theory of? It seems to have developed out of a desire to classify the ontological status of mathematical objects, so as to avoid the over-application of process-object theories. However, from these roots it has developed into a threefold categorisation of mathematical modes of operation. I suggest that the example of the imaginary sphere indicates that these two purposes are mutually exclusive: either the theory is about the nature of mathematical objects, or it is about modes of operation; if it is about both it must be conceded that the worlds overlap in unexpected ways, and that they are not distinct.

Tall, in recent expositions of the theory, seems to have concentrated on 'modes of operation' rather than 'types of object' (e.g. Tall, 2005a, 2005b), and this is to be

welcomed. It is unclear, however, how this change in emphasis fits with Tall's (2004b) classification of the imaginary sphere as an embodied object. If the 'three worlds of mathematics' is to be developed further as a useful theory, it's purpose needs to be explicitly clarified: what empirical observations does the theory seek to explain, and to what types of analyses does it intend to be applicable?

The three worlds of mathematics has clearly struck a chord with many mathematics education researchers, and has proved helpful in their work. However, as I believe Tall himself would accept, for the theory to become a more widely accepted framework within which to base research, further clarification and development is necessary.

A WORD ABOUT 'A WORD ABOUT WORDS'

Tall (2004b) closes his response to my earlier article with a half page reaction to a short comment made in a footnote. I had written, in reference to Gray and Tall's (2001) use of language:

> [Gray and Tall] have the frustrating habit of using the words 'concept' and 'object' interchangeably. (Inglis, 2003, p.26).

Since Tall (2004b) devoted half a page to responding to this one sentence, it may be worth clarifying the point here.

Gray and Tall (1994), when defining the notion of a procept, had consistently used the words "object" and "concept" interchangeably. For example, underneath the heading "Process becoming conceived as Concept" they had written:

> The notion of actions or processes becoming conceived as mental objects has featured continually in the literature. (p.116)

In the heading the word used was "concept", but in the text the terminology had reverted to "object". In a later paper this was explained in the following terms:

> Although other theorists (including Dubinsky, 1991, and Sfard, 1991) use the term 'object' we prefer 'concept' because terms such as 'number concept' or 'fraction concept' are more common in ordinary language than 'number object' or 'fraction object'. (Gray, Pinto, Pitta & Tall, 1999, p.3)

Essentially the same argument was made by Tall (2004b) in response to my footnote:

> In colloquial language we speak of counting processes and number concepts not number objects. For instance we use the term 'concept of number' not 'object of number'. (p.32)

But this argument is suspect. Surely the reason that "concept of number" is a more common phrase than "object of number" is that "number" is not an object. At the very least it is certainly not an object in the same sense that 4, 2π or $\sqrt[3]{23}$ are objects.

Is "number" a procept? Surely not. What is its process? It seems slightly peculiar to claim that "$f(x) = x^2 + 2x + 1$", "$g(x) = \sin^2 x$" and "function" are all the same type of thing. And further, seems quite incorrect to say that they are all procepts: f and g

have natural processes associated with them, but "function" does not.[2] It seems that, at least subconsciously, this is accepted by Gray and Tall (1994) as the definition of procept studiously avoids using the word "concept". However, soon afterwards they again lapse back into the use of "concept" by talking about the "procept of sum". "Sum", at least at the junior school level[3], is not an object (and therefore not a procept), it is a concept.

The idea of a procept is an undeniably good one, it is insightful and useful. Unfortunately it is also unattractively named. As the definition makes clear, a procept is a combination of a process, a symbol and an *object* not a *concept*.[4]

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[2] As a rough and ready test to distinguish between objects and concepts, it seems sensible to consider whether you would be happy to consider sets of these things. I would be happy dealing with $\{2, f, g\}$, but not $\{2, f, \text{fairness}\}$. The concept of fairness, according to this test, is not an object.

[3] Of course, once one is capable of reifying functions, sum does indeed become an object.

[4] It seems clear that if the portmanteau idea was to have been carried off successfully the term "obcess" would have to have been used (pro-ject being already taken). Although slightly odd sounding, this terminology would have had another big advantage over that adopted by Gray and Tall (1994): they could have described at length how, if they are to be successful at arithmetic, children must bridge the obcessive divide.

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THE SLIPPERY ROAD FROM ACTIONS ON OBJECTS TO FUNCTIONS AND VARIABLES

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Functions are all around us, disguised as actions on concrete objects. Furthermore, composition of functions too is all around us, because these actions can be performed in succession, the output of one serving as the input for the next. In terms of Gray & Tall's 'embodied objects' or Lakoff & Núñez's 'mathematical idea analysis', this 'embodied scheme' of action on objects may serve as intuitive grounding for the function concept. However, as Gray & Tall and their colleagues have shown, such embodied schemes can also lead to serious 'epistemological obstacles' in later stages of concept development. In the same vein, our own data shows that the intuitions about change and invariance entailed by the action-on-objects scheme may clash with the modern concepts of function and of composition of functions.

INTRODUCTION

In this article we investigate the relation between the intuitions of actions on familiar objects on the one hand, and the formal function concept on the other. We focus on the 'algebraic' conception of functions, which is based on the image of an input-output machine, as opposed to the 'analytic' conception, which is based on the image of co-variation of two magnitudes. In terms of the theoretical frameworks of Gray & Tall's (2001) 'embodied objects' or of Lakoff & Núñez's (2000) 'mathematical idea analysis',[1] we propose that the function concept, at least in its algebraic conception, may be grounded in the 'embodied scheme' of actions on objects.[2] This idea will be developed in the first part of the theoretical background. In the second part, we will briefly introduce the functional programming paradigm, which helped us uncover some deep-seated pre-conceptions about functions. In the research section, we bring data to show how the same intuitions of actions on objects can also get in the way of further understanding of functions.

THEORETICAL BACKGROUND

How functions can be conceived (and misconceived) as actions on objects

One of our most basic intuitions is *perceiving the world in terms of objects and actions on them.* Indeed, according to Piaget (1983/1970) this is one of the basic mechanisms through which the developing child comes to know the world:

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[1] Gray & Tall point to important differences between the two theoretical frameworks, especially their use of 'embodiment'; however, these differences will not affect the present argument.

[2] For our purposes it is useful to consider the embodied objects together with the actions (also embodied) which they naturally afford, hence the 'embodied *scheme*'.

> Actually, in order to know objects, the subject must act upon them, and therefore transform them: he must displace, connect, combine, take apart, and reassemble them. From the most elementary sensorimotor actions (such as pulling and pushing) to the most sophisticated intellectual operations, which are interiorized actions, carried out mentally (e.g., joining together, putting in order, putting into one-to-one correspondence), knowledge is constantly linked with actions or operations […]. (p. 104)

> [Logicomathematical] experience also involves acting upon objects, for there can be no experience without action at its source, whether real or imagined, because its absence would mean there would be no contact with the external world. (p. 118)

Thus, a child learns to make sense of the world by interacting with its environment, specifically, by gradually coming to perceive objects in its environment and by performing various actions on these objects. We shall refer to this universal trait as the Actions-on-Object Scheme (AOS). The essence of the AOS for us – and the one we will invoke in explaining our findings about functions – is the conception that when an action has been performed on an object, *the object has undergone some change, but it is still* **the same** *object before and after the operation*. Or, in Piaget's (1983/1970) words: "[All children agree] that when a ball of clay is changed into a sausage, it is the 'same' lump of clay even if quantity is not preserved." (p. 123)

Our study fits well within the theoretical framework developed by Gray and Tall (2001) and their colleagues, on the *relationship between embodied objects* (or embodied schema in our case) *and symbolic procepts*.[3] Like the present paper, they point out that these embodied objects are helpful in early stages of learning, but can cause epistemological obstacles in later stages:

> We observe that a practical, real world understanding of simple mathematics can very well benefit from a focus on the operations on the base objects, and such a perspective is satisfactory, even insightful in everyday situations. However, an *exclusive* focus at this level can act as an epistemological obstacle barring the way to the more sophisticated theory that is required for subtle technical and conceptual thinking. (p. 66)

The empirical study of the relationship between functions and the AOS (of which this study is a mere beginning) falls under the branch of cognitive science that Lakoff & Núñez (2000) call 'Mathematical idea analysis':

> For the most part, human beings conceptualize abstract concepts in concrete terms, using ideas and modes of reasoning grounded in the sensory-motor system. The mechanism by which the abstract is comprehended in terms of the concrete is called conceptual metaphor. Mathematical thought also makes use of conceptual metaphor, as when we conceptualize numbers as point on a line. (p. 5)

Lakoff & Núñez (2000) also discuss briefly some of the conceptual metaphors for functions (e.g. on p. 386), but not for the algebraic conception, which is our concern here. In our case, to be specific, the metaphorical mapping would map *action* to

[3] "When the symbols act freely as cues to switch between mental concepts to think about and processes to carry out operations, they are called procepts." (Gray & Tall, 2001, pp. 67-8)

function, *object* (or the *state* of the object) to *variable*, and the *initial and final state* of the transformed object to the function's *input and output*.

Our empirical findings below demonstrate the influence of the AOS – in particular the conception mentioned above that *the object is changed but still remains the same* – on students' conceptions of functions. In a way, these findings help substantiate the theoretical analysis of the function concept as grounded in the AOS metaphor (or embodied scheme). On the one hand, teachers and textbooks commonly use some version of the AOS to engage students' intuitions when thinking about functions; witness, e.g., the prevalent use of the function machine metaphor, or expressions like 'the function that multiplies x by 2' (see more examples below, in the context of functional programming). On the other hand, on the 'proceptual' level, this same image clashes with the formal function concept, since it invites the erroneous conclusion that if we apply the above function to (say) x = 3, then x will become 6. The general AOS-induced image (which, as has been said, is sometimes useful and sometimes not) is that the input-object is being transformed by the function into the output-object, but still it remains 'the same' object.

A very brief introduction to functional programming

The discipline of computer science comprises several major programming paradigms. The research described below focuses on Israeli high school students learning the functional programming paradigm, where the basic entities are functions and composition of functions. The functional paradigm is usually studied as an 'additional paradigm', after or in parallel to studying the procedural paradigm in Pascal. This paradigm has originated with the LISP programming language and its various offsprings, including later dialects such as Logo and Scheme – the language used by the research population. In these languages the basic data objects are lists, i.e. an ordered set of objects of the language. For example, the following is a list with 4 elements (the last being itself a list): *[all we need [is love]]*. We can create *variables* in the language by assigning a name to an object. For example, suppose the name *L* is assigned to the above list. Then a variable has been created whose *name* is *L* and its *value* is the list *[all we need [is love]]*.

In addition to lists, functional programming consists of operations (functions) on lists. For example, the operation *first* inputs a list and outputs its first element; thus, for the list *L* defined above, *(first L)* will output the word *all*.[4] Similarly, *rest* inputs a list and outputs the list without its first element; thus *(rest L)* will output the list *[we need [is love]]*. Finally, *cons* inputs any object and a list and adds the object to the beginning of the list. More precisely, *cons* constructs a new list whose *first* is the input object and whose *rest* is the input list. Two functions can be *composed*, as in mathematics, by taking the output of one as the input to the other; thus, *(first (rest L))* will output *we*. Similarly, (*cons (first L) (rest L))* will output the original list *L*.

[4] Note the slight difference in syntax: In Scheme we write *(first L)* instead of the customary *first(L)* from mathematics.

We remark that in order to ensure that this course will indeed introduce a different way of thinking and programming, in particular, to emphasize the difference from Pascal, a didactical decision was made to avoid the use of direct assignment (similar to 'let x equal 3' in mathematics, or x := 3 in Pascal). Students still worked a lot with *indirect* assignment through the input variables of functions.

THE RESEARCH

Research Setting

The study population consisted of five 11-grade classes (about 20 students each), who studied functional programming in a DrScheme environment.[5] Observations were also made in additional classes selected from three schools in Northern Israel.

Our qualitative research approach has been characterized by a flexible developing research setup, gradual refinement of the research foci, and parallel processes of data collection and analysis. The research data, collected mainly through observations and interviews, were analysed in order to draw conclusions and develop a 'grounded theory' that would reciprocally inform the subsequent course of research. During the initial stages of the research, classroom observations were the main tool for data collection. Later on, special interviews were conducted with students, in which the students were observed and taped as they were working on programming tasks or asked to explain given functions.

Research finding: The clash between the AOS and the function concept

From a general perspective, this paper can be viewed as contributing additional evidence to the familiar and prevalent phenomenon of a clash between people's intuitions and the cognitive requirements of contemporary formal systems, such as mathematics, science and computer science (e.g., Fischbein, 1987; Stavy & Tirosh, 2000; Geary, 2002; Leron & Hazzan, in print). Specifically, we study this clash in the case of the function concept in mathematics and computer science. As mentioned above, we propose that in certain circumstances the AOS intuitions clash with the formal function concept. Our data demonstrates two kinds of such clashes: One, the clash between the AOS intuition of *change* vs. the formalism of function as *correspondence*: Under the influence of the AOS, the students perceive the function as changing the input-object into the out-put object, yet in the formal function concept nothing really changes – the output-object simply corresponds to the input-object. Two, the clash between the AOS intuition of performing a *chain of actions* on the same (continuously changing) object, vs. the *composition* of functions.

[5] DrScheme has been developed at Rice University, USA, with the objective of offering a pedagogical environment for functional programming (Felleisen et al., 2001). The software can be downloaded from http://download.plt-scheme.org/drscheme/ .

Do functions change their inputs?

Here is a typical example of a phenomenon we have frequently observed among our students.

Mili is working on the following task:

Write a function which inputs a list L *and a number* x*, and inserts the number both before and after the first element of* L*. For example, if the input list is* [a b c] *and the number is* 7*, the output should be the list* [7 a 7 b c] .[6]

Mili writes in her notebook *rest L* and stops. She calls the researcher for help.

Researcher: What's the problem?

Mili: I ask, is it possible to do *first L* and then I get, like, a?

Researcher: Is *first L* something the computer knows how to do?

Mili: Yes.

Researcher: So what's the question?

Mili: Is it allowed?

Researcher: Why shouldn't it be allowed to do *first L*?

Mili: But no, because I already chopped off the list. The question is, is it allowed to do it again to the full list?

Researcher: When you did *rest* here, does it mean that the list was spoiled?

Mili: Yes.

Researcher: Why?

Mili: Because it is now left with only *b* and *c*.

Clearly, Mili thinks that a function transforms its input into its output; that is, if *L = [a b c]* and *rest L* has been performed, Mili thinks that *L* has become *[b c]*. Needless to say, this is not what really happens: In modern mathematics and computer science the input value *corresponds* to the output value, but nothing really changes; and in the case of programming, this conception leads to programming errors. We emphasize that this intuition is strong and prevalent: it has been found in many students and in every class, and even among mature students and teachers. In our interpretation, this is a case of the clash between the AOS – which leads to the view of a function as an agent of change – and the formal definition of function which denies this view.

Some readers may feel that the formulation of the task (the function *inserts the number* in the list *L*) helps create the misconception, since it actually describes the function as changing its input *L*. Indeed, a more precise (but more clumsy) formulation would be 'a function that input a list and a number, and outputs a new list which is identical to the input list except…'. Yet, most experienced teachers

[6] Typical solution: *(cons x (cons first L (cons x rest L)))* .

prefer the first formulation, precisely because they prefer intuitive formulations. This is also true about textbooks, including the textbook that our subjects learned from. Incidentally, the same formulation is also common in mathematics teaching, as in the example of a function machine that takes in a number and 'multiplies it by 2'. We do not believe that choosing the more precise formulation would help avoid this 'bias'. Instead, we believe that the changing-the-input bias itself should be discussed with the students when the problem actually comes up, drawing their attention to the clash.

Chaining or composition?

In teaching programming of complex tasks, we use the technique of first asking students to describe the algorithm in natural language. This encourages them to use their intuition and everyday experience, and usually they have no problem coming up with the desired description. But now the question arises, how do we go from here to the formal code? One step is translating from natural language to programming code, and often this is all that's necessary, especially in procedural languages (such as Pascal or C), where direct assignment is routinely used. In our functional programming classes, however, direct assignment was not allowed (as explained above), and the translation has not been so straightforward. Specifically, when the verbal description consists of a sequence of actions on objects, this cannot be translated intuitively as a chain of assignments, but has to be formulated as a composition of functions, which is known to be hard.

For example, consider the case of Sharon, who was working on the following task:

Write a function which inputs a number and a list, and replaces the last element of the list with the number. For example, if the number is 15 *and the list is* [a b c]*, then function should output the list* [a b 15].

In describing the algorithm in natural language, the first thing the students would want to do is remove the last element of the list. However, the given programming language has an instruction (*rest*) for removing the first element of the list, but not one for the last. What one usually does instead is reverse the list and then remove the first element. The remaining actions in the present case are to add the new number to the beginning of the reversed list and reverse again. Note that this is a description within the AOS since it is given in terms of actions on concrete objects.[7] The translation to functional programming code (which is the same as mathematical formalism except for slight syntactical differences) is at the heart of this paper, since it shows how students negotiate the gap between intuitive actions and formal functions. Here is how Sharon managed this translation.

[7] Concrete is a relative term, and a list of numbers or words is very concrete for these students.

Sharon's code	Explanation
(define (replace-last n L)	Define a function called *replace-last*, with a number *n* and a list *L* as inputs.
(reverse L)	Reverse the list
(rest L)	Remove the first element
(cons n L)	Add n in the beginning of the list
(reverse L))	Reverse the list

This indeed is a precise rendering in the programming language of the above AOS description, but unfortunately it doesn't work. Sharon, presumably guided by the intuition of the AOS, writes a chain of actions, believing that each works on the result of the previous one (as in real life). However in functional programming, as in mathematics, what is needed here is composition of functions, not chaining, thus:

(reverse (cons n (rest (reverse L)))) .
This demonstrates another facet of the clash between AOS and the modern concept of function; that is, the intuition of chaining actions on objects clashes with the required formalism of composition of functions.

In two minds

The case of Sharon indirectly highlights an interesting phenomenon. Sharon seems to assume that, like real-world objects, L is changing all the time, each L being the result of the previous operation. Yet, our data concerning programming with recursion (not shown here for space limitations) indicates that students who made this kind of error in an advanced stage of their learning mostly knew, when asked explicitly, that functions did *not* change their inputs. However, when their attention was otherwise engaged in complex tasks, they behaved, despite that knowledge, as if the function *did* change its input. This shows that the students' learning during the course did not eliminate the intuitions of the AOS, just drove them underground. This may have been the case with Sharon too: When she was working on a task that demanded her full attention, the AOS intuitions re-surfaced and took control of her behavior.

This phenomenon can be neatly explained in terms the influential *dual-process theory* from contemporary cognitive psychology (Kahaneman, 2002; Kahneman & Frederick, 2005; Stanovich & West 2000; Leron & Hazzan, in print). According to this theory, our cognition operates *in parallel* in two quite different modes, called System 1 (S1) and System 2 (S2), roughly corresponding to our common sense notions of intuitive and analytical thinking. These modes operate in different ways, are activated by different parts of the brain, and have different evolutionary origins (S2 being evolutionarily more recent and, in fact, largely reflecting *cultural* evolution). S1 processes are characterized as being fast, automatic, effortless, 'cheap' in terms of working memory resources, unconscious, and inflexible (hard to change

or overcome). In contrast, S2 processes are slow, conscious, effortful, fully engage the working memory resources, and relatively flexible. In addition, S2 serves as monitor and critic of the fast automatic responses of S1, with the 'authority' to override them when necessary. In many situations, S1 and S2 work in concert, but there are situations in which S1 produces quick automatic *non-normative* responses, while S2 may or may not intervene in its role as monitor and critic. See the heuristics-and-biases research by Kahneman & Tversky (e.g., Kahneman, 2002), or Leron & Hazzan (in print) for similar phenomena in advanced mathematics thinking.

In terms of dual-process theory, we might say that the intuitions of the AOS had initially been part of the students' S1 knowledge, and remained so even after the students learned that functions do not change their inputs – the latter having become part of their S2 knowledge. (It is quite common for S1 and S2 to simultaneously hold conflicting pieces of knowledge.) Since S2 relies heavily on the limited resources of working memory, it is less likely to do its S1-monitoring job while being engaged in another complex task, as explained in Kahneman and Frederick's (2005):

> The effect of concurrent cognitive tasks provides the most useful indication of whether a given mental process belongs to system 1 or system 2. Because the overall capacity for mental effort is limited, effortful processes tend to disrupt each other, whereas effortless processes neither cause nor suffer much interference when combined with other tasks […] People who are occupied by a demanding mental activity […] are much more likely to respond to another task by blurting out whatever comes to mind […]. (p. 268)

We can now offer the following dual-process interpretation of Sharon's data (or, more precisely, the data of students who exhibited this kind of behavior when studying recursion). Usually, when given a task that directly tests her knowledge of whether a function changes its input, the fast automatic S1 response (that it does) first comes up in her mind, but is subsequently being overridden by her S2 knowledge (that it doesn't). In the case reported here, her S2 was busy working on the complex task, hence S1 could 'hijack' her attention with its automatic intuitive response, without being 'caught' and corrected by the busy S2.

CONCLUSION

We have seen that the AOS can serve as an effective intuitive support for learning about functions, but that it can also clash with the formal concept of functions, leading to some stubborn programming (and thinking) errors. We believe that more research may reveal similar phenomena in other mathematical concepts, such as limits and continuity. We conjecture that this is typical of the (cultural) evolution of mathematical concepts: While they may have their origin in everyday intuitions (Gray & Tall, 2001; Lakoff & Núñez, 2000), their modern formal incarnation may often clash with these very same roots. We further conjecture that the reason for this clash is that the modern version, in order to achieve utmost rigor, power, and consistency, has to suppress all traces of time and process. But our basic intuitions about the world are rooted in action, hence are inseparable from the very same

processes that are eliminated in the modern formalism. In terms of Gray & Tall (2001), we might say that our findings add more evidence to the great difficulty of acquiring 'proceptual fluency'.

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GIVING MEANING TO EQUATIONS: AN ANALYSIS OF STUDENTS' WORK

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In this paper we present data collected form a questionnaire and conceptual maps designed by 15-16 year old Brazilian students in an attempt to better understand their conceptions of equations. Analysis of both instruments shows us that these students do not relate equations to either conceptual embodied aspects of equations or to symbolical meanings. We hypothesise that excessive emphasis on procedures may guide students to the use of meaningless rules to solve equations.

INTRODUCTION

Many are the concerns about teaching and learning of algebra. Students' difficulties with this subject have been reported in many research studies. In the case of equations, findings show that students do not understand the equals sign as an equivalence relation (Kieran, 1981) nor the equivalence between two equations in terms of their roots (Linchevski and Sfard, 1991). Also, the rules to solve equations are turned into meaningless procedures that should be used to get a result (Cortés and Pfaff, 2000) and procedures are confused and misused by students (Freitas, 2002).

In an attempt to understand why students are faced with difficulties reported in those studies, we have developed a research aiming to look through students' conceptions of equations and the solving methods they use. For this, we have worked with six Brazilian mathematics teachers that agreed to collaborate with us in order to design instruments and collect data from their students.

Data will be analysed with the theory of the Three Worlds of Mathematics, which hypothesise that there are three different kinds of cognitive mathematical development that inhabit three different worlds.

We claim that the absence in learning of aspects from at least two of those worlds may prevent students from giving meaning to equations and from having flexible thinking to solve them.

THE RESEARCH

As part of a broader project on the teaching and learning of mathematics, we discussed pedagogical and mathematical aspects related to equations and their solving methods in weekly sessions with six mathematics teachers. Those discussions resulted in an attempt to understand what the conceptions about equations those teachers' students would hold. For this, researchers and teachers together designed three instruments to collect data: a conceptual map, a questionnaire and interviews. In

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this paper, we wish to relate data from conceptual maps and the first four questions of the questionnaire in order to raise those students' conceptions of equation.

Data have been collected from three classes: one with 32 fifteen-year old first graders (denoted as C15) and another with 28 sixteen-year old second graders (C16), both from a public school. The third class consisted of 18 sixteen-year old second graders from a private school (P16). Both schools are located in the Greater São Paulo area.

The conceptual map consists of the design by the students of a scheme which is based on the words from a brainstorming session: students were asked by their teacher to say a word that comes to their mind when they see the word EQUATION and the teacher would write all words on the blackboard. Figure 1 shows how a blackboard would appear after the brainstorming. After that, working in groups of four or five, students had to separate those words in at least three different categories, name them, make a scheme using those names and write a few sentences to explain the scheme.

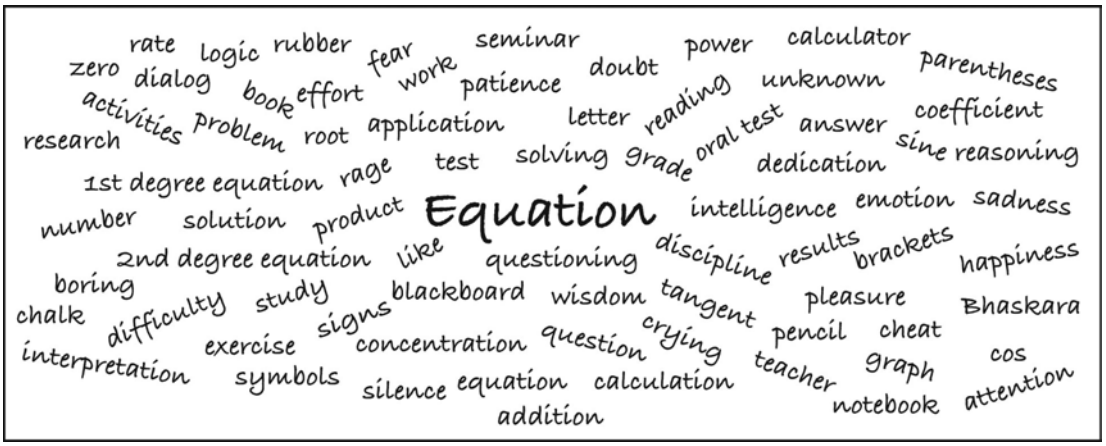

Figure 1: Words from class P16.

We believe that saying words related to equations and arranging them in categories will raise different ideas, concepts and aspects students relate to equations.

The questionnaire used had eight questions related to the concept of equation. We aimed that data collected from it could show us how students solve equations, which mathematical concepts they use in their solving methods and also whether they could model a given problem using a quadratic equation. The four questions we are analysing for this paper are presented in Figure 2. They were chosen among the others because, in order to answer them, students should explicitly write about their understanding of equation.

What is an equation?

What is an equation for?

Give an example of an equation.

What does the solution of an equation mean?

Figure 2: The questionnaire

With those questions and the conceptual maps designed by students, we believe it is possible to make a picture of what these students' understanding of equations might

be. We are also comparing data from both instruments to observe whether they show the same picture.

THEORETICAL FRAMEWORK

From the understanding that there are three different kinds of mathematical concepts – the ones that come from the study of objects, the ones from the study of actions and the ones from the study of properties (Gray & Tall, 2001) – Tall (2004) claims that there are also three different types of cognitive development related to those mathematical concepts which inhabit different worlds of mathematics.

The first world emerges from the perceptions individuals have from the physical objects and their properties. By observing and manipulating objects, it is possible to grasp mathematical concepts which are related to them. With experience, the individual can also develop mental experiments and extract meaning from them instead of only rely on physical ones. This is the *conceptual-embodied world*, where actions are performed in order to give mathematical meaning to properties of objects.

While individuals grow in sophistication, they feel the need to represent the actions they experience in the embodied world. For this, they use symbols that inhabit the *proceptual-symbolic world*. In this world, symbols compress both procedures and concepts associated to the actions. This duality of symbols as processes and concepts is called *procept* by Gray and Tall (1994) and they hypothesise that the flexibility of moving from the process to the concept helps individuals to give meaning to mathematical symbols.

The third world of mathematics is the *formal-axiomatic world*, composed by the axioms, definitions and properties and theorems which are deduced from them that build the body of knowledge of mathematics. This world is not a focus in the teaching before undergraduate levels, as mathematics is not axiomatically constructed in lower school levels.

When the teaching approach is based on conceptual embodied relations, like the balance model (Vlassis, 2002), it may give embodied meaning to equations, as the balance must be kept by performing the same operation in both sides. Nevertheless, it is necessary to transfer this understanding to symbols, giving them symbolical meaning, so the student can use the concepts from the embodied relations in more sophisticated situations which it is impossible to represent with a balance, such as when negative numbers are used. Alternatively, a functional embodiment (related to procedural meaning) may occur when students pick a symbol from one side of the equation and put it to the other with an additional changing of signs, or underneath another symbol. In this way, symbols are devoid of meaning and a use with such aspects may prevent students from understanding symbols as procedures and concepts in a flexible way.

In this research study, we search for aspects from both embodied and symbolic worlds that may be presented in students work and also try to get insights of what kind of meaning they give to equations.

RESULTS

In this session, we are first attempting to find what insights we can draw from words in the brainstorming sessions and then looking more closely at what their categorisation and final schemes suggest. Then, we present some findings from the questionnaire and compare them to the previous ones.

Conceptual maps

The brainstorming sessions from the three classes are alike. All of them contain words like “number”, “signs”, “symbols” “calculation”, “addition”, “subtraction”, “multiplication”, “division”, “unknown”, “rules” and “solution”. These words may mean that these students are likely to understand equation as a calculation with numbers and unknowns in order to get the solution. Words for the operations and the word “rule” may represent the actions performed on symbols that inhabit the proceptual world. Nevertheless, the lack of words related to the structure of an equation, such as equality, equivalence or balance, may mean that the symbols that were emphasised here do not carry a conceptual meaning. This leaves them with only procedural ways of thinking.

Categories created by students were useful to show how they perceived the words. For instance, the word “equal”, which appeared only in the brainstorming of the class C15, was used in five of the eight small groups that were formed in this class. Four of them placed this word in categories called *Signs* or *Symbols* and one group placed it in the one called *Operations*. This may be evidence that students are likely to see the equals sign as an operational symbol (Kieran, 1981), which entice them to perform an action, a calculation, instead of representing equivalence. Another example is the word “unknown”, which sometimes was placed in rather peculiar categories, like *Not Useful*. Students who did that may not know either the meaning of the word “unknown” or its role in an equation. Finally, words for subjects, like “Biology”, “Physics” and “Chemistry”, which appeared in the brainstorming of class C15, were placed in categories named like *School* and *Subjects*. We claim that they do not mean theses students may understand equations as modelling problems that pertain to different subjects and therefore to real life situations, but that they are part of the school life only.

One example of conceptual map designed by students can be seen in Figure 3. It is from a group in class P16, and their brainstorming also contains words related to their feelings about previous experiences with equations, like “anger”, “sadness”, “panic”, “doubt”, “fear”, which were placed in the Feelings category of this group. The Actions category contain words that show the need of abilities, like “dialog”, “interpretation”, “attention” and others. The Working Instruments category contains

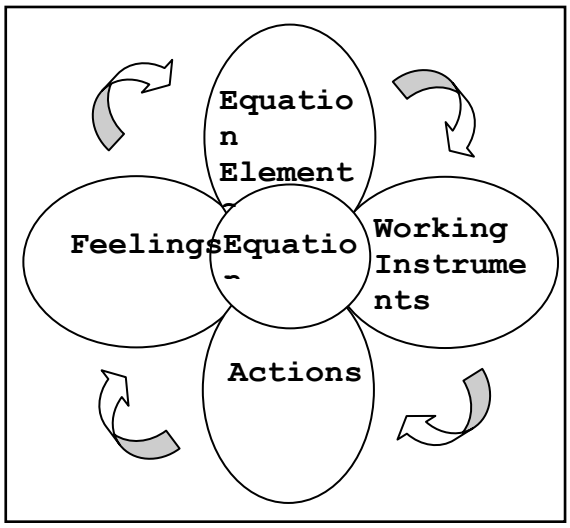

Figure 3: Conceptual map from one group of P16

objects like "blackboard", "calculator", or even words related to the classroom, like "tests" and "exercises". Lastly, the Equation Elements category contains all words related to mathematics that appeared on the brainstorming. Our interpretation of this map leads us to conclude that students relate mathematical elements mostly with school, which is represented in the category *Working Instruments*. To succeed in the learning of equations, they need to make efforts but also to overcome their fears and difficulties.

The absence of some words is also meaningful to our research. As there are no words related to real-life problems or embodied approaches to the teaching of equations like the balance model (Vlassis, 2002), we hypothesise that they were not introduced to the concept by physical or embodied relations to equation, which prevented them from giving embodied meaning to symbols. As individuals start giving meaning to mathematical concepts from the embodied world, we cannot expect these students to be comfortable with symbol manipulation. Also, concepts like equality and equivalence where not mentioned. It seems to be evidence that symbols do not play the role of representing concepts, which leaves students without any symbolical meaning to equations as well.

The Questionnaire

Analysing students' work in the questionnaire guided us to believe that students from all classes probably understand equations either as a common calculation like addition or subtraction or as a calculation in which it is necessary to have an unknown and to find its value. It can be seen in answers to Question 1 (*What is an equation?*), like *"It is a calculation in which there are additions, multiplications and many symbols like +, ÷"* or *"Equation is a set of calculations"* and *"It is a mix of numbers and letters aiming to find the value of the unknown"*. Other answers to this question also show difficulties students are faced with while solving an equation and the fear of not being successful (*"I cannot understand them nor solve them"*). Characteristics of equations, like *"they have two members"* or *"they can be linear or quadratic"*, and in the search for an unknown number were also emphasised.

The general agreement among students in answers to Question 2 (*What is an equation for?*) is that it is useful to find the solution, meaning the value for the unknown. Students in P16 also refer to equation as a mathematical calculation that does not have a practical use. For instance, one student wrote *"I will rarely or never*

use equations to solve daily life problems" and other wrote *"in life and everyday, it is not useful for many things, but maybe it is useful for people who likes mathematics"*.

Examples students give in Question 3 (*Give an example of equation*) are basically linear and quadratic equations. There are a few different answers such as students in classes C15 and C16 who wrote, together with a linear equation, a calculation like $12 \times 7 = 84$ or $2 + 2 = 4$, written vertically. Numerical and algebraic expressions like "$30 - 200 + 15 \cdot 5 - (-5 + 1) =$" and "$2 + [5x + 13x\{16 - (55x - 10x \div 2)118\} \div 75] =$" were also present in students work. In addition, 47 of the 77 students who answered the questionnaire made an attempt to solve the example they gave. It seems that students are more able to apply rules and procedures to solve equations than to perceive concepts that underlie those solving methods. We hypothesise that the procedure is more important than the concepts behind it.

Lastly, for Question 4 (*What does the solution of an equation mean?*) the usual answer is *"value for the letter"* (meaning the unknown), or *"the end of the calculation"*. Again, students in P16 showed their feelings, saying that it is *"the end of a great effort"*. In this case, the focus is on the final product of an action. As their understanding and use of equations is mainly addressed to mathematical issues, the solution attached to the value of the unknown, without any meaning from the situation that might have generated the equation.

Although referring very often to symbols, none of the answers to these questions show that they are meaningful to students. This emphasis given by students may be part of their learning that seems to have been based on symbolic manipulation only.

DISCUSSION

The conceptual map and questionnaire give us very similar characteristics of what these students understand by equations. Apparently, they focus more on procedural meaning than any other ways of giving meaning to equations. The fact that more than half of the students actually solved examples they gave is evidence of their understanding of equations as a process. The lack of symbolic meaning to equations may be evidence of the absence of conceptual understanding of equations. Both these characteristics prevented them from having proceptual thinking and seeing symbols in a flexible way. Further data (Lima & Tall, 2006a, 2006b) show that this led them to a meaningless use of rules to solve equations.

Vlassis (2002) claims that giving embodied meaning to equations by the use of the balance model as approach to the teaching of equations made it possible for students to give meaning to the equals sign as an equality symbol. We hypothesise that giving embodied meaning to equations helps them to reach proceptual thinking to equations by transferring embodied to proceptual meaning.

In addition, the absence of meaning to the equals sign may be a reason for students to understand equation as a regular calculation with no difference from multiplication or others. The equals sign just shows them where to place the solution.

For further research, we suggest a teaching approach based on aspects of embodied and proceptual worlds. We believe that, by starting with embodied characteristics, it is possible for students to recognize the equals sign as an equality symbol that balance both sides of an equation. Giving meaning in this world, it would be more reasonable for them to understand symbols as procedures and concepts that give flexibility to the mind and would enable them to grasp the use of rules to solve equations not as a meaningless procedure that it is necessary to follow, but as a mathematical reasoning that guides them to successful solutions. In addition, links between embodied and proceptual worlds should be emphasised by the use of equations to represent real life situations.

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PROCEPTS AND PROPERTY-BASED THINKING; TO WHAT EXTANT THE TWO ARE CO-OPERATIVE?

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The paper examines the relationship between procept and property-based thinking. At the first sight, property-based thinking is different to thinking in terms of procepts, as the former involves a property that must be a-priori associated with some category of objects, whereas for the procept the identity of the objects or conceptual input is negotiated through processes, and vice-versa. Despite this difference, the paper will argue through some examples that, to some degree, the two can be reconciled, and can be made to act productively in tandem. The illustrations concern the limit of a real sequence, the Fundamental Theorem of Calculus and the prime decomposition of positive integers.

INTRODUCTION

The seminal paper by Eddie Gray and David Tall (1994) introduced the notion of procept. The term was conceived as a convenient word to employ when discussing the relationship between processes and concepts in mathematics in cognitive terms. The main feature to their framework is the role of symbolism that can evoke both a process and a concept. The concept is considered as the conceptual output of the associated process. For example, the symbol

$$\Sigma\, a_i$$

suggests both a process of addition and a sum concerning the numbers a_i. (Many more examples are given by David, Eddie and their colleagues, e.g. Tall, Gray et al., 2000.) The sum of course can be regarded simply as a particular real number, but the conceptual weight was on how the result was obtained from the process. The conceptual side can incite the student to negotiate certain ways to act on the process without affecting the outcome (ibid), so that the process can be thought of as a collection of different procedures, for example

$$\Sigma\, (-1)^i \qquad i = 1,\ldots, n \text{ where n is a natural number}$$

can be re-formulated as the summation of sums of pairs of two successive terms; these would cancel if n is even, or leave the result 1 if n is odd. Focusing on the symbolism would seem to encourage this processing. This illustrates that thinking about the kinds of actions incited by the conception, allied with symbolism, can be extremely powerful. Because of this, the notion of procept is very wide in application, and it encourages flexibility in thinking. The symbolism is the key to how the student can compress the conceptual complexity that is implicit in the given situation. If the students cannot use symbolism as an effective channel to make a synthesis between their use of process and conceptual understanding then they will

tend to perform worse than those who can; Eddie and David have referred to this phenomenon as the 'proceptual divide' (Gray et al, 1999).

However, David and Eddie recognize that procepts are not adequate to account for all conceptualization in the doing of mathematics. David in particular has built a more general framework that he calls the "The Three Worlds of Mathematics" where thinking in terms of procepts constitutes the second 'world' (Tall, 2004). The first world is related to concept acquisition from direct perception of physical or geometric systems and how these are interpreted. The third is called the "formal-axiomatic" world; here properties determine objects rather than the vice-versa. The consequent mathematical activity would be to further the classification of the objects, and in order to forward this, it is likely to derive new properties from the ones given (ibid.). This 'third world' could be a useful criterion in characterizing 'advanced mathematical thinking' (or AMT), as intimated by David's introductory article in the book titled by the same term. However, as the mathematics becomes more abstract, extracting meaning remains crucial not only in forming mathematical argument, but in deciding what are the mathematical issues that should be chosen to examine, e.g., Dubinsky (2000). Thus, most educators agree that all mathematics is embodied, and this opinion is often stressed in David's work. Some confusion can arise about the cognitive issue of 'objects' in this respect, as the word 'object' might suggest that we are projecting our thoughts to exterior sources. However, the notion of (mental) object is crucial psychologically as actions presuppose the idea that 'things' have to be operated on (e.g. Asiala et al, 1996), and there is no real clash here with embodied thought. But the status of object remains a rather tenuous theme; for example in both procept and property-based thinking the 'entity' that is highlighted in the mathematical activity sometimes seems more suitably called as an object, or as a concept, (or perhaps even as a system).

There are other theories concerning conceptualization and the so-called 'objectification'. These tend to be, as the notion of procept, influenced by Piaget's well-known categorization of empirical, pseudo-empirical and reflective abstraction. The theory most often used in educational research is APOS espoused by Ed. Dubinsky and his colleagues (e.g., see Asaila et al, 1996) and the notion of 'reification' formed mostly by Sfard, for example Sfard (1994). There are many papers that include discussions comparing these frameworks, as in Gray et al (1999). Given the space limitations for this paper, we will say nothing more about these other theories, apart from pointing out that the procept notion allows a more bi-directional stance between concept and process.

The aims of this paper will be to explore some issues concerning the interplay between procepts and property-based thinking. This is done by commenting on certain mathematical themes, discussed theoretically in the sense that we do not refer to specific fieldwork results. The issues that will be raised and illustrated are the following:

- Can a procept be unsound, but be made reliable through the intervention of property-based thinking?
- How symbolism can be made to change in response to a more abstract approach?
- Can there be cases where the 'result' of a process is understood as a property first, only determining conceptual weight by its utility after?

PARADIGM 1: THE CONCEPT OF LIMIT

Here we shall consider limits of real sequences. There is a 'process' already suggested in an infinite sequence; you take the first value, then the second and so on. An intuitive reading is that you consider whether in the long run the values 'settle down' to a particular value, and if so this value is regarded as the limit. It has been observed in the literature that a consequence is that the limit is considered as the 'last entry' in the infinite sequence; sometimes students would denote it a_∞ if the sequence is denoted by (an); other student behavior related to this belief is listed in Davis and Vinner (1986). The feeling that the limit is integrated with the sequence can be restrictive in cognitive terms; it can lead students to believe that the sequence acts to 'produce' a limit in certain circumstances. However, in the definition of limit, the situation is different in that the sequence is acted on when obtaining the limit. Given $\varepsilon>0$, one has to provide a positive integer N for which all $n\geq N$, an must satisfy a particular condition involving ε. This means that the concept of limit involves a second process beyond the one inherent in the sequence itself. Also the definition entails explicitly the symbol l for the limit (for all n 'sufficiently' large, $|a_n - l| < \varepsilon$), so it cannot be straightforwardly interpreted as a result of the second process. This leads to a somewhat complicated case concerning how processes are related to objectification/conceptualization. Further, because the second process is implicit in the definition, the limit can be regarded also as a property-based concept. We shall discuss these two issues further.

Initial intuitions addressed to what the limit is (if it exists) will naturally focus on what is given, that is a sequence. Either the intuition can be expressed by descriptions in colloquial language that is too vague for mathematical modeling, or it is possible to consider the difference of sequence values in the 'long run'. (The latter can lead students to formulate some properties that should be expected for the definition of limit; the most sophisticated would be the condition characterizing Cauchy sequences, though this would be a demanding task.) However, both a process and some interpretation of what the process 'yields' is in the student's mind, and it is likely that the students have been exposed to the symbolism, $\lim_{n\to\infty} a_n$. Hence we have all the ingredients for a procept, yet it is dysfunctional. How can this situation be retrieved?

Well, the crucial step, mentally, is to shift your attention primarily to the object, the limit; then secondary how the given sequence 'fits in' to this. Psychologically, this shift is very important. From the limit being regarded as produced by some process,

we turn everything on its head and we say that a limit is just a number, but with a context; the number (as a limit) is recognized as a 'result' of a process that has to be formulated consonant to the original intuitions. However, the '(ε, N)-process' that we construct is complicated and has to be expressed with great care. This suggests that the precision that it requires has to be presented formally, and so the notion of limit needs a formal definition. At this stage, one formulates a suitable defining property. A limit l, then, is determined by a certain property that the original sequence must satisfy. The exact form of the property has to be guided by the first intuitions, by the reformulation of these (probably involving the resolution of epistemological conflicts), and by taking the primary status of a limit as a real number. Here we see that the consonance between intuition and the forming of the definition is a prerequisite for devising the (ε, N) process. This process with the enriched understanding of how a limit l should be regarded could be thought of as a grounded procept, where property-based thinking has intervened.

Now the extracting of a process from the mathematical form of the definition may well require a 'new reading'. For example, the definition has a quantifier 'for all ε', whereas one might expect for the process to consider a particular ε, then (for example) ε/2, and so on, and this indeed is sufficient to check for the limit. Such differences of the definition with a natural process have been expounded in Mamona Downs (2001). A study by Pinto & Tall (2002) features a case study that chronicles the stages a mathematics undergraduate goes through whilst understanding the concept of limit of a sequence. At the end of the sessions, he is able to state (without intermediate 'working') an acceptably rigorous statement for a characterization for a sequence not to have a limit (a task with which he had no previous experience). Most of the other students failed. The successful student showed himself particularly able to reconcile the process and property aspects in the definition; and it is likely that because he could use his sense of property, he was able to understand also what it meant to contravene that property, and from then what this signifies for the suitable adaptation in process and formal statement. The more general point illustrated here is that property-based thinking can comprise a conceptual core that can be allied with processes, in which case property-based thinking and procepts are closely related.

In the discussion above, we have referred much to procepts but have largely left out the role of symbolism that supports them. The situation here is interesting. The symbol $\lim_{n \to \infty} a_n$ suggests a process on the given sequence, the stark assignation of a letter, usually l, for the assumed limit encourages a formal and property-based frame of mind, whereas the symbolism $a_n \rightarrow l$ as $n \rightarrow \infty$ could be thought as a composite of the other two, that can be effective if it is read consonant to the enhanced procept but not otherwise. This theme of symbolism taking different forms for procept and abstract modes of thinking will be taken up further in the next section in terms of consequences for problem solving.

PARADIGM 2: INVOKING THE FUNDAMENTAL THEOREM OF CALCULUS

The definite integral is defined as a limit concerning Riemann sums. Being a limit, much of what we said for the first paradigm can be paralleled. However, we take a different tack, and consider the significance of the notion of procept to the so-called Fundamental Theorem of Calculus. We will not be examining the cognitive difficulties for students in understanding the 'fabric' of the proof itself (though this theme seems rather under-represented in the AMT literature). Instead, we use it as an illustration of when a concept (the definite integral) is motivated by a process, yet its channel of calculation is through another concept (the anti-derivative). There is no essential problem in linking the limiting process of Riemann sums with the idea of 'area under' the graph of the given real function between the limits as a procept. However, the procept might be finally regarded as seemingly being incidental when a new process that produces explicit answers is introduced. This new process demands ways to determine which functions when differentiated give the original. It is difficult to take such a program to have a conceptual character, but it fits in well with a property perspective. In effect, integration is linked with a (solution) process that is not directly in tandem with its a-priori conceptualization, and so what started as a procept does not remain one because of the re-structuring of the cognition due to the Fundamental Theorem. Very quickly, students relate the symbolism $\int_a^b f(x)\,dx$ with the task: find the anti-derivative and apply the limits of integration. This is done despite this symbolism is clearly motivated by the original definition, not by its formal equivalence, the anti-derivative. It is interesting how the retention of this symbolism can cause problems to students in proving more or less trivial identities of integrals, such as

$$\int_{a+c}^{b+c} f(x-c)dx = \int_a^b f(x)dx$$

$$\int_a^b g'(x)f(g(x))dx = \int_{g(a)}^{g(b)} f(x)dx$$

The task of proving the first identity is included in a list of problems for which it is claimed that the 'average' calculus student would not succeed in performing (Eisenberg, 1992), even though the identity is open to informal argumentation using visualization. Both identities are easily seen if you use an alternative form of symbolism, i.e. instead of $\int_a^b f(x)\,dx$ we write $[F(x)]_a^b$ where F is understood as a differentiable function whose derivative is equal to f:

$$[F(x-c)]_{a+c}^{b+c} = F(b) - F(a) = [F(x)]_a^b$$

$$\int_{g(a)}^{g(b)} f(x)dx = [F(x)]_{g(a)}^{g(b)} = F(g(b)) - F(g(a)) = [F \bullet g(x)]_a^b = \int_a^b (F \bullet g)'(x)dx = \int_a^b g'(x)f(g(x))dx$$

The notation $[F(x)]_a^b$ stresses (at least) two aspects involved in obtaining the proofs. First, it emphasizes that an integral has a functional basis, where its value is determined by substituting the limits in the usual way. The rather fabricated symbolism $\int_a^b f(x)\,dx$ that reflects the limiting process underlying the concept of integration seems to interfere with handling this aspect; in particular the first proof is simply a special case of a property that a function (with domain |R) has generally. Second, introducing the symbol F is done without any expectation to determine what it represents. The idea is we know that something, F, exists with some characterizing properties, and once F is introduced, its properties are exploited to remove F again but in such a way we have produced a new expression in the givens, thus obtaining an identity. Though we do not have a procept, we do see something analogous. The symbolism acts first to highlight a certain focus, second a set of conditions, that allows both the interpretation of the conditions as fundamental properties of what has been focused on, and the working out of theorems. Perhaps how this scheme most essentially differs from a procept is that with a procept, there is an assumption that the objects involved have a potential to be determined explicitly. Thus, a student with psychology restricted to proceptual thinking could be disadvantaged in adopting the symbol F when it could be of use in problem solving.

PARADIGM 3: THE PRIME DECOMPOSITION OF A POSITIVE INTEGER

This theme brings out well the relativity between process and property. Certainly, there is a process in determining the decomposition, and there is property because there is a theorem that says that any natural number has a (unique) prime decomposition. But is it appropriate to call prime composition a procept? In this issue, the question whether the property qualifies as a concept is central. At the first sight the case seems poor; a 'literal' reading is that the decomposition gives information about the highest power that any prime divides the given natural number n. Such information would seem 'neutral' in conceptual terms. However, when the decomposition is read in another way, i.e., it characterizes the divisors of n, its conceptual significance is suddenly enhanced. Some intriguing results can be formulated very easily, such as the squares are characterized by their having an odd number of divisors. In the framework of a procept, one has a process, pivotal symbolism and conceptual <u>output</u>. In the present case, the first and third exist and there is standard symbolism

$$n = p_1^{e_1} p_2^{e_2} \ldots p_r^{e_r}$$

that can raise suggestive linkage to both, hence there is a good case to regard prime decomposition indeed as a procept. However, there is a need for a certain mental reformulation for students to acquire the procept, so some students may well not integrate the process and conceptual factors, resulting in the proceptual divide to widen. Such reformulation via structural considerations means that some procepts will require considerable reflection for obtaining the requisite new interpretations.

CONCLUSION

The notion of procept is a very wide and useful tool to describe how much of the doing of mathematics is influenced by associating processes with consonant conceptual output via symbolism. However, quite often meaning has to be extracted from a formal definition, where no a-priori sense of process accompanies the entities satisfying the definition. This difference can be regarded as a basis for characterizing advanced mathematics thinking. Nonetheless, in this paper we illustrate several ways in which procept and property-based thinking can be reconciled, as in our first and third example. The second example, though, illustrates how easily the consonance and directness that is expected between the process and concept aspects of a procept can be disturbed when confronted with a property-based reformulation. All the examples had similarities in the sense that there was some reformulation of the original conception towards a more workable or utilitarian meaning. In the first, a procept had to be negotiated from realizing the weaknesses of another. The second and third both involved influences from utility; the difference was that in the third, the aspect of utility was open to spontaneous re-interpretation, whereas the second was not, explaining why the third was regarded as a procept, the second not. The importance of symbolism has always been stressed in the theory of procept, but our examples illustrate how important it is to adopt different forms in the face of switching between procepts and property-based thinking.

The argument that procept and property-based thinking interact naturally sometimes, but not at others, may help in deciding what material is suitable, and how to teach it, for the 'transitional' level between school and university mathematics. This is just one compelling reason that the impetus given by David and Eddie by their ideas should endure and proliferate in the future.

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DISTINCTIVE ONTOGENY: ONTOLGICAL MUSINGS INSPIRED BY TWO DISCERNING GENTLEMEN OF DISTINCTION(S), DAVID TALL & EDDIE GRAY

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INTRODUCTION

Where is mathematics education headed? Are we making progress, or are we going round in circles, re-discerning and re-labelling each others distinctions? Is mathematics education developing as a coherent articulation of increasing insight, re-articulated and refined in each generation, or is it turning into a morass of the same or similar insights repeatedly put forward as new and different, often in fresh language but otherwise the same?

In this tribute I wish to demonstrate my high regard and admiration for the work of my two friends and colleagues, David Tall and Eddie Gray, by musing on their place in history as a case study in the ontogeny and ontological status of distinctions in mathematics education: how certain distinctions come into existence and, at least for a time, are taken up and used, become embedded in the fabric of 'taken as shared' by the community at large, while other distinctions wither into obscurity. I shall focus primarily on distinctions signalled by *procepts, concept images* and *advanced mathematical thinking* whose origins and development have been due largely to David and Eddie's ongoing programme of enquiry, aided by colleagues and students over the years.

I want to see how these distinctions manage to bridge the supposed chasm between theory and practice, that unholy distinction which is, in my view, wholly unhelpful in making sense of a complex world. In other words, some distinctions are helpful by informing future action, while others trap us in artificial tensions. Some are distinctive in their longevity; others vanish quickly without a trace. Apart from socio-cultural forces which advantage some people's distinctions over others, I suspect that the reason that some survive is pseudo Darwinian: they enrich people's sense of meaning and understanding of complex situations, and they inform actions.

LASTING LEGACIES

I suggested that many distinctions and their labels are forgotten within a few years, much less a generation. Today however we are celebrating contributions of two discerning scholars whose distinction is that some, at least, of their distinctions have already influenced several generations of researchers, and are in almost common parlance within mathematics education. I refer to the notions of *concept images*, *procepts,* and *advanced mathematical thinking*.

Procepts

The label *procept* signals a distinction between a technical mathematical term used as a label for a process and for a concept. Classic examples include adding (or any other binary or unary operation, perhaps most spectacularly division), angle (experienced as rotation), the quadratic formula (result of algebraically completing a square), and rotation (or any other transformation). Some people argue that since all concepts are ultimately based on sense-based metaphors, all mathematical concepts have a proceptual basis (see for example Lakoff & Nunez 2000). Others, myself included, would suggest that a concept such as *ring* might be a counter-example, because it arises as an abstraction through identifying and isolating properties derived from relationships discerned in particular instances, rather than being process based. The abstraction comes about through what I believe is a peculiarly mathematical move: taking the identified properties and making an almost ontological assertion that what is to be studied are objects with those properties (and what that can be deduced from them). Of course mathematically it is necessary to verify the ontology by proving that at least one such object exists.

The notion of a *procept* is naturally and appropriately self-referentially itself a procept, for it is a label which signifies that a process (of reification) has itself become a concept. A process has been discerned and recognised *as* a process, which could apply to situations other than the immediate experience. The possibility that the immediate experience could be an example of a phenomenon, a family of similar situations is at least implicitly recognised. The process is isolated from its immediate context. As a rich we of associations grow around the seed-crystal of one or two experiences, the label triggers awareness (possibly below the level of consciousness) which sensitises discernment. Having the label come to mind can inform practice through alerting the teacher to a process of maturation which make take considerable time, as for example the technique of 'counting on'. It can also inform task and interaction design by alerting the teacher to supportive experiences and suggesting possible ways of fading the initial scaffolding so that learners are encouraged to become explicitly aware of, and to reify the process.

This raises what I consider to be an important question: what occasions this emergence? What initiates the transformation from a succession of experiences into an experience of the succession? What mediates that transformation, and on what prior experience does it act? In other words, how do pertinent and long lasting distinctions come into existence?

Concept Images

The notion of a *concept image* (Tall & Vinner 1981) has been taken up by many generations of mathematics educators. The reason, I think, is because of the useful distinction it makes between formal definitions and the gradual development of a *sense* of what the term means to them. For example, the distinction speaks to many people's experience of learners displaying an apparently cavalier appreciation of the

significance of details of formal definitions, and are often stymied when expected to reason on the basis of those definitions. Thus the label identifies a phenomenon. As such, it can alert teachers to the underlying distinction, and thereby inform their choices as to what experiences and tasks to offer learners at each stage (see for example Mason & Johnston-Wilder 2004, 2006).

Because it crystallizes features of a complex phenomenon based on a recognised distinction, the label has entered our vocabulary. The label helps us discern aspects of learner behaviour, and brings into existence a phenomenon to which we are more sensitised than previously. Through reification and the corresponding ontological commitment it opens up the possibility for probing in more detail the constituents of concept images and the ways in which these arise and are integrated (to some degree or other) in individuals in and through their interaction with peers, with expert-others, and through reflection on their own experience. It is important to bear in mind that distinctions and associated phenomena are only a human social construction.

Advanced Mathematical Thinking

David's *magnum opus* (Tall 1991) which arose from years of collaborative work with many colleagues through PME and elsewhere, alerts the reader to a distinction, but a distinction with a distinctive difference. In this case there does not seem to be a precise border or clear cleavage between ordinary or school-based mathematical thinking and advanced mathematical thinking. Every time you isolate some feature which might be thought of as advanced, such as axiomatisation, or multi-step reasoning, you find that it is or could be presaged by, and even found, in the thinking of young children. Every feature of mathematical thinking is within the reach and experience of virtually all learners, no matter how they are classified by the educational system. It is the tragedy of mathematics education research that as a community we have not managed to infuse and infect national curricula and national teaching standards with an ethos in which mathematical thinking, advanced or otherwise, is an integral part of every mathematics lesson.

For me the main contribution of this distinction is that it has supported and promoted the growth of attention to issues in teaching and learning mathematics in the tertiary phase of education. With the decline of interest in mathematics amongst learners, and attempts by many disciplines traditionally dependent on mathematics to develop curricula less dependent on mathematics, there are now good economic as well as socio-cultural-educational reasons for interest in and concern about the educational experience of learners of mathematics in universities.

One of the features, perhaps, of advanced mathematical thinking, at least within the domain of pure mathematics, is that learners need to distinguish between, and keep in touch with both the formal and the informal. The informal is the source of intuition, the domain of inductive and abductive reasoning. It informs choices for exploration or application. The formal is the domain of deductive reasoning. Concept images form a major component of what it means to appreciate and understand a

mathematical topic (Mason & Johnston-Wilder 2004). In my own work I am finding that some at least of the difficulties of learning mathematics and of reasoning within mathematics may come about because of a potential confusion between on the one hand perceiving properties of objects and reasoning about them, and on the other hand reasoning on the basis of selected and explicitly enunciated properties. These are quite different ways of perceiving and reflecting on the world of experiences and images. They signal for me distinctively different structures for attention (Mason 2003, 2004). I mention this simply to try to give a flavour of how distinctions arise and develop.

INTERLUDE ON REFLECTIVE ABSTRACTION

Having reviewed just three of the distinctions that David and Eddie have offered our community and that have had a major and lasting influence, I am drawn now to engage in a process of *reflective abstraction*. Jean Piaget (1971) used this term to label the act of 'learning from experience', of making sense of situations and observations so that future practice is informed. It seems to apply not only to sense making arising from experience of process as a contribution to the achievement of a procept, but also to sense making that is not process based.

Reflective abstraction underpins a good deal of David and Eddie's work (e.g. Gray & Tall 1994, Tall 1995, 1999, Tall *et al* 2000 amongst many others). It involves much more than simply 'thinking back over what happened', because it is based not on individual actions but on coordinated actions amongst colleagues construing collectively. The import of *coordinated actions* has been explored and developed by Humberto Maturana and colleagues (Maturana & Varela 1972, 1988, Maturana 1988, Maturana *et al* 2004) to show that language can be seen as the concensual coordination of the concensual coordination of action. It is through language that we weave the narratives which justify actions in our own eyes and in the eyes of others, and so coordinate our sense of self with the observed behaviour of others. Through narratives we construct a sense of ourselves as continuous entities rather than as bundles of fragmented memories. Coordination brings the process into the social domain, as a component of a complex action undertaken by individuals within and through their social milieu (Brousseau 1997) to make sense of mathematics and to make mathematical sense.

Although it makes use of natural powers possessed by all learners, *reflective abstraction* certainly belongs in the Vygotskian category of *higher psychological processes* (Vygotsky 1978), accessible to learners mainly through being in the presence of others for whom it has become an established practice. Reflective abstraction requires a standing back from participation in action, a shift of attention away from acting to becoming aware of that action. It is probable therefore, that reflective abstraction is an attempt to describe or capture the essence and core of all learning.

It is typical of many pertinent and robust distinctions that it is difficult if not impossible to formalise the distinction without losing its force. Furthermore it is non-trivial to introduce a distinction to others. It is, for example impossible to force someone into reflection, much less into reflective abstraction, but the process can be demonstrated.

DISCERNING DISTINCTIONS

Of course the remarks in the previous interlude and in the first section have already begun the process of addressing the question of ontogeny and ontology of distinctions. Indeed the notions of *procept* and *reflective abstraction* already constitute a partial response, by drawing attention to the way in which processes become objects through reflective abstraction. But to be perceived *as* a process, processes depend on distinctions being perceived and acted upon. The process itself has to be distinguished from the surrounding environment, and the process will necessarily involve the discerning of distinctions in order to act upon chosen elements in order to carry out the process. For example, to add two fractions requires distinguishing those fractions from other marks on the paper, and to discern and identify two numerators, two denominators, and the symbol which indicates division. If this seems trivial, then try adding the following fractions, and perhaps try to catch what you do with your attention:

$$\frac{\frac{2+\sqrt{3}}{5}}{\frac{2-\sqrt{3}}{3}} + \frac{\frac{6}{\sqrt{5}-1}}{\frac{1}{\sqrt{5}+1}}$$

It takes a little bit of conscious work to identify and retain the numerators and denominators as objects while at the same time being aware of their substructure as fractions.

Distinctions are made as a result of experiencing a disturbance, because that is how our sensory apparatus works. Animals and even plants react or respond to perceived disturbance to the *status quo* within the ecology of their environment Thus distinction-making is fundamental to all life forms.

Distinctions are in a sense vaccination doses based on previous disturbances, triggering reactions which were made in the past as well as offering possibility of less habitual responses. So a disturbance which attracts a need for a fresh response (accommodation rather than assimilation) generates what we call a distinction. When there is enough similarity to trigger a similar response, or as Marvin Minsky (1975) put it, if there are sufficient parameter values in addition to default parameters so as to fire a 'frame of mind', we detect relative invariance in the midst of other changes. We become aware of 'similarity': we become aware of aspects of a situation which it

is possible to vary without changing the essence of the situation. This is what gives birth to our sense of a phenomenon: we perceive a potential property which generalises a relationship detected in a particular instance. Variation as the essence of learning has been developed particularly by Ference Marton and colleagues (Marton & Booth 1997, Marton & Trigwell 2000, Marton & Tsui 2004). It is no surprise then that detecting and characterising relative invariance in the midst of other change is a fundamental theme of mathematics.

How do we detect or determine similarity rather than difference? In one sense, no two experiences are the same, or as Heraclitus put it, you cannot step into the same river twice. Yet if every instance where entirely novel, organisms could not function. Attention would be totally absorbed by constant novelty. Practices are only possible where there is repetition, and hence perceived sameness. Our perceptual apparatus does an enormous amount of classifying and rejecting before our cognitive functioning even gets a look in (Norretrander 1998). This is perhaps the principal source of many philosophical conundra, for it is very easy to fall into the trap of believing that the perceptions of which we are aware somehow truly reflect or present some objective material world, rather than being the tip of the iceberg of sensory processing below or beyond the level of consciousness.

Minsky's notion of neural states which fire when sufficiently many parameters are instantiated is certainly attractive as an explanation of perceived sameness. Caleb Gattegno's notion of stressing and consequent ignoring (Gattegno 1987) as the origin of generalisation serves a similar purpose in explaining the experience of sameness. So *phenomena* are what we call the experience of 'sense of sameness' of having 'been here before', and around which is then possible to theorise, making use of distinctions to discern elements to which we are currently sensitised, amongst which we recognise relationships, which I turn become properties that might obtain in other situations in the future, and on the basis of which we can reason about the worlds we inhabit.

Reification

Numerous authors have addressed the question of how reification comes about, perhaps most notably (other than David and Eddie), Anna Sfard (1991, 1992, 1994), Ed Dubinsky (1986, 1991a, 1991b) and Lakoff & Nunez (2000), not to say Zoltan Dienes, Jean Piaget, Hans Freudenthal, and of course Plato through his Socratic dialogues. In a sense all enquiry into learning of mathematics addresses this issue, however peripherally. It remains an action guaranteed will necessarily take place for any particular learner or learners at any particular time, however, trust in the organic and developmental thrust of life in a challenging but not entirely hostile environment is usually rewarded. It is the lack of guarantee which makes me doubt the efficacy of cause-and-effect as a suitable mechanism for discussing learning or designing teaching.

Ference Marton offers an example of how we might probe beneath the surface of reflective abstraction and reification. He argues that exposure to limited variation over a short period of time is, if not necessary, particularly advantageous, especially in an educational context (see Marton & Booth 1997, Marton & Tsui 2004). Here Marton and colleagues depend on natural powers of learners which seem to be hardwired into the brains which 'we construct' as our organism grows and develops. Gattegno (1987) put it most challengingly as "I made my brain" during development in the womb. But not everyone uses these powers in the same way. Reflective abstraction as a label crystallises ancient wisdom that something more is required for learning than merely suffering experience, and whatever this is, it depends on a suitably supportive classroom rubric (Floyd *et al* 1981), or as it has come to be called, a *community of practice* (Lave & Wenger 1991), or a *local community of practice* (Wimborne & Watson 1988). It seems to me that how to stimulate learners to engage in reflective abstraction is one of the abiding and core issues which requires more, and more detailed research, in order to inform practices as well as policies.

Ontogeny

Distinctions which inform behaviour in a positive or acceptable manner reinforce the pathways which triggered them. For our conscious awareness to become involved and to participate in choosing, several elements are required. First, the distinction needs to be encrusted with a rich web of experiences. It is not just that individual experiences need to come to mind, because the pathways triggering action need to become robust and stable. Second, it helps conscious participation if there is a suitable label. What functions well is a few chosen words which have metonymic associations with making the distinction. It is much harder to make use of an unfamiliar word or sound as a label than a familiar one, but it is also difficult if there are contrary associations which trigger other actions and discernments. One of the problems with introducing labels into the mathematics education discourse is that unhelpful or extraneous associations are either already present or soon develop. One has only to think of the misuse of associations with *zone* in the zone ofn proximal development, or the ways in which *concrete* ties people's thinking to the physical, material world when discussing mediating tools.

The subtle move in the ontogeny of distinctions seems to be the reappearance of the same or similar distinction on a subsequent occasion. Sometimes this awareness is well below the surface of consciousness and so beyond the level of cognitive control. Sometimes the distinction emerges into conscious awareness. When our tendency to language throws up a label, a distinction comes into social existence, enabling communication with others. We are already being socialised into recognition of and reaction to repeated or recurring instances of this phenomenon, according to characteristic samenesses of which we may or may not be consciously aware.

Communicating Insight & Jargon

The act of naming has consequences. The up-side is that naming makes it possible to communicate with others about what we discern, what relationships we are stressing, and hence what properties are salient for us. A label makes it possible to communicate efficiently and effectively with yourself and with others. The down-side is that naming seduces us into ontological commitment that the distinction being made is a natural cleavage in the world, rather than simply an indication of an instance of the coordination of our perceptual and languaging systems. Thus it is important to raise questions about distinctions which we try out. Do they inform future practice? Are they communicable to others who likewise find them informing their practice? What do they obscure and what do they illuminate?

A classic example of the negative effects of a distinction can be experienced in the terms *theory* and *practice.* This distinction pervades not only the mathematics education literature, but philosophic and psychological literature stretching back as far as Plato. The very fact that this distinction is so well established makes it difficult to generate an alternative frame of mind. One such attempt is through *enactivism* (Kieran 1988, Varela *et al* 1991) in which knowledge and action are seen as synonymous. The power of an established distinction can be experienced by trying to encompass *enactivism* as a stance: it is remarkably difficult to accommodate to a strongly enactivist perspective starting from a traditional perspective which distinguishes so essentially between knowledge and action.

Mathematics and Distinctions

One of the features of mathematics is that it works explicitly on similarity and difference through its pervasive theme of *invariance in the midst of change*: most mathematical theorems can be seen as statements about something which is (relatively) invariant, and aspects that are permitted to change and still retain that invariance. For example, the sum of the angles of a planar triangle is invariant, while the shape and dimensions of the triangle are permitted to change; a triangle has to have three 'angles' formed by three line segments meeting at three points, but those points can be anywhere in a plane, including lying on a single straight line, or even coincident; the product of the lengths of the two segments from a point to the two points of intersection of a line through that point with a given circle, is independent of the line, as long as it meets the circle!

Mathematics goes further, distinguishing and drawing attention to relationships which connect different features (e.g. where two straight lines cross, the pairs of opposite angles are always equal). Relationships in particular situations are generalised to properties, and a good deal of mathematics is about locating the logical interdependencies between properties. An extreme form of this is axiomatisation, in which certain properties are isolated and identified as axioms, and then other properties are derived from them. Although axiomatisation as a formal process is usually mentioned explicitly only at undergraduate or perhaps sixth form level as if it

involved advanced mathematical thinking, the process itself is woven through the entire school curriculum. The fact that this is often overlooked or unnoticed by learners and by teachers may contribute to the way I which school mathematical experiences often seem to miss or overlook the essence of about what mathematics is about.

Robustness

Distinctions which serve a purpose and inform behaviour naturally become more robust and stable with repeated use over time. However, after a while the may actually blunt perception and obscure rather than sharpen and inform, because their quick triggering inhibits different disturbances leading to fresh distinctions and further sensitivity.

Very often, the more a distinction is used, and the more people who use it, the fuzzier and less distinct it becomes. This might be happening to *advanced mathematical thinking*, and it often happens during the ontogeny of an individual's discernment of a distinction that the label encompasses rather too much before it narrows again and is used consistently with the way others use it. This is a natural feature of acquiring language (Brown 1973). It is not always easy to stand aside and let others mangle what for you are clear and useful distinctions, as they struggle ton accommodate their interpretations into their own thinking.

BEYOND DISTINCTIONS

Distinctions may be the origins of perception and fluent participation in thought and action, but they are far from the end. There is more to life than distinguishing this from that, foregrounding and consequently backgrounding, stressing and consequently ignoring (Gattegno 1987). It is possible to discern much finer structure in attention than merely discerning boundaries and variation. Without becoming aware of relationships and the emergence of properties and reasoning based on those properties, distinctions would soon become an unmanageable collection. It would be impossible then to describe or explain how some distinctions dominate others, and how distinctions coagulate into larger structures which can be discerned in the complexity of experience.

Ockham's razor is an admirable tool, but every attempt to reduce complexity to simplicity leaves itself open to mechanisation. The present trend to record and count every action, and to value only that which can be explicitly observed, recorded and counted, will eventually have to give way to a resurgence of appreciation of subtlety of that which is left unsaid and implicit, of that which maintains complexity. In short a revivifying of trust in the organic. Just as we trust plants to grow given suitable conditions, we have to trust children to learn to read, count, and use their undoubted powers to make mathematical sense of the world, and to make sense of mathematical thinking given suitable conditions. As long as we try to ensure learning, to ensure proceptual development, to ensure the growth of concept images, I suspect we will

fail with a significant proportion of the population: cause and effect is not a suitable mechanism for analysing and accounting for educational progress. I suspect however that the notions of procepts and concept images will continue to be an integral part of the awareness of mathematics teachers and researchers for a long time to come.

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PROCEPTUAL THINKING IN GEOMETRY

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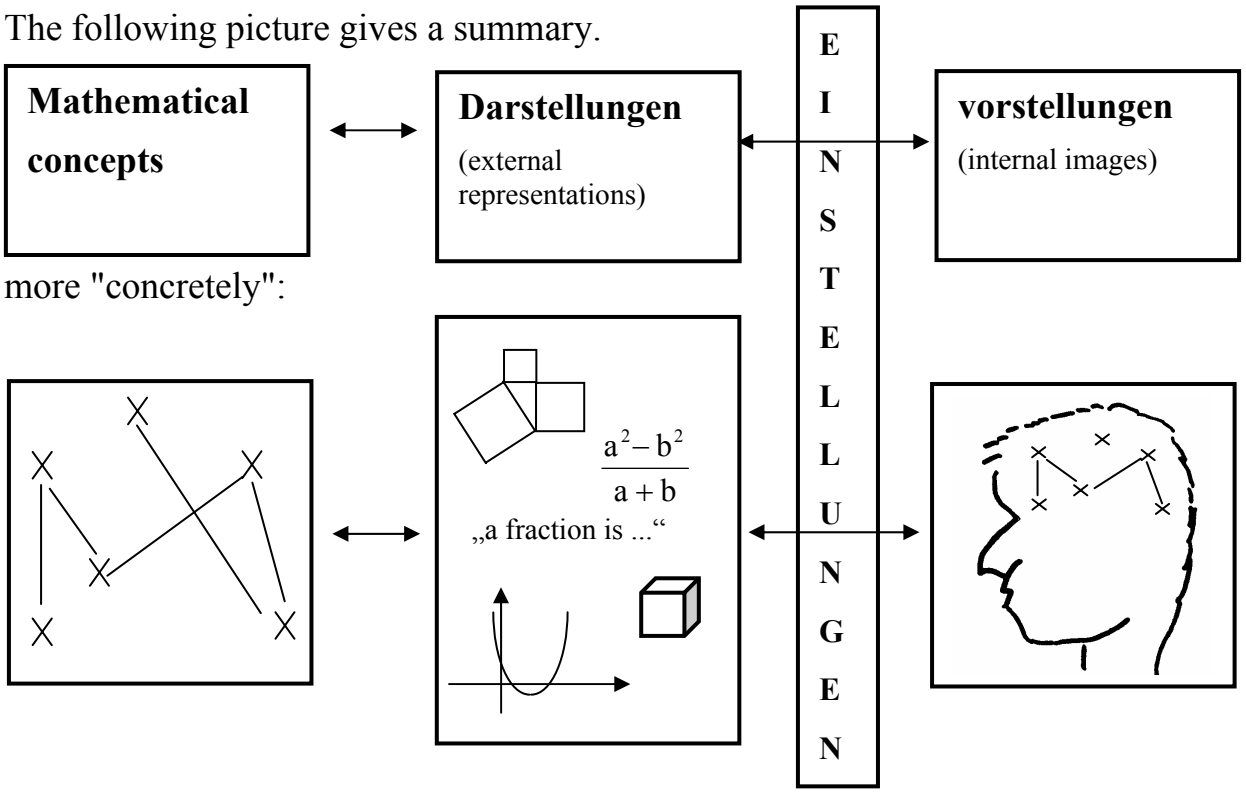
It is a pleasure and an honor to the author of this paper to contribute some ideas to the extended work of Eddie Gray and David Tall. Both have stimulated my research, through many discussions at conferences as well as in private atmospheres. I remember several visits in Muenster, singing communities of math educators in London and at other places, and fruitful and vivid exchanges of strong arguments in pubs at many places in many countries, especially also here in Prague. So I appreciate very much to be here now again, to celebrate together with all of you the work of David and Eddie.

DARSTELLUNGEN AND VORSTELLUNGEN

Much research has been done by Eddie and David in the domain *Psychology of Mathematics Education*, especially in the field of advanced mathematical thinking. They have studied intensively the interaction of teaching and learning and how students develop a mathematical understanding. Thus please allow me to start this paper with our view on the main aspects of *learning* and *understanding* mathematics (Meissner 2002b). I will concentrate on the interaction of ***Darstellungen*** and ***Vorstellungen***.

The following picture gives a summary.



The problem of teaching and learning mathematics is that we cannot see or feel or touch mathematics. Mathematical concepts and theories and theorems are abstract ideas. Mathematics seems to be ‘something’ which exists independent from human

beings. We only can get representations of mathematical ideas, which in German we call ***Darstellungen***. These Darstellungen can be objects, manipulatives, activities, pictures, graphs, figures, symbols, tags, words, written or spoken language, gestures, ... It is nly Darstellungen of mathematics we can read, or see, or hear, or feel, or manipulate, ... In a Darstellung, the mathematical idea or example or concept or structure is hidden or encoded and there is no one-to-one-correspondence between a mathematical idea, concept, etc. and a Darstellung.

Human beings are able to 'associate' with these objects, activities, pictures, graphs, or symbols a meaning. That means each Darstellung evokes or recalls a personal internal image, a concept image (cf. Tall & Vinner 1981), which in German we call a ***Vorstellung.*** A Vorstellung is a personal internal representation. It also includes reflections on actions and perceptions and it constantly can get modified by the individual. A Vorstellung in this sense is similar to a cognitive net, a frame, a script or a micro world. That means the same Darstellung may be associated with many individual different internal representations or images. Each learner has his/her own Vorstellung. And again here, there is no one-to-one-correspondence between a Darstellung and a Vorstellung[1].

This makes the process of teaching and learning mathematics very difficult: Do the Darstellungen we see from the learner really reflect the Vorstellungen we expect the learner should have developed? We only can judge a Vorstellung by a corresponding Darstellung. That means communication is necessary. To prove understanding, the learner must transform their individual Vorstellung into a Darstellung. And when the learner's Darstellung corresponds with one of the expected Darstellungen we may assume that the learner did understand. The problem is obvious. We do not judge a Vorstellung but we interpret a performance.

To distinguish the Vorstellung from the performance we will speak of a ***conceptual understanding*** when the learner has an adequate Vorstellung, i. e. his or her internal representations seem to correspond appropriately to the given situation. A conceptual understanding also may be intuitive or unconscious. And of course, a conceptual understanding only can be demonstrated indirectly up to a certain degree, consciously or unconsciously[2].

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[1] The process of building up a Vorstellung also depends very much on the basic mentality of the learner, i.e. on his or her ***Einstellung***, which includes affective components like attitudes, beliefs, emotions, values, etc. Einstellungen work like a filter or a catalytic converter in the transformation processes Darstellungen $\leftrightarrow$ Vorstellungen.

[2] Already Skemp (1978) had pointed out that there are performances which can be interpreted quite differently. He distinguished between *instrumental understanding* and *relational understanding*. In addition to Skemp we try to differentiate if we concentrate on Darstellungen or on Vorstellungen. Thus we distinguish:

- ***conceptual understanding*** (obviously adequate Vorstellungen seem to exist),

To study the student's conceptual understanding we still need *Darstellungen* from the student. But we must allow flexibility to interpret his/her Darstellungen. Not the Darstellung itself is important but the Vorstellung behind that Darstellung. When there seems to be a misunderstanding concerning a specific Darstellung, just change the Darstellung to clarify if the misunderstanding originates from the Vorstellung or from the Darstellung. Of course, also non-standard Darstellungen should be accepted. Clinical interviews with experienced interviewers and experienced class room teachers try to identify the student's conceptual understanding. Written tests like TIMSS or PISA usually cannot help to prove conceptual understanding[3].

PROCEPTUAL VORSTELLUNGEN

Which are the cognitive processes when children develop their individual mathematical Vorstellungen? Analyzing many mathematical concepts we discover a "conflict between the structure of mathematics, as conceived by professional mathematicians, and the cognitive processes of concept acquisition" (Vinner in Tall 1991, p. 65ff). Mathematical definitions mainly describe objects or a static view while the process of acquiring new insight often runs in parallel with activities or procedures or mental processes in time. Already Piaget and Davis have discussed this phenomenon (e. g. Davis 1984 or Piaget 1985).

But how can these inconsistent views, an object on the one hand and procedures on the other hand, grow together to form appropriate mathematical Vorstellungen? Here we can learn from Eddie and David and from some other authors[4]. Gray & Tall (1991, p. 72ff) identify a "duality between process and concept in mathematics, in particular using the same symbolism to present both a process (such as the addition of two numbers 3+2) and the product of that process (the sum 3+2). The ambiguity of notation allows the successful thinker the flexibility in thought to move between the process to carry out a mathematical task and the concept to be mentally manipulated as part of a wider mental schema." They hypothesized that the successful mathematical thinker uses a mental structure "which is an amalgam of process and

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- ***communicable understanding*** (similar to Skemp's relational understanding: "knowing both, what to do and why", i. e. communicable understanding combines both, the conceptual understanding with the ability to communicate in a wanted or given format flexibly using different types of Darstellungen),
- ***instrumental understanding*** (similar to Skemp 's "rules without reasons", correct Darstellungen are performed, but obviously without an appropriate conceptual understanding).

[3] Hospesova & Ticha (2002) found through interviews examples for both, good conceptual understanding, but no expected TIMSS-Darstellung and correct TIMSS-Darstellung, but no adequate conceptual understanding.

[4] Sfard (1987, 1992) identified a constant three-step pattern in the successive transitions from operational to structural conceptions (interiorization, condensation, reification). Dubinsky and his colleagues (in Tall 1991, Cottrill et al 1996) also studied the encapsulation phenomenon and developed the APOS theory.

concept". Tall (1991, p. 251ff), reflecting the dual roles of several symbols and notations: "Given the importance of a concept which is both process and product, I find it somewhat amazing that it has no name. So I coined the portmanteau term 'procept' ". And in Gray & Tall (1994) Eddie and David proposed the following definitions:

"An ***elementary procept*** is the amalgam of three components: a ***process*** which produces a mathematical ***object***, and a ***symbol*** which is used to represent either process or object. A ***procept*** consists of a collection of elementary procepts which have the same object."

symbol	process	concept
4	counting	number
3+2	addition	sum
-3	subtract 3 (3 steps left)	negative 3
3/4	sharing/division	fraction
3+2x	evaluation	expression
v=s/t	ratio	rate
y=f(x)	assignment	function
dy/dx	differentiation	derivative
$\int f(x)dx$	integration	integral
$\lim_{x \to 2} \left(\frac{x^2-4}{x-2}\right)$ $\sum_{n=1}^{\infty} 1/n^2$	tending to limit	value of limit
$(x_1, x_2, \ldots, x_n)$	Vector shift	point in n-space
$\sigma \in S_n$	permuting {1, 2, ..., n)	element of S_n

Table 1: Symbols as Process and Concept

In Tall et al (2001) we find the above table of examples for symbols which can provoke both a process as well as a concept. That means those *Darstellungen* stimulate or recall *proceptual Vorstellungen*, i. e. Vorstellungen where (dependent on the momentarily situation) either procedural or conceptual aspects will become dominant. Also Dubinsky (2000, p. 43) lists mathematical concepts, that have been treated on the use of APOS theory: Functions, binary operations, groups, subgroups, cosets, normality, quotient groups, induction, permutations, symmetries, existential and universal quantifiers, limits, chain rule, derivatives, infinite sequences, mean,

standard deviation, central limit theorem, place value, base conversion and fractions[5]. To summarize, for a constant interaction between Darstellungen and Vorstellungen it is necessary to develop, unconsciously and/or consciously, proceptual thought.

PROCEPTS IN GEOMETRY

It is interesting to read the above list from Dubinsky or to study the examples from Tall in table 1. We miss geometry. Are there special reasons for that? Are there no procepts in geometry in general? As already mentioned, the problem of teaching and learning mathematics is that we cannot see or feel or touch mathematics. This is a bit different when we regard the teaching and learning of geometry. Another reason might be that in many school books there still is a method of teaching and learning geometry which is similar to an axiomatic approach: Those books start with 'definitions' and properties (line, point, circle, square, ...), discover relations and prove statements. Of course, it will be difficult then to discover (like in arithmetic) "processes which may produce a mental mathematical object". Then there also is no necessity in geometry for getting symbols which are used to represent either a process or an object.

Tall et al (2001) formulate the hypothesis that there are three types of mathematics (space & shape, symbolic mathematics, axiomatic mathematics) and that each of them is accompanied by a different type of cognitive development. They consider - before focusing on the growth of symbolic thinking – "briefly ... the very different cognitive development in geometry". There are perceptions of real objects initially recognized as whole gestalts and classifications of prototypes. Reconstructions are necessary to give hierarchies of shapes and to see a shape not as a physical object, but as a mental object.

Struve (1987) also analyses how Vorstellungen in geometry develop. He summarizes, that children in primary and lower secondary classes learn geometry like a natural science, they describe and explain and generalize phenomena. Thus for them geometry is like an empirical theory. Points, lines, triangles, squares, etc. become *perceived* objects. In this stage children *believe but cannot prove* that the properties discovered will be true also tomorrow. There are big similarities between physics and empirical geometry: Given certain assumptions mathematics becomes a tool to predict future events.

What does that mean for geometrical procepts? When we analyze in "3+2" possible step-by-step procedures of the children we also observe "empirical mathematics" with real objects. And like in geometry the children generalize and learn to predict future events. They trust but cannot prove, that '3+2' always "will be the same". For them in this stage also 'five' is a 'perceived' object, like 'triangle'. Only we as

[5] For details see http://www.cs.gsu.edu/~rumec/index.htm.

mathematics educators or researchers, we regard a different 'five', a *conceived* object based on 'counting principles' (similar to the axioms in geometry). In this view an elementary procept in the meaning of Tall et al (2000) just can be seen as the shift from the empirical stage to the theoretical stage. Following these ideas we argue that there are no fundamental obstacles also to find procepts in geometry. Before we will start doing this we first will summarize what we are looking for.

We need activities or procedures. Then we have to identify some kind of mental development like:

- carry out accurately the/a given procedure/technique,
- several procedures/techniques are possible, select one (procedures become a process),
- several procedures/techniques are possible, make an efficient choice and
- explain,
- carry out the process flexibly and efficiently, i.e. determine and select an appropriate procedure/technique ***and discuss possible alternatives***,
- ***use abbreviations (symbols, tags, key words)*** for discussing and arguing (process becomes an object).

We think communication and especially the process of tagging or naming is a necessary part to develop proceptual thinking, see the above aspects printed in ***bold***. Via these steps we reflect on the actions and related perceptions, we move from unconsciously performed activities to conscious activities, and we shift from physical processes to conceptual Vorstellungen. Importantly, there is only one symbol or key word to summarize that dual meaning. And we should add that it is not important what type of symbol or key word it is. We also may use words, figures or even gestures[6]. 'Tags' just are abbreviations to name or to recall these ambiguous proceptual Vorstellungen.

EXAMPLES

In Meissner (2001, 2002a) we already described our observations of third graders who were working with developments of geometrical solids. A basic activity was to construct and to discuss nets of pyramids, of rectangular solids, of houses, a church, a garage, … We found that some students during these activities and discussions had developed a proceptual thinking on 'nets'.

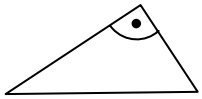
Here we will concentrate on three other examples. Let us begin with the example ***triangle***. Triangles are things we can see, touch or feel in our environment. These Darstellungen provoke Vorstellungen, perceptions as well as conceptions. When we work with triangles empirically (gluing together to nets, tessellations, folding, …) we probably will remain in the field of empirical mathematics. But the Vorstellung

[6] In one of our classes the students used in discussions their hands to show "rectangular" instead of using the word "rectangular" or the symbol ⊥.

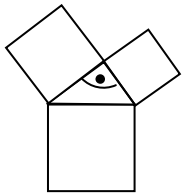
triangle gets changed when we start discussing properties of triangles, when we have to construct a triangle by three given properties, when we discuss congruence or when we must decide which types of triangles are necessary to design the roof of a certain building or to measure distances between two geographic points on the earth. Especially when using Dynamic Geometry Software (DGS) we will need an explicit proceptual thinking when working with “triangles”.

Another example of a procept is the ***Pythagoras'*** Theorem. There are several types of ‘tags’:

PYTH

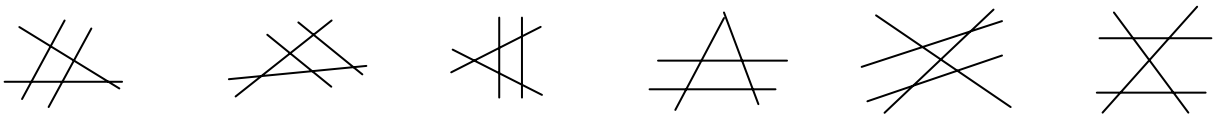


$a^2 + b^2 = c^2$



The tag shall recall a mathematical theorem to analyze a geometric situation, to discover geometrical structures, to identify geometrical properties, or it recalls a tool to perform specific constructions or calculations. But often our students do not achieve a proceptual ‘pythagorean’ thinking. They ignore or they do not see the property ‘perpendicular’ or they have fixed mental images of how to name the sides of a triangle.

Our third example is one of the main theorems in school geometry, the ***Strahlensätze***: A figure of four lines, where two intersecting lines cross two parallel lines, leads to three or four basic statements about the ratio of according lengths:



When I was a teacher I used the first diagram as a tag. By the use of this tag (or the verbal tag ‘Strahlensätze’) every student got the key word to remember processes how to construct specific figures, especially how to construct specific triangles with given properties. Or they switched to *Strahlensätze* as a tool which can be used also to analyze a geometric situation, to discover geometrical structures or to identify geometrical properties. But this proceptual thinking of "Strahlensätze" only is possible when we are able to regard the above figures as an entity. In this “entity” the angle between the two intersecting lines or the width of the two parallels can vary or the point of intersection of the two lines may be between or outside the two parallels. Thus the procept "Strahlensätze" is encapsulated in each of the above figures.

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AN ANALYSIS OF THREE MODES OF PROOF

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During the last four years, Tall (2002, 2003, 2004) and some of his colleagues (e.g. Watson, 2002; Watson, Spirou, & Tall, 2003; Tall & Mejia-Ramos, 2004) have been discussing differences in the ways in which statements are validated in what they have termed "the three worlds of mathematics". Using Toulmin's (1958) argumentation scheme, this paper presents a structural analysis of three arguments that Tall (2002) uses to exemplify the corresponding distinction between three "modes of proof". This analysis provides further insight regarding this distinction.

INTRODUCTION

This paper examines the notion of "mode of proof" described by David Tall in two recent conference presentations (i.e. Tall, 2002, 2003). Although this short paper indicates a way of producing a comprehensive analysis of this notion, its aim is simply illustrative and focuses on *one example* given by Tall. A detailed analysis of the *general* notion of "mode of proof" is left for a longer paper.

THE THREE WORLDS OF MATHEMATICS

David Tall's differentiation of three worlds of mathematics is the product of a search for a theoretical account of the variety of (successful and unsuccessful) ways in which individuals operate in mathematics and understand mathematical notions, not only at different stages of their cognitive development, but also through their use of different representations and ways of validation (Tall, 2002, 2003, 2004; Watson, 2002; Watson, Spirou, & Tall, 2003; Tall & Mejia-Ramos, 2004). These three worlds of mathematics are described by Tall (2002, p.5) in the following manner:

1. The *embodied world* of perception and action, including reflection on perception and action, which develops into a more sophisticated Platonic framework,

2. The *proceptual world* of symbols, such as those in arithmetic, algebra and calculus that act as both processes to do (e.g. 4+3 as a process of addition) and concepts to think about (e.g. 4+3 as the concept of sum) as formulated in the theory of procepts (Gray and Tall, 1994).

3. The *formal world* of definitions and proof leading to the construction of axiomatic theories (Tall, 1991).

More generally, these worlds categorise conceptions of mathematical notions (visuo-spatial in the embodied world, proceptual-symbolic in the proceptual world, and formally defined in the formal world), ways of conceptualising these notions (through perception, action, and reflection in the embodied world; through encapsulation in the proceptual world; and through the deduction of properties in the formal world), and ways of operating with these notions (physical manipulation and thought experiment in the embodied world, symbol manipulation in the proceptual world, and axiomatic

structuring in the formal world). This paper focuses on the corresponding categorisation of three "modes of proof and belief" (Tall, 2002), i.e. the ways of validating statements in each one of these worlds of mathematics.

"Warrants for truth"

Following Rodd's (2000) interpretation of Plantinga's (1993) notion of *warrant*, Tall (2002, 2003, 2004) uses the term *warrant for truth* to denote *that* which secures someone's knowledge in what is claimed to be known. The notion of *warrant for truth* is mainly used to contrast mathematical proof to *that* which engenders personal understanding and internally convinces[1] someone of the truth of a given statement. Indeed, as Rodd (2000) attractively summarizes in the title of her paper, a proof may fail to internally convince someone of the truth of the proved statement, and people (certainly young ones!) often understand through, and are internally convinced by, evidence that would not qualify as mathematical proof.

Tall (2002, 2003) notes that each world of mathematics can be further characterised with a particular kind of *warrant for truth*, and illustrates this distinction by presenting three different ways of *warranting the truth* of the following statement:

The sum of the first n whole numbers is $\frac{1}{2}n(n+1)$.

Regarding these three *warrants for truth* (presented in Figure 1) and their association to the three worlds of mathematics, Tall (2002) remarks:

> What this shows are three distinct ways of convincing, one embodied, one proceptual and one formal. Each has a different warrant, the embodied one uses a pattern to allow us to 'see' the truth of the statement, the second uses the same pattern in a numerical calculation, the third uses a formal proof by induction. (p.8)

In general, the way of operating in each world (as well as the nature of the conceptions dealt with in that world) is related to a particular way of warranting truth:

1. In the *embodied world* something is initially true if one knows it is true through perception and action in the physical world. Later on, thought experiments and deduction on generic examples become the main source of *warrants for truth* in this world.
2. In the *proceptual world* something is true if it is warranted by a correct calculation or symbolic manipulation. In this world, "algebra gives us a warrant for truth, based on implicit use of 'rules of arithmetic', by manipulation of symbols" (Tall, 2002, p.13).

[1] The distinction between internal conviction and external conviction is discussed by Mejia-Ramos and Tall (2005) and aims at differentiating conviction that rests on personal belief systems (internal conviction) from conviction based on socially-established belief systems (external conviction).

3. In the *formal world* something is true if it can be deduced from a set of axioms and basic definitions. In this world, mathematical proof is the main *warrant for truth*.

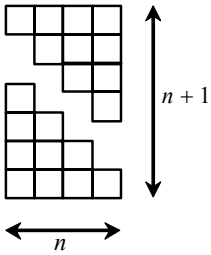
EMBODIED	 "A picture of the sum of the first n numbers as a staircase, and the putting together of two staircases to give a rectangle of area n by n+1." (p.7)
PROCEPTUAL	"He simply imagined the numbers in the reverse direction and, on adding them in succession, he obtained 101 for each sum a total of 100 times, so twice the sum is 100x101 and the sum of the first 100 numbers is half of this." (p.8) *add columns* ↓ $1 + 2 + \ldots + 99 + 100$ $100 + 99 + \ldots + 2 + 1$ $101 + 101 + \ldots + 101 + 101 = 100 \times 101$
FORMAL	A proof by induction.

Figure 1: Three warrants for truth (diagrams and wording from Tall, 2002, p.7-8).

However, some questions arise from the example presented in Figure 1. Certainly, a proof by induction of the given statement may be produced syntactically (Weber and Alcock, 2004) through symbolic manipulation, with no awareness of the structure of mathematical knowledge that characterises *warrants for truth* in the *formal world*. While it could be argued that in such a case the student is only acquiring an external sense of conviction (possibly based on what an authority believes), it can also be argued that for some students the truth of statements *is secured* through this kind of procedures, particularly when these procedures have been used successfully and confidently many times in the past. Therefore, this syntactically-produced argument may provide a *warrant for truth*, but it is unclear whether it constitutes a *proceptual proof* or a *formal proof.*

Another question is: does this description of the three worlds of mathematics account for mathematical argumentation that carries different degrees of confidence in the conclusion? The *embodied proof* in Figure 1 may *reduce* someone's uncertainty about the truth of the statement, without necessarily *securing* that person's knowledge of its truth: in what sense is this argument a *warrant for truth*?

In order to provide some answers to these questions, this paper focuses on the structure of *warrants for truth* and their differences in the three worlds of mathematics.

TOULMIN'S SCHEME

In *The Use of Arguments*, Toulmin (1958) presents a layout for analysing arguments, a scheme that departs from the classic syllogism and formal logic approach in an attempt to model *practical* arguments (which may lack logically necessary conclusions). The complete scheme classifies the statements of an argument into six distinct categories (Figure 2):

1. **Conclusion** [C]: The claim whose truthfulness the arguer wishes to establish.
2. **Data** [D]: The particular evidence that is put forward as the foundation of the conclusion [C].
3. **Warrant** [W]: The general proposition that bridges the data [D] with the conclusion [C].
4. **Backing** [B]: The field-dependent assurance set to support the general authority of the warrant [W].
5. **Qualifier** [Q]: A statement of the level of confidence and general strength of the conclusion [C].
6. **Rebuttal** [R]: The circumstances in which the conclusion [C] is not supported by the data [D] and warrant [W] provided.

The scheme is used to model both explicit and implicit statements of a given argument. Often, only some of these six types of statements are explicitly stated by the arguer. In these cases, the analysis of the argument may include statements that, while not being stated explicitly by the arguer, act implicitly in the argument as one of the statement- types of the full scheme.

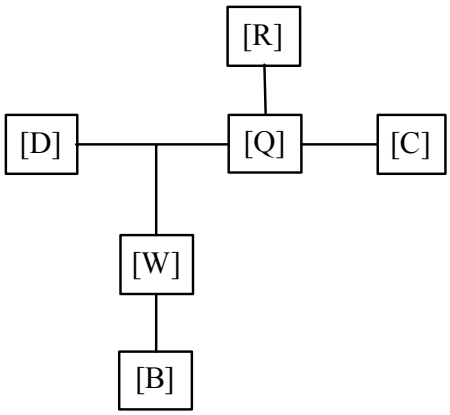


Figure 2: Toulmin's scheme of a general argument. The argument would read "[D], and since [W] (on account of [B]) we can [Q] conclude [C], unless [R]"

A reduced version of this scheme has been used by Krummheuer (1995), and other mathematics education researchers after him to analyse arguments emerging in mathematics classrooms. However, in line with Inglis, Mejia-Ramos and Simpson

(submitted), this paper illustrates the usefulness of using the full scheme in the analysis of arguments in mathematics.

"WARRANTS FOR TRUTH" AND TOULMIN'S SCHEME

Initially, it becomes clear that one difficulty of using Toulmin's scheme alongside Tall's worlds of mathematics lies in the differing use of the term *warrant* in their descriptive accounts of argumentation. While Toulmin allows *warrants* to be qualified, rebutted, and backed in a particular way, Tall's *warrants for truths* "secure knowledge", implying that the conclusion *surely* follows from the data, and the argument allows no rebuttal. Furthermore, in Toulmin's terms Tall's *warrants for truth* seem to encompass both *the data and the warrant* of an argument. This difficulty is addressed by making a consistent distinction between Toulmin's *warrant* and Tall's *warrant for truth*. Once this distinction is made, Toulmin's scheme reveals interesting characteristics of Tall's "modes of proof".

Considering the three different *warrants for truth* summarized in Figure 1, and their structure in Toulmin's scheme (Figures 3, 4, and 5), it becomes clear that Tall's criteria for discriminating between them takes into account a combination of different types of data, warrants, and backings. Certainly, the data of the *embodied proof* (Figure 3) is primarily visuo-spatial. However, the data in both the *proceptual* and the *formal proof* emerge mainly from calculation and symbolic manipulation. Therefore, in this particular case, the kind of data used by the arguer does not reveal a distinction between the *proceptual proof* (Figure 4) and the *formal proof* (Figure 5).

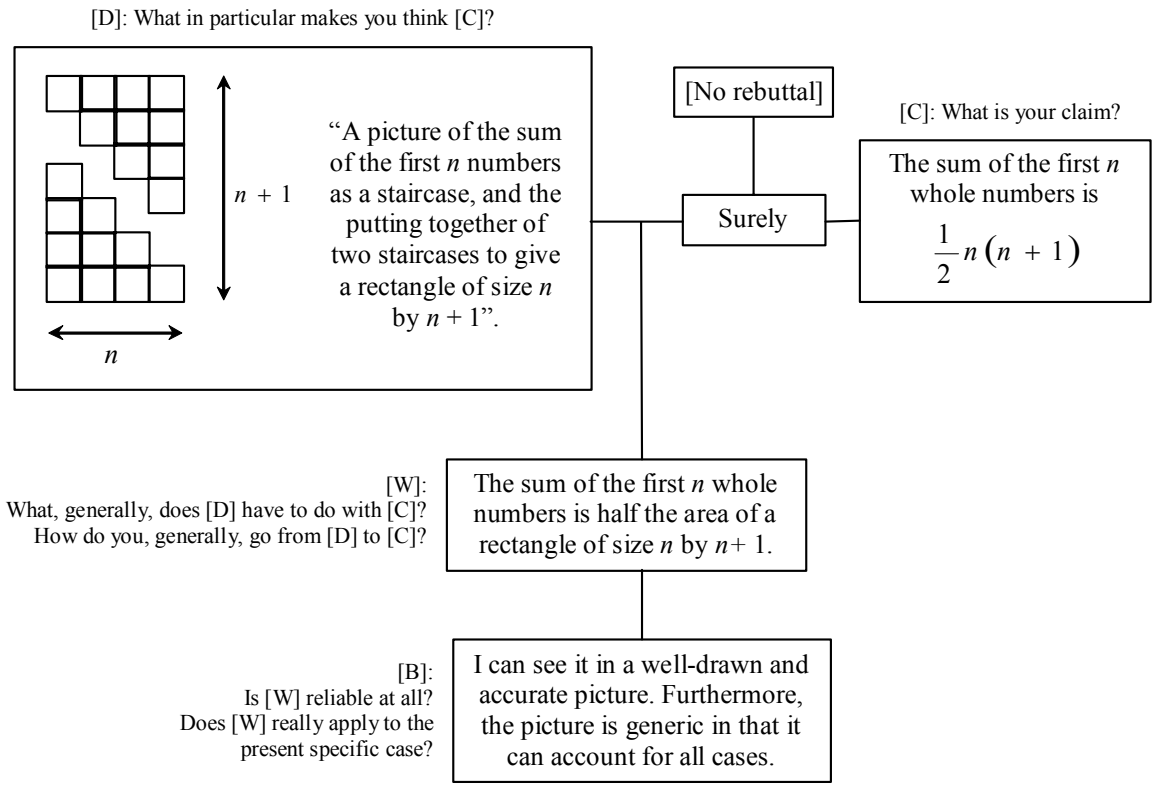


Figure 3: Tall's (2002) *embodied proof* laid out using Toulmin's scheme.

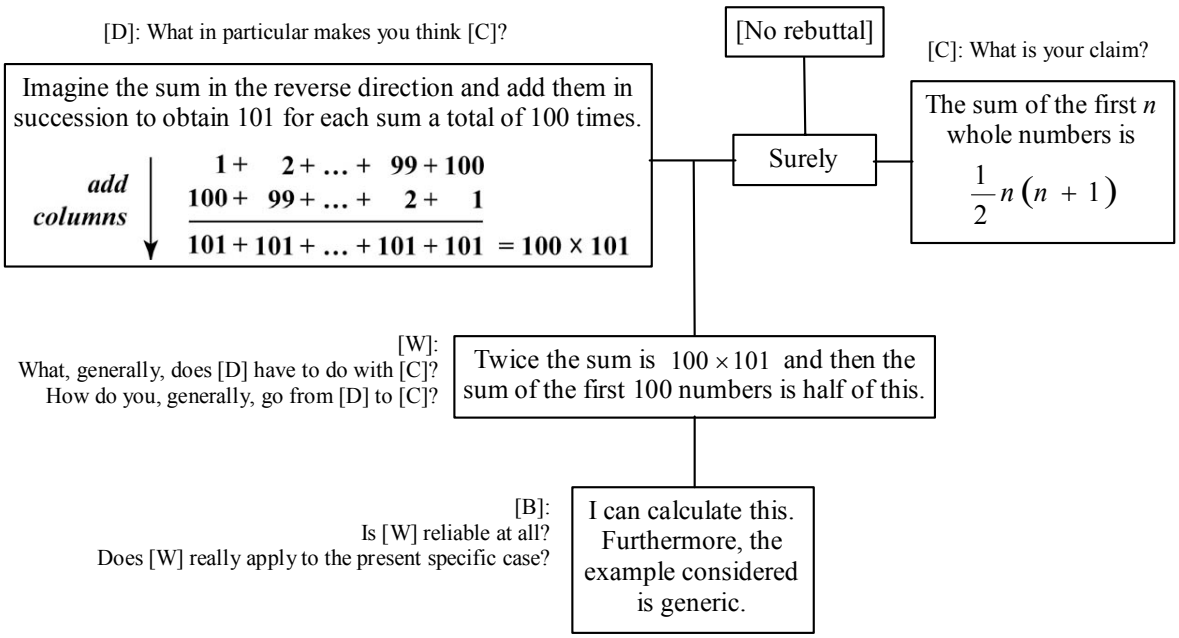


Figure 4: Tall's (2002) *proceptual proof* laid out using Toulmin's scheme.

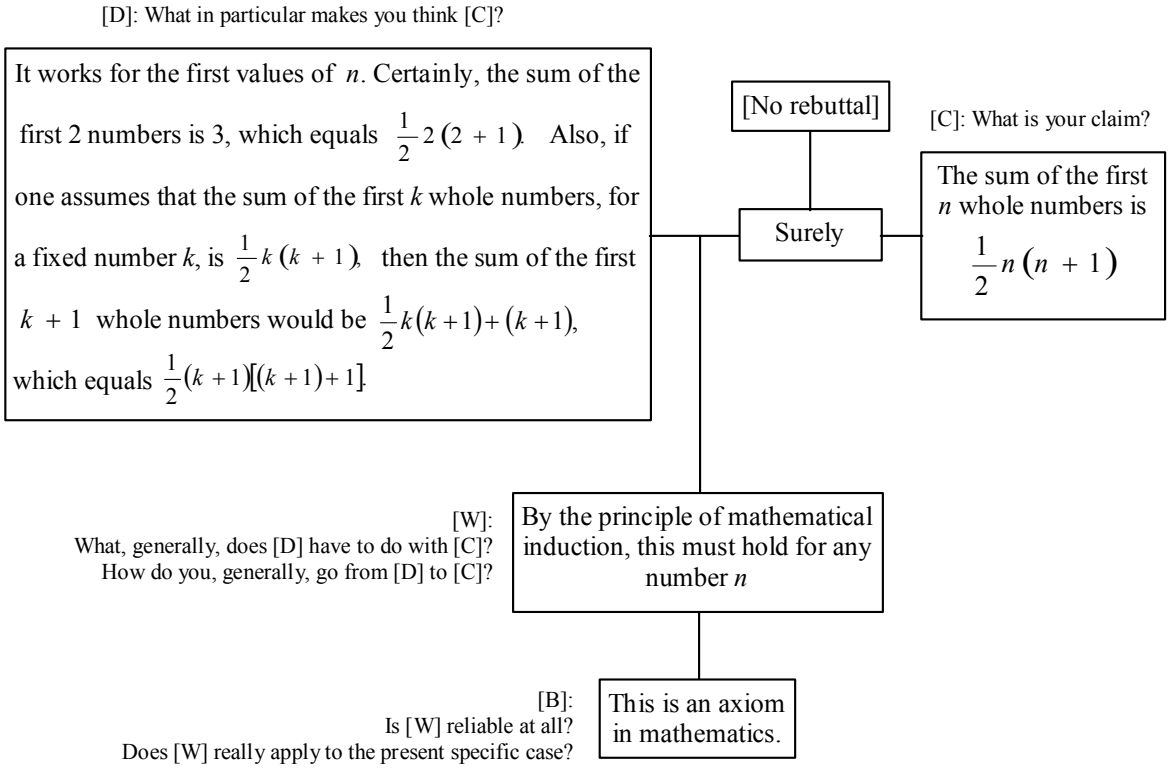


Figure 5: Tall's (2002) *formal proof* laid out using Toulmin's scheme.

The *warrants* for the *embodied proof* and the *proceptual proof* are similar in that they both link generic cases with a general conclusion; and their main difference is the kind of data that they link with the conclusion. On the other hand, the *warrant* of the *formal proof* is a formal mathematical justification based on symbolic manipulation and deduction from the Peano axioms; and is particularly interesting because it can be backed both as an axiomatic principle of mathematics or as a symbol-manipulating procedure learned in school.

Therefore, the essential criterion used by Tall (2002, 2003) to discriminate between these three arguments emerges only with an analysis of their *backings*. The *warrant* of the *embodied proof* is clearly backed by the fact that *one can see* the rectangle and the size of the two triangles forming it. On the other hand, the *warrant* of the *proceptual proof* is backed by the fact that *one can calculate* $1+100, 2+99, 3+98,$ etc. to find out that they all add up to 101. Contrastingly, the *warrant* of the *formal proof* is backed as an axiom. This, I claim, is what may differentiate this last argument from a *proceptual proof* consisting of the same data and warrant, but with a backing referring to a procedure learned in school. In other words, the *formal proof* in Figure 1is *formal*, not because it is warranted using the principle of mathematical induction, but because Tall (2002) explicitly backs this warrant as being an axiom in mathematics. This, of course, reveals a level of subtlety which may obscure the categorisation of arguments that lack *explicit backings*.

Finally, the *qualifiers* of these three arguments are very strong. Certainly, this is a defining characteristic of *warrants for truth*, as they are meant to "secure knowledge". However, as discussed by Inglis et al. (submitted), many mathematical arguments are given, not to secure knowledge, but to increase confidence in the truth of a statement. For a sceptic mathematician, conclusions reached by *embodied proofs* and some *proceptual proofs* (through backings like "one can see it", or even "one can calculate it") would tend to be *qualified* as "plausible" or "reasonable", and *rebutted* by statements like "the picture may be misleading". The strong qualifiers attributed to the three arguments in Figure 1 are the product of assuming that, in a certain "community of practice" (Tall, 2002, p.8) this kind of backings would be enough to "secure knowledge". However, higher levels of expertise in a given field give rise to an awareness of possible rebuttals, and consequently, the appropriate use of weaker qualifiers. This feature of mathematical growth does not seem to be accounted for in Tall's (2002, 2003) description of the three worlds of mathematics.

CONCLUSIONS

The use of Toulmin's (1958) scheme to analyse Tall's (2002) example of three different "modes of proof" reveals some common characteristics between them (e.g. symbolic manipulation in the *proceptual* and the *formal proof*; warrants based on generic cases in the *embodied* and the *proceptual proof*; and strong qualifiers in all three arguments), as well as an essential difference in the type of backing used in each argument. However, a more detailed analysis of the general notion of "mode of proof" is needed, as the differences/similarities between these three *warrants for truth* are more subtle than the ones revealed by the analysis of one particular example:

> Each world of mathematics has its own sequence of development and warrants for truth are not static notions that are fixed for all time. As the individual grows in sophistication, the need for subtler arguments becomes apparent. (Tall, 2002, p.8)

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MATHEMATICIANS AND CONCEPTUAL FRAMEWORKS IN MATHEMATICS EDUCATION … OR: WHY DO MATHEMATICIANS' EYES GLINT AT THE SIGHT OF CONCEPT IMAGE / CONCEPT DEFINITION?

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There is a pressing need for addressing issues of learning and teaching mathematics at undergraduate (and any educational) level through close collaboration between the communities of mathematics and mathematics education. And that includes theory building – a quintessential element of the overall aims of mathematics education research. Introducing mathematicians to already existing theoretical constructs of mathematics education research is one step in the often long, arduous but eventually rewarding road of collaborative work. Here I offer evidence from a study in the context of which an opportunity arose for mathematicians to be introduced to, and adopt the 'vocabulary' of, the Concept Image / Concept Definition framework; and, I reflect briefly on the epistemological and pedagogical significance of this evidence.

> 'I have little experience of mathematics education conferences, but the one occasion I seriously attempted attendance [names PME session] was a thoroughly discouraging experience. There was this curious argument between [names session leader] and someone in the audience and the tenor and the content of the argument left me thinking I was wrong to attend that talk and right to miss all the others. Which is unfair… but you know what I mean? This is just nonsense. This is egotistical nonsense. And I thought of somehow connecting what I was hearing with what I was doing in a lecture theatre or in a seminar group… but, to do that, first I would want them to kind of not be arguing about what they are meaning. If they don't know, I am not going to pay attention to what they are saying because they should know. And they seemed to be arguing about what they meant which was unfortunate…'. *M, mathematician*

The above 'quotation'[1] originates in a series of interviews with mathematicians in the context of which opportunities arose to discuss the solidity, consistency, relevance etc. of theoretical constructs of mathematics education research (as known, if at all, and perceived by the interviewees). M's words above consolidate the longstanding and ongoing suspicion mathematicians hold against the capacity of mathematics education researchers to 'deliver' on the theory-building front (Goldin, 2003). Here I offer a few examples from the study which, I believe, demonstrate how the participants' 'acquaintance with' one of the (seminal) theoretical constructs of mathematics education research, *Concept Image / Concept Definition* (Tall & Vinner, 1981), was followed by its rather seamless absorption into their 'vocabulary'. I then

[1] See next page where I explain briefly how verbatim quotations from interviews with groups of mathematicians have been consolidated into 'quotations', utterances of a single mathematician -'character' named M.

briefly comment on the potential significance of these examples. First however I outline the study.

ENGAGING MATHEMATICIANS AS EDUCATIONAL CO-RESEARCHERS

The work I am exemplifying from in this paper originates in a series of studies that have been taking place at Oxford University and the University of East Anglia in the UK for several years. These studies represent a certain type of research into the teaching and learning of mathematics at the undergraduate level that aspires to address at least two issues that are significant to educational practitioners at this level: the longstanding traditionalism of university mathematics teaching – usually coupled with the fact that colleagues in mathematics departments are rarely 'provided with incentives to seek out the findings of research in mathematics education', (Nardi et al, 2005, p284) – and the often fragile relationship between educators and mathematicians (Nardi & Iannone, 2004). This type of research aims to be a collaborative (Jaworski, 2003), non-deficit (focusing not on only on problematic but also on successful current pedagogical practices) and non-prescriptive (Dawson, 1999) exploration of pedagogical issues at university level. In the context of this type of research practitioners and researchers are offered opportunities for reflection on current practice and consideration (as well as implementation/evaluation) of reform that is collaboratively conceptualised and applied.

In this paper I draw on the latest of the above mentioned studies that engaged mathematicians from six universities in the UK as educational co-researchers[2]. In particular, the study engaged university lecturers of mathematics in a series of group interviews (11 interviews, approximately half-a-day long each) each focusing on a theme regarding the teaching and learning of mathematics at university level that the literature and my, and my associates', previous studies acknowledge as seminal (see a more detailed rationale for this selection in Nardi et al, 2005 and Iannone & Nardi, 2005). The focused group interviews revolved around six themes: two on Formal Mathematical Reasoning (*Students' Perceptions of Proof and Its Necessity*; *Students' Enactment of Proving Techniques and Construction of Mathematical Arguments*); two on Mathematical Objects (the *Concept of Function Across Mathematical Topics*; the *Concept of Limit Across Mathematical Contexts*); one on *Mediating Mathematical Meaning Through Words, Symbols and Graphs*; and, the meta-theme: *Collaborative Generation of Research Findings in Mathematics Education*. Discussion of the theme in each interview was initiated by a Dataset that consisted of: a short literature review and bibliography; samples of student data (e.g.: students' written work, interview transcripts, observation protocols) collected in the course of the previous studies; and, a short list of issues to consider. Analysis of the interview transcripts drew theoretically on constructivist, sociocultural and enactivist ideas

[2] Conducted at the University of East Anglia in 2002-04 with Paola Iannone and with a grant from the Learning and Teaching Support Network in the UK.

(see, e.g., Nardi et al, 2005 for details), largely followed Data Grounded Theory (Glaser & Strauss, 1967) techniques and resulted in sets of Episodes arranged in terms of the following five broad Categories: Learning; Teaching; School Mathematics; Culture of Professional Mathematics; Relationship with Mathematics Educators.

MATHEMATICIANS AND *CONCEPT IMAGE / CONCEPT DEFINITION*

The examples I offer here are samples from a book I am currently working on[3] and which builds on the study's analyses. In it the participating mathematicians' perspectives are presented in the slightly unconventional but not unusual (e.g. Lakatos, 1976; Davis & Hersh, 1981; Sfard, 1998; etc.) format of a fictional yet data-grounded dialogue[4] between two characters, M and RME, a mathematician and a researcher in mathematics education. Typically the discussion between M and RME starts out from commenting on samples of Year 1 mathematics students' written responses to questions that had been handed out as 'homework' in the context of Calculus, Linear Algebra and Group Theory courses (and are the examples the mathematicians had been asked to reflect on in the study's group interviews).

The two Episodes I draw on below are entitled *Concept Image(s) / Concept Definition* and *Concept Image Construction*. The latter is the Episode cited in (Nardi & Iannone, 2003). I summarise it here and supplement it with a 'meta-scene', an Episode in which M, after reading (Nardi & Iannone, 2003), discusses the presentation of himself and his ideas in that paper (and along the way refers to the issues that are germane to the discussion here).

Episode I: *Concept Image(s) / Concept Definition*

The excerpts below are from two scenes of the Episode, one entitled Domineering presences (function-as-formula), conspicuous absences (domain-range) and another entitled The turbulent relationship with the concept definition.

In the former M focuses on two dominant, and interrelated, images of function students arrive at university with: the absence of reference to domain and range as constituent elements of a function,

> M: It is difficult for them to perceive the importance of sets and axis labeling. It is the measurement of a quantity in terms of another that is the students' focus. Observe and measure is what they have associated functions with: if I measure the temperature of

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[3] Excerpts from Chapters 5 and 7 in (draft of) *Amongst Mathematicians: On the Teaching and Learning of Mathematics At University Level (…and everything meta-mathematical in-between)*.

[4] Processing of the interview transcripts followed a combination of Narrative Approach holistic-content analysis techniques such as *Re-storying* (Clandinin & Connelly, 2000) in which the 'stories' told by participants in the original raw data are 're-told' in thematically ordered units of Narrative. In this sense M's utterances in the dialogues are a synthesis of the views of the participating mathematicians on the particular issue that each Scene of the dialogue focuses on. Analogously the (far shorter, intended to have a probing/auxiliary role) utterances of RME are a synthesis of the contributions made in the interviews by the researchers conducting the interviews.

every point in this room – or the electric field in every point in this room – then I have a function of three variables (for the location of the point in space). Range/domain are almost irrelevant to them because they are defined by the general problem you look at. Say, if you are solving a differential equation, your range and domain are somehow given by the context. Temperature will not be smaller than, say, 280 degrees Celsius. So range and domain are not so important in a way. And this kind or approach marks the beginning of a battle for me because they have to see the importance of range, domain and co-domain sooner or later. For example, they have to start seeing the mechanism of Mathematical Induction as a function running across *N*.

…and of function-as-formula:

M: In fact [students see] one formula for every function. A function like $1/x\sin x$ defined as zero at zero is interesting because understanding its definition involves seeing this expression in terms of the various values of x in R, realising the need to define it via a different formula or value at a point where this formula doesn't work, $x = 0$. The problem is that they would see this as more than one function because it is defined in terms of more than one formulae or relationships.

In the latter M explores what 'the intrinsic difficulty of the concept of function is'. 'A Bourbaki definition of this concept is too weak a building for the students at this stage', he proposes, as 'students have too few previously acquired building blocks in order to deal with it'. To deal with the urgency of helping students build an understanding of one of the most 'basic notions' they will ever encounter in mathematics 'maybe students' ideas that they have fixed already need to be challenged a bit':

M: If they write down a formula of a graph and that is part of the problem, we may then have to challenge them: if you want to find the inverse of this function, in what circumstances can you do that? And get these things hard at them and things start to break down. It's no good if they go through all this first year stuff and they come up thinking, well, that was pretty tough, wasn't it, but I will just carry on with what I was doing at A level. It would be very interesting to ask them *what is a function?* on the first day of the first year and then the second year because I am sure that on the first day of the first year they would know and on the first day of the second year they wouldn't know. They have seen this building collapse several times. All their previous notions of function would be declared invalid. […]

at the concluding part of that scene M focuses on the role pictures (graphs etc.) can play in forming students' concept image at this stage:

M: You see a graph as a concept image of function is perfectly suitable in Analysis but rather useless in Algebra. Nobody would ever draw a graph of a morphism or a permutation. And the student is supposed to jump between these paradigms at great speed and use the essence in each paradigm to the best effect. Is that reasonable? I don't think it is reasonable. A picture, a graph in Analysis is a fundamental tool, mathematically it is absolutely essential that you can think in those terms. In Algebra it is almost meaningless. The word graph is a completely superfluous notion. I guess the expectation from the students is that they learn how to distinguish between what are the

important images that are supposed to work in one field but not in another. From a teaching point of view I would of course question whether it is acceptable to call all these things functions in the different contexts but not discuss that in fact they are all the same thing. Can we reasonably expect that the commonality across all these uses of the concept will naturally emerge in the students' brains? Then again you may say that prematurely indoctrinating students about that commonality is futile. And we have seen what happened to the Bourbaki approach, haven't we?

Episode II: *Concept Image Construction* (and a '*meta-scene*')

In Episode I M and RME discuss students' concept images and how these relate to the students' handling of definitions in the context of examining students' emerging understandings of the concept of *Function*. Episode II takes place in the context of a discussion of a Linear Algebra problem which involved the concept of the *Adjoint Matrix*. This triggers discussion of the concept of *Determinant* (one way to define $Adj(A)$, the *Adjoint Matrix* of a matrix A, is as $det(A)A^{-1}$). Below I present the Episode in summary – see (Nardi & Iannone, 2003) for full citation of the transcript.

M initiates discussion of the concept of *Determinant* with citing possible images one initially may hold of the concept: 'just a bit of garbage coming your way when you have to apply certain rules'? 'a volume'? 'as something to be worked out'? In the context of our problem – which involves the Adjoint – one could see, he says, 'the Adjoint as a cute way of getting the Inverse of a matrix'. This would be at this stage one of those 'simple justifications' a student 'needs to have at hand for new things':

> M: Probably these simplifications take away some of the true power of the concepts but, to start with, they are good enough for me. So I am looking at the student's work and I am thinking: what were the pictures in her mind when she put this or that down? In a sense I am trying to understand and appreciate the student's landscape. Everyone has their own personal landscape. For example, regarding determinants, I wouldn't be surprised that, like an intergral, a determinant is seen by most students as a number to be worked out. And then I am contemplating whether this is what I would like to instill in them about determinants or whether I can give them a more structural *raison-d'-être* for the concept.

RME then asks whether 'there is some sharing of landscapes between him and the students'. After all he has been 'working with these things for a long time and his landscape is probably richer than theirs'. How does one then conduct this imparting of landscapes? M lists several examples from his lectures (e.g. linking 3x3 matrices to transformations in the context of Analysis) and stresses that students are constantly surprised by these links, e.g. between Linear Algebra and Analysis. They find them difficult, even 'incongruous', he claims and stresses that establishing these links is to him absolutely crucial:

> M: The question we are looking at is in fact a good example of an opportunity to make that link between determinants as something which just saves you writing down a large number of elements. For instance, you can write down the adjoint of A in terms of $det(A)$. I see this business of sharing landscapes as my main business as a lecturer. We

are not just communicating facts, we are saying that this is one way you can view it and that is another way you can view it, let's put these together somehow. And it's not an easy job, believe me! Say you are thinking of determinants as volumes, what the heck is a 4x4 determinant then? And, of course, mathematically you want them to think of an *nxn* determinant. Plus the fact that making the connection is a personal issue because you need to establish links between your pictures, not somebody else's pictures. […] You need to have your own tailor-made brain version of what the thing is, ordered according to your view, in your own time, without anyone's landscape imposed on you… but of course with some guidance. […]

What concept images of a determinant would he expect students to have at the end of their degree? RME then asks.

M: Volume, images from physics, integration etc: any image would be welcome as long as the students don't say this is a thing that I can compute. If that happens then I would say that I failed. If they say it is volume, or it helps to solve systems of linear equations… I would sort of be surprised but yes, this is also good. I recognize that the emphasis on calculational tasks is sometimes a necessary evil of devising exam questions and making the exam passable for students but I also happen to believe it is a mistaken attitude. Rather than algorithmic ones, maybe we should be setting questions that foster understanding where students need to…showcase their concept image, rather than algorithmic understanding. I can recall incidents where students stared in bafflement about going from one idea to another and were very uncomfortable suddenly going from numbers to vectors. This crucial navigating between these things is often very difficult – and it shouldn't […]

A 'meta-scene'. Once publication of material from the earlier parts of the study was under way M was asked to read and reflect on (Nardi & Iannone, 2003). His comments on the above Episode, which constituted the main example from the study's data presented in the paper included the following:

M: […] the excerpt is an example of us talking about shared or less shared notions on a mathematical concept – there seems to be surprisingly little common ground amongst us about the notion of a matrix as I would expect an algebraist, for example, to first situate a matrix in a group. I wonder actually whether enough of this is being done between teachers and students. Sharing an exactly common ground is not in fact the issue at all, indeed it is interesting that we differ but at least we are talking about something that is coming from a common experience that we have. Hunting for some common ground with the students would be great but we don't.

M then 'elaborates a bit upon the algebraist's arguably preferred perception of a matrix' and juxtaposes it, for example, to that of a physicist and cites several contexts in which connection across perceptions is possible (Physics, Economics, Transformations in Analysis and Algebra). Returning more specifically to the concept of *Determinant* M cites several contexts (Integration, Jacobeans etc) where the concept is encountered. He juxtaposes the potent image of a determinant as 'a distortion of volume' to an initial image a student may acquire, for example in the context of Computer Science, of a determinant 'just as an array of numbers'.

M: And then maybe go through the whole degree without dealing with them as anything else. And we should never forget the difficulty of learning to deal with all this multiplicity of contexts or that the things we want to connect are actually not known to the students. You see, it would be nice at one stage to say, ok, this is what matrices do in geometry and this is what they do in economics and so on and now we can potentially do it in a unified way, but we don't know each one of the topics we want to connect. And I don't mean to underestimate the knowledge with which the students arrive from GCSE but some have never seen these things before at all; or seen them at GCSE and then not at A level so by the time they are here they haven't seen them for a long time. So do not be surprised if you ask them about the rotation of R cubed, what is that in terms of a matrix, and what you receive is a very blank look. *(Discussion turns to the influence of school mathematics, a different Episode)*

INTEGRATING M'S AND RME'S PEDAGOGICAL 'GENRE' SPEECH[5]

In the above6 M resorts to the Concept Image / Concept Definition theoretical construct – either literally or in apparently different but akin words such as 'landscapes', 'pictures', 'brain versions' etc. – in order to reflect on students' trials and tribulations in their encounter with new (or re-newed) concepts. Amongst the mathematicians I have worked with, and in the midst of general resistance and suspicion towards what they often perceive as RME's wanting attempts at the construction of viable, consistent and relevant theoretical frameworks, CI/CD has always been the one theoretical construct that 'catches on' in interviews, group discussions etc.. I believe this is significant for at least three reasons:

- It reinforces the often faltering confidence that mathematicians have in the capacity of mathematics educators to generate theoretical constructs that are valid, relevant (and accessible!) to the concerns of practitioners. It thus contributes to the highly needed building of respect and trust between the worlds of M and RME.
- It reinforces the validity of the construct itself (as its seamless adoption 'proves' that in some way it helps respond to questions about mathematical learning and teaching practitioners strongly care to answer). Its becoming part of a mathematician's vocabulary reflects its power to 'direct thought', to become a 'category of mind' (Sierpinska, 1994).

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[5] Bakhtin (1986) proposes that 'genre', a construct typically used towards distinguishing between types of literature, can be also used towards distinguishing between ordinary language and specialist languages such as mathematics, science, law etc.. Bakhtin makes the distinction between primary genres and secondary genres: primary genres are those which legislate those words, phrases, and expressions are acceptable in everyday life, and secondary genres are those which legislate which words, phrases and expressions are acceptable in these other extra-ordinary types of language, for example, mathematics. In this sense the students' contact with textbooks, their lecturers' writing etc is their contact with the secondary genres that legislate what are the acceptable ways of writing and speaking mathematically. Here I employ the term to denote the secondary genre speech of *mathematics education*.

[6] At the Prague event and/or in later and longer versions of this paper, it would be perhaps advisable to revisit the Episodes with the intention to highlight, and qualify more strongly, the evidence on which the claims made in this concluding paragraph are grounded.

- It opens up the possibilities of enriching said theoretical construct (see, for example, how M links concept image construction with multiple representations, interrelatedness of mathematical topics, the pedagogical potential of cognitive conflict etc.) in ways that may have not hitherto been adequately explored.

In other words it seems to me that the acceptance the CI/CD construct enjoys has emerged from its capacity to tell a part (yes, a part only, but a good part) of the 'story' of learning mathematics (in particular, of how learners construct newly introduced concepts and of how they employ the resulting, and constantly evolving, constructions). Of course the part of the 'story' the CI/CD construct tells is the more 'cognitive' part – and, perhaps far from coincidentally, the part M consistently chose to focus on. There are other parts too, such as the affective and the socio-cultural, that are implicit in M's words and are not necessarily covered by the CI/CD construct or his other similar metaphors. I believe that subsequent phases of theory-building around this (and other) theoretical construct needs to focus on re-conceptualising it in ways that its initial inception could not possibly relate to back in the 1980s, when it made its first 'public appearance', and re-embed it in the remarkably richer contemporary theoretical landscapes of the field[7].

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[7] In this sense I find the recently developed discursive approaches to research in mathematics education (e.g. Sfard's (2002) *Focal Analysis* - Pronounced, Attended and Intended Focus - and *Pre-occupational Analysis*, complementary tools for examining mathematical communication, with self and with others) as truly potent ways to engage in this re-conceptualisation and re-embedding. Perhaps at the Prague event part of the discussion can turn to this notion of 'upgrading' seminal theoretical constructs of mathematics education research, such as CI/CD and Procepts, in the light of recent theoretical developments?

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SAYING THE NAMES: VOICE AND PERFORMANCE IN MATHEMATICS EDUCATION

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In many North American aboriginal cultures, there is a strong and significant tradition of acknowledging and naming one's ancestors at the beginning of an event, drawing on highly refined linguistic registers which explicitly marks different potential forms of relationship. Academics get to claim at least two primary sets of ancestors – biological and intellectual – and the 'generational' gestation time of the latter kinships is often much shorter than the former. So I want to start this paper by happily claiming David Tall, now an elder of the entire tribe, as my intellectual ancestor, someone who has influenced my development and subsequent life in mathematics education, a field I still call home.

As with any biographer, however limited and partial, in the selection and retelling of tales from particular points of view, in the recounting and reshaping of the mutual past in the service of the present, I reveal elements of myself as much as of David. These 'rationally reconstructed' tales come from the parts of our lives which ran closest, when influence, in some ways like the phenomenon of coil inductance, is at it strongest, despite often being unrecognized at the time. Biography is a Janus-like undertaking in this respect, also looking back at the would-be looker.

I am currently reading David Lodge's recent novel, *Author, Author*, which presents the life of Henry James written in the style of Henry James. Lodge, who spent much of his professional life at Birmingham University just down the road from Warwick University where David Tall spent virtually all of his, is a master literary mimic as well as a significant literary theorist. But the word 'mimic' suggests that the actual author's voice is necessarily muted as someone else's is being taken on, being mimed out loud.

In formal mathematics, there are significant issues of voice (both in its muting and its mutating), a notion of increasing interest to me as I continue to work in the area of interrelations between language and mathematics, a topic to which I have remained faithful for most of my academic life. And towards the end of this piece, I shall same some further things about it and about time as well. While David did not address such issues directly in either his research or his teaching, his embodiment of them as a mathematics lecturer from home I learned attuned me early to other, richer possibilities.

I spent three years at Warwick as a mathematics undergraduate (from 1971 to 1974) and it was there that I first encountered David Tall as a mathematics lecturer on complex analysis (as well as, even then, a mathematics lecturer who was intensely interested in school and university mathematics learning). Unlike almost all the other courses I had taken, then or since, David staged his lectures like a subtle piece of

theatre (the potential etymological link between 'theatre' and 'theorem' is alluded to in Melissa Rodd's 2003 article on university mathematics lecturing as a performance art), one in which he spoke with two quite different voices. He developed two different spaces in the auditorium for mathematical talk and writing, a 'two-boards' approach, where the left-hand board resulted in a conventional set of 'notes', the familiar linear mix of definitions, lemmas, theorems, proofs, examples, corollaries and formal 'remarks', recited in the familiar deadpan style of the stage mathematician.

The right-hand board, however, was quite a different space in terms of the affordances it offered. Here, David no longer addressed the board in the customary if curious manner of mathematicians' soliloquies and films like *Shirley Valentine*: 'O wall,' says our Shirley, evoking *A Midsummer Night's Dream* in passing; 'O Board', says the mathematician's apparent vocative address. And in each case the audience simply overhears. In front of the second board, David turned to address us directly and continuously. He would talk about motivating or suggestive examples, curious similarities and startling differences between the real and complex numbers; he would converse about the history, about mathematicians and problems. And he drew and drew and drew – sketches and drawings rather than formal pre-prepared diagrams – on this second board. He discussed mathematics in terms of something that people *did*, for a range of reasons and in a variety of ways. He probably spent at most a quarter of each fifty-minute hour in front of this second board, in a sense improvising there: in retrospect, I feel I learnt far more about mathematics while attending there than when my gaze perforce was turned to the other side, populated with its differentiability conditions, its contour integrals and its poles.

This separation of boards came about in a typical DOT manner, him responding to unease from some students in previous years who wanted different sorts of things from a lecture course. By separating out these two strands, those of talk and meta-talk (talk about the talk), and establishing them on distinct but adjacent stages, I learnt both mathematics and about mathematics.

It was quite a bit later, in the mid-1980s, after I had fetched up at the Open University for a long stay, that John Mason introduced me to a small stanza from the *Rig Veda*.

> Two birds, fast-yoked companions,
> Both clasp the self-same tree:
> One eats of the sweet fruit
> The other looks on without eating.

In John's hands, this verse became a powerful condensation of a view of mathematical problem solving, where developing a second bird, one who watched the first one eating of the sweet fruit of mathematics, was seen as a core necessity. Leaving aside (and hopefully excusing) the awful pun, I see how David developed a similar pair of birds in myself and some of my compatriots by means of his two boards, equally fast-yoked.

That same academic year, I also took from David one of first mathematics education courses in the UK in a mathematics dept (the genesis of the one taught by Brian Griffiths and Geoffrey Howson at the University of Southampton might have been earlier), that he had started the previous year. This evoked a world of children, but also a realm inhabited by Piaget and Dienes (the latter's *Building Up Mathematics*, known affectionately then by its vaguely scatological *majuscule* acronym BUM, was a set book for the course), but also of the work of Richard Skemp, including his book *The Psychology of Learning Mathematics* which had been published the previous year. It would only be in 1974 that Skemp himself would arrive at Warwick from Manchester, as Professor of Education, one with a very strong interest in mathematics.

There are a couple of other figures of mutual relevance in these early years, Rolph Schwarzenberger and David Fowler. Both were colleagues of David's (and also teachers of mine) in the Mathematics Institute. In Pimm (2004), I explored on David Fowler's significance to me as a teacher, and some of the things I wrote there also apply, *mutatis mutandis*, to David Tall. David and Rolph had started to do research on first-year undergraduate mathematics students' understanding of the real numbers and limits. In the later 1970s, after I had parted for North America for the first time (as a graduate student), the University absorbed Canley College of Education and both David and Rolph Schwarzenberger moved over to Department of Science Education, part of a newly and massively expanded Faculty of Education. The Mathematics Education Research Centre (MERC) was born in 1977 and Richard Skemp was its first Director.

David took his retraining in a new field very seriously, not only undertaking a second doctorate (with Richard Skemp), but also spending a year teaching in a middle school. In those halcyon days before the UK National Curriculum, David enjoyed a freedom in terms of what he taught. My suspicion is that it would have been a classical *quadrivium*-based curriculum, heavy on the mathematics and music, harking back to a very different sort of basics. David was also the musical director and, I believe, conductor of the Beauchamp Sinfonietta, and a considerable arranger of music, in particular that of Percy Grainger. One mathematics education essay of mine came back bearing the marginal note "Chop it off, Koko!" Mikado Act II next to an offending sentence over ten lines in length.

I returned to Warwick from the US in June 1979 to work with Richard Skemp on a research project (one that also brought Janet Ainley to Warwick that year, doing her masters degree with Richard in addition to being a project teacher, before becoming Richard's Research Associate when I moved to the Shell Centre in 1981). Richard was finishing his book *Intelligence, Learning and Action* (1979). And the previous year, David, in conjunction with Alan Bell and Kath Hart, had been instrumental in setting up the British Society for the Psychology of Learning Mathematics (later renamed as BSRLM, with 'research' replacing 'psychology', a move PME has only just taken I gather).

I could go on and on and on. Remembering, certainly, yet also moving beyond remembering, in terms of paying tribute to people and places that have shaped me, teachers and colleagues and friends, too many of whom are now dead: Rolph Schwarzenberger, Richard Skemp, David Fowler, Christine Shiu, John Fauvel, … – members of my academic ancestral lineage and my kin. It feels very good to me to say the names once more here, to and for David who is decidedly still with us.

'Say the names' is the title of a poem by British Columbia poet Al Purdy, a poem full of resonant place names like Tulameen, Osoyoos and Similkameen, places alive with moment and meaning for my wife, Eileen Phillips, and her father. The poem starts:

> – say the names say the names
> and listen to yourself
> an echo in the mountains

and ends:

> till the heart stops beating
> say the names

David Fowler once said to me that a good university experience is in some ways independent of place; it offers a home away from home. And as David Seymour has commented, 'Leaving home is the beginning of resemblance'. Warwick was such a homeplace for me and other universities I have been to since evoke resemblance in the same way.

I wish to say, David, now turning to you directly in person in print, and adjusting my form of address to you as you once did for us, that you as a teacher were a central part of my Warwick. And as I am about to become a landed immigrant in Canada, I take particular pleasure in finding reason to say these names aloud once more, with yours at the centre today. In thinking long about what it means to me to become an immigrant rather than an emigrant, I have travelled back to my English past many times. This occasion has given me the opportunity to do so once more, remembering your own good company, and recalling once again your, Sue's and your children's openness and accessibility to students, those perpetual travellers in search of whatever they may find.

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LEARNING AND EXPERIENCING THE MATHEMATICAL ENTERPRISE

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This paper explores a perspective of becoming mathematical built by David Tall and other researchers who collaborate with him or were supervised by him, in particular, his colleague Eddie Gray. Tall´s perspective moves from an initial focus on mental representations to the actual different modes of operation or mathematical practices with representations, and to the variations on the language used to carry out actions and to communicate results.

INTRODUCTION

In this paper I discuss a framework supporting theories and teaching projects built by David Tall during his academic life. Part of the empirical research conducted by several authors who collaborate with him or were supervised by him are organized in order to compose a coherent whole, rather than a chronological report on his publications and research. I hope I have in part captured some of his main ideas, or, I hope I have offered a useful interpretation of elements of his important and extensive work.

At the centre of his academic life and research projects is his attraction for new technologies and an appreciation of the use of visualization and visual software to conceptualise mathematical ideas. His interest in visualisation began with his intent of teaching calculus in a meaningful way, assisted by new technology. Despite the power of technological tools widespread throughout the world, even when used for educational purposes, Professor Tall acknowledge that simply using technology does not mean we understand what is happening. There is still a need to help students to construct meanings for the mathematical concepts that not only work mathematically but also are meaningful personally. In his projects, he implicitly uses mathematical notions which would support formal mathematical ideas, devising cognitive roots as grounds for abstractions.

Having these orientations in mind, Professor Tall started an enterprise through mathematics education, which has influenced many researchers in his country and around the world. His initial research was broadened out to include the role of visuospatial and dynamic ideas in experiencing the whole mathematical enterprise. The role of language and gestures in mathematical growth is explored empirically through investigating the relationships between iconic-enactive information, verbal descriptions, and concept construction.

A PERSPECTIVE ON BECOMING MATHEMATICAL

Tall´s theoretical perspective on mathematical growth draws on Bruner's (1966) three modes of mental representation – the sensori-motor, the iconic and the symbolic –

which he considers as a sequence in the cognitive growth of the individual. Bruner was broadly referring to a person's effort to "translate experience into a model of world" (Bruner, 1966, p.10), with an educational hypothesis that "any idea or problem or body of knowledge can be presented in a form simple enough so that any particular learner can understand it in a recognizable form" (ibid, p.44). He suggests three ways of doing it: through action, followed by the use of "summarising images" (p. 10-11) and finally in words or language. This corresponds to enactive, iconic and symbolic modes of representing an experience.

Referring specifically to mathematical representations, Tall (2003) reinterprets these categories and proposes three fundamentally distinct modes of operation:

- Embodied: based on human perceptions and actions in a real-world context including but not limited to enactive and visual aspects.
- Symbolic-proceptual: combining the role of symbols in arithmetic, algebra and symbolic calculus, based on the theory of these symbols acting dually as both process and concept (procept) (see Tall et al, 2001).
- Formal-axiomatic: a formal approach starting from selected axioms and making logical deductions to prove theorems.

Here Tall (2003) proposes grouping Bruner's two first modes of mental representation – the sensory motor and iconic, which he named *embodied*.

Thus the use of the term *embodied* does not correspond to its use in some other recent theories on the nature of human thought (see for example, Maturana and Varela, 1987; Lakoff and Johnson, 1999; Lakoff and Nunez, 2000)[1]. In the case of Tall´s theory, the author gives the term a more focused meaning in mathematical thinking, using it "to refer to thought built fundamentally on sensory perception as opposed to symbolic operation and logical deduction." (Tall, 2003). Comparing the different theories, Mejia (2004) observes that, in essence, they aim at different descriptions. Tall uses the term to refer to conscious activities in a theory of mathematical growth, while, for example Lakoff and Nunez (2000), who are mainly concentrate on unconscious thought, are revealing (their) conceptions of today's mathematical knowledge. Tall's notion does not refer to the general idea that all knowledge, in particular mathematics, is embodied.[2]

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[1] In those theories, the term refers to the assumption of a biological and cultural interdependent relationship between animal and environment (Maturana and Varela, 1987), where sensori-motor experiences shape human thoughts through the use of metaphors (Lakoff and Johnson, 1999).

[2] In its turn, once concentrating on unconscious thoughts with no mention to the conscious ways in which we do and think about mathematics, Lakoff and Nunez misses historicity and also important cultural and contextual aspects which interfere in knowledge construction. For instance, the roots of today´s complex number theory as retraced by Lakoff and Nunez (2000, p.[missing page number]) presents them as we are nowadays presented with complex numbers – as points in the plane. However, in solving cubic equations in the sixteenth century, Tartaglia and Cardano (in his Ars Magna, 1545) performed calculations which led to the square roots of negative numbers that happened to cancel and, in the end, give a genuine real solution. In mathematicians' practices at that time, such 'numbers' were initially devoid of any link to the individual's geometric imagination. This shows that, over time, using symbol manipulation can lead, in

Bruner´s third mode- the symbolic, is split in two: the symbolic-proceptual and the formal-axiomatic.

The term *proceptual* for the *symbolic-proceptual* mode of operation is a reference to processes which are symbolised in such a way that the symbols dually stand for both the process to be carried out and the concept to be thought (Gray and Tall, 1991).

Gray and Tall observe that mathematical practice involves evoking a symbol as a process or as an object, whichever it is appropriate to the specific situation. Most of the time this is unconsciously done by individuals during a mathematical activity which is familiar to them. To account for this idea the researchers formulate the term 'procept' to mean:

> …the amalgam of process and concept in which process and product is represented by the same symbolism. (Gray and Tall, 1991)

In these terms, grasping the mathematical activity leads individuals to think of a procept in a way which makes explicit duality (as process or as concept), flexibility (adapting its use to the context) and ambiguity (not always making it explicit which is being used). Empirical research at different school levels have been focusing on the construction of mathematical objects from mathematical processes, pointing out such dichotomy or duality of mathematical objects. In general, theories emerging from such a perspective claim that first it is necessary to acquire the ability to carry out a procedure, which after practice is then compressed mentally into a mathematical object. Gray and Tall (1991) are among those who stress that the use of symbolism enables the objects and their relationships to be on the focus of attention at more abstract level. Therefore conquering a procedure would be the initial task. Research by Gray (1991) indicates that encapsulating procedures and the use of appropriate symbolism seems to result in deriving facts and new knowledge from old. In the route described by Gray and Tall, the recognition of many procedures to carry out a mathematical task is understood as a process level, a different and higher stage from the initial procedure. Reflecting on processes, their encapsulation into objects represented by symbols could result in flexibility in thought and proceptual thinking[3].

The transition from an elementary mode to the formal-axiomatic mode is characterised by a change in focus from the discovery of *properties* in elementary mathematics that are found through exploration and operation, to the *specification* of properties as a basis for formal *definition and deduction*. (Tall, 1991).

From an epistemological point of view, Tall relates each of these three modes of operation to different worlds of mathematics, "each with its own world of meaning and distinct methods of justification." (Tall, 2003).

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the end, back to a meaningful conceptual metaphor (in Lakoff and Nunez´s terminology), and not the only other way round.

[3] Research has shown that students may have more than one procedure to carry out a process but may not have a flexible concept of the process itself (Ali and Tall, 1996).

> The *embodied* world is the (world of) fundamental human mode of operation based on perception and action. The *symbolic-proceptual* world is a world of mathematical symbol processing, and the *formal-axiomatic* world involves the further shift into formalism… (Tall, 2003)

I personally find useful to consider the above notions by just referring to the specific modes of operation, meaning the quite different ways of reasoning and establishing truth in each case:

> There are some 'mathematical truths' such as the commutative law or associative law for addition which we 'know' are true from our experience in the embodied world. When we operate on numbers, the results we get confirm these as facts. In general, we naturally assume these results hold in algebra. Other truths such as the formula for the difference of two squares ($a^2 - b^2 = (a-b)(a+b)$) are shown to be true by carrying out the algebraic manipulations. In the formal world, axioms will support our arguments.
> (Tall, 2003)

From this perspective, the above notions of the three worlds of mathematics still have an explanatory power on the mathematical experience and one may avoid the discussion on the nature of permanent mathematical objects emerging from each of these modes of operation (Watson, Spyrou and Tall, 2002; Gray and Tall, 2001; Tall, Thomas, Davis, Gray and Simpson, 1999). This could be indecisive, in particular, if taking into account the point of view of learners (see Inglis, 2003; Tall, 2004).

For instance, Inglis's (2003) example of Lambert's imaginary sphere is a case where the object may be thought as arising out of the symbolic-proceptual world in which the algebra of spheres $x^2 + y^2 = r^2$ is applied to the case where $r^2 = -1$. Inventing his result by analogy (Inglis, 2003, p.26) and exploring it symbolically, Lambert formulates his conjecture which could later be definitely explored in embodied mode through the use of Poincare´s disk model of hyperbolic spaces, and could also be made axiomatic by Riemman in 1980. In our empirical research we found that individuals are capable of successfully experiencing even advanced levels of formal mathematics following different routes (Pinto, 1998; Pinto and Tall, 1996, 1999, 2001), which are supported by other than just a single mode of operation. It could include thought experiments and continual refinement of imagery. Our findings suggest that the expected shift from visual and enactive actions to the efficiency of symbol manipulation in arithmetic may also be done successfully by students who retain links with perceptual notions and embodied modes of operation. (Gray, Pitta, Pinto, and Tall, 1997). Therefore in my view it would be unfruitful to expect that mathematical objects we talk about have a permanent status, confined within one of those three worlds.

The fuzzy boundaries between the three worlds are also discussed by Gray and Tall (2001). They observe that a symbolic representation such as 2/3, which might have emerged from embodied modes of operation such as cutting a cake into three pieces and taking two, may also be reinterpreted in formal set theory suggesting embodied modes of operation such as dividing a set into three equal pieces and two of these

being selected. These examples are also similar to the case of complex numbers latter explored in the complex plane, which had arisen from symbol manipulation. Yet Tall (2004) recalls that Cayley's theorem suggests to us to think of an axiomatic group as a subgroup of a group of permutations, returning operations on formal group theory to the embodied mode of permuting elements of a set.

Tall observes that results from symbolic manipulation and also from formal definitions and formal proof can produce *structure theorems* stating the structural properties of formal or symbolic systems (Tall, 2002). Such theorems yield new journeys into the embodied or symbolic worlds, through speculations in terms of thought experiments or calculations[4].

In synthesis, once focusing on modes of operation, the *embodied world* is a world of sensory meaning, where "warrant for truth is that things behave in an expected way" (Inglis, 2003; Tall, 2004). It refers to thought which is fundamentally built on sensory perception, encompassing visual and spatial imagery which includes Bruner's enactive and iconic modes. The *symbolic-proceptual world* refers to mathematical symbol-processing. Symbols allow calculations and manipulations to establish truth. The *formal-axiomatic world* relies on formal definitions for concepts, from which deductions are made. The notion of truth is now established through formal deduction.

RECONSTRUCTIONS AND A NEW PERSPECTIVE ON THE SYMBOLIC MODE OF OPERATION

As in most developmental theories, Bruner accounts for an individual's growth in stages, from dependence on sensory perception through physical interaction to sophisticated modes of thought through the use of language and symbols. Solo Taxonomy (Biggs and Collis, 1991) offers a perspective that when more sophisticated modes of operation are attained by the learner, earlier modes remain available to be used when appropriate.

The three worlds of mathematics operate similarly to the last proposal. As each mode becomes available it remains available so that in later life all three may be used in any appropriate sequence. Tall observes that the child first encounters embodiment, and through actions on objects builds symbolic ideas in arithmetic and later focuses on the properties and their relationships that lead to proof. For the more mature

[4] Structure theorems may suggest results that turn out not to be true. For instance, the 'completion' of the rational numbers to add the irrationals to give the real number line is often conceived as being the ultimate destination, with the real numbers filling out the whole line, banishing the possibility of infinitesimal quantities on the line. Yet this conceptualization limits our imagination and is simply untrue in the formal world of mathematics. It is very simple, mathematically, to place the ordered field R in a larger ordered field (e.g. the field of rational functions consisting of quotients of polynomials in an indeterminate x) which can be mentally imagined as a more sophisticated line that can be magnified to 'see' infinitesimal quantities (Tall, 2002a).

individual, the three worlds can support each other (Tall, 2003) and boundaries between them are not clearly defined.

For the representations of the individual's experiences into a mathematical model of the world, Tall (2003) is splitting Brunner's last mode of mental representation – the symbolic – into two other modes or categories: the symbolic-proceptual and the formal axiomatic.

Bruner (1966, pp. 18, 19) had touched on such a formulation when explaining that symbolism would include both "language in its natural form" and "the two artificial languages of number and logic". Tall reconstructs these last two categories to include algebraic and functional symbolism, other than just number, and the language of axiomatic mathematics.

In doing that, Tall's perspective moves from Bruner's initial focus on mental representations to the actual different mathematical practices with representations or modes of operation, and to the variations on the language used to carry out actions and to communicate results.

Mathematical language is, most of the time, identified with the mathematical symbols used to perform mathematical calculations. In contrast, Tall's research comes from a perspective that symbols alone cannot provide a total environment for mathematical thinking: the concept image and concept definition notions (Vinner & Hershkowitz, 1980; Tall & Vinner, 1981; Vinner, 1983) oriented several of his researches.

Tall also recognizes that doing mathematics in any of the three worlds of mathematics is also supported by an extensive use of the mother language, of pieces of the mother language in use in mathematics - such as the word 'limit' with a technical meaning when referring to a mathematical notion. (see, for example, Monaghan, 1991). Visualisation is considered as an intermediate representation which enriches the mathematical experience with additional concrete exploration of the object (see, for just a few examples, Tall, 1986a,b; 1989, 1991b; Tall and Thomas, 1989; Tall, Blockland and Kock 1990). In recent research, gestures bring iconic representations and enaction to the process of exploring and communicating ideas.

For instance, experiments described in Watson and Tall (2002) focus on the relationship between embodied and symbolic modes of operation with the concept of vectors. In a special reference to an embodied mode of operation, Watson and Tall observed that the range of physical experiences gives very different meaning to the concepts, playing a role in the process of knowledge construction.

I perceive an analogy among these results and those explored on the proceptual-symbolic mode of operation: we could think of a situation where learners distinguish processes based on the different procedures to carry them out. For instance, 'vector as a journey' leads more naturally to the use of the triangle law for addition and 'vector as a force' leads more naturally to the parallelogram law. Watson and Tall found that

a natural interpretation of vector addition by using the parallelogram law may be experienced by our bodies being pulled forward by our two arms, which causes the individual to move in a forward direction representing the combination of the two forces. The two rules are theoretically identical, but students many times appear to conceive of them as different. In more general terms, a flexible idea of vectors as a free vector with given magnitude and direction seems to be another obstacle. From interviewing students, the researchers came to the notion that a set of actions in the embodied world could be identified if they have the same *effect*. From this perspective, the triangle and the parallelogram law for addition of vectors would be identical. An experiment based on such reconstruction of the notion of 'identity' within the embodied world was carried out to build a notion of free vector. Watson and Tall developed an experiment physically translating an object (for example a triangle) on a table, pushing it without turning it. Encouraging students to imagine the beginning and the end position of each moving point, focusing on the hand as a whole or on a single finger that moves during the action, they agreed that the shift could be represented by arrows. The notion of free vector emerged from the various possibilities of representation which could be identified by recall of effect.

In a further step, Watson and Tall realized that the idea of a set of actions having the same effect in the embodied world could correspond to the notion of different procedures having the same output at the process level within the symbolic-procedural world. This observation led not only to the design of many experiments relating the two modes of operation but also to the reinterpretation of earlier experiments.

LANGUAGE, GESTURES AND MATHEMATICAL DEVELOPMENT

In respect to the latter, different hypotheses emerging from psycholinguistics have been feeding a debate on the relation between gestures, language and conceptual processing. Amongst those, Kita (2000) recognises gestures as involved in the conceptual planning of the message to be verbalised, which gives to gesture a role not only in speaking, but also in the thinking processes and, therefore, a role in a learning context. Kita understands gesture and speech as related to two complementary modes of thinking he named spatio-motor thinking and analytic thinking.

In mathematics education, research by Edwards (2003) has draw upon McNeill's (1992, 2000) ideas, which acknowledge a distinction between the two complementary modes of thinking, though sees them as integrated into a multiple representation of a unified task. Edwards has adopted McNeill´s identification of *deictic gestures* as those pointing to an object, *metaphoric gestures* which represents an abstract idea with no physical form and *iconic gestures* which recall a semantic context. Further analysis in the mathematical learning context led Edwards (2003) to refer to *iconic-physical gestures* and *iconic symbolic*-gestures when distinguishing between those gestures associated with symbolic inscriptions or with procedures on these inscriptions. Assuming the view point of those researchers, we could say that gestures

come into play in the mathematical development through revealing a changing knowledge, and, to a certain extent, integrate the embodied mode of operating on mathematical notions into all these three worlds of mathematics. I suggest all those views are aligned with Tall's empirical descriptions on the role of enactive-iconic modes of operation, which in fact extensively supports his entire life research.

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COGNITIVE STYLES, DYNAMIC GEOMETRY AND GEOMETRICAL PERFORMANCE

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The aim of this study is twofold, first to investigate the relationship between individual cognitive styles and performance in geometrical tasks, and second to explore the impact that a piece of dynamic geometry software may have on students with different cognitive styles. The research was conducted with 49 sixth grade students. We hypothesized that students who tend towards a more visual cognitive style may have more abilities in geometry and also that they may be more positively affected by a dynamic geometry environment that matches their cognitive style. The main findings of this study provide evidence that verbalizers appeared to gain more than imagers from the graphical setting of dynamic geometry due to the development of their abilities in constructing geometrical shapes.

INTRODUCTION

Since learning is a primarily cognitive activity, it is likely to be influenced by the cognitive styles of learners. Although much work has been done in this area, little attention has been afforded to the effects of cognitive styles of students on their learning of mathematical concepts. One of the main themes that interested a number of researches in the past and today, in regard to cognitive styles, is the fact that cognitive styles may have important implications for educational theory and practice (Sternberg & Grigorenko, 1997). This becomes even more crucial now that multimedia and technology based instruction is more widely used in the mathematics classroom. One of the domains in which technology has been more widely used has been the use of dynamic geometry software. It will thus be of interest to investigate the impact that dynamic geometry environments may have on different cognitive style students.

The aim of this study is twofold, first to investigate whether certain cognitive styles are related to performance in geometry and secondly to investigate whether teaching the area of triangle and parallelogram with the use of dynamic geometry software is more beneficial for a certain style of students. It is hoped that this will shed some light whether such teaching is beneficial to all students.

THEORETICAL BACKGROUND

Definition of cognitive style

Cognitive style is an individual's characteristic and consistent approach to organizing and processing information. Cognitive styles are a subset of the general subconstruct of style, which is defined as "a distinctive or characteristics manner… or method of acting or performing" (Guralnki, 1976, p.1415). The concept of cognitive styles goes

back to Jung (1923) who proposed a theory of psychological types. Later on Allport (1937), defined cognitive styles as the habitual way in which an individual processes different information. However, modern research actively began in a number of laboratories within a short period with the work of Witkin (1964) and Gardner (1959). Since then a number of researchers have investigated cognitive styles. Friend and Cole (1990) have expanded the definition of cognitive styles to include the way in which the individual perceives, codes, saves and recalls information, while Riding and Rayner (1998) added to Allport's definition that cognitive style is an individual's preferred and habitual approach to organizing and representing information and subsequently affects the way in which one perceives and responds to events and ideas.

Types of cognitive styles

Different researchers identified different types of cognitive styles. Witkin (1965) distinguished between field-dependent and field-independent individuals, Gregorc (1982) between concrete-sequential, abstract-sequential, abstract-random and concrete-random. Sternberg's theory (1997) postulated 13 thinking styles. These are legislative, executive, judicial, hierarchical, oligarchic, monarchic, anarchic, global, local, internal, external, liberal and conservative. Riding and Cheema (1991) reviewed over 30 methods of defining cognitive style and concluded that most could be grouped within two fundamental independent cognitive style dimensions, the Verbal-Imagery dimension and the Wholistic-Analytic dimension. An individual's position along the Verbal-Imagery dimension reflects the manner in which they represent information while thinking, whether as words or mental pictures, while the Wholistic-Analytic dimension reflects whether they understand a situation as a whole or see things in parts.

Within the field of mathematics education the verbalisers/imagers distinction was the one that attracted most of the attention. However, it needs to be noted that this distinction was not always referred to as a cognitive style but as preferred mode of thinking, or type of students (Lean & Clements, 1981; Presmeg, 1986). Mathematics education researchers often tried to link the verbalisers/imagers distinction to mathematical performance (for example, Kruteskii, 1976, Fennema & Tartre, 1985; Presmeg, 1986; Gray, Pitta, & Tall, 2000; Pitta-Pantazi, Gray, & Christou, 2002a, 2002b, 2004; Gray & Pitta-Pantazi, 2006; Pitta & Gray, 1996, 1997, 1999, 2000; Gray & Pitta, 1996). Nevertheless, the results of the relationship between visualization and mathematical performance is actually not so clear, or at least there is a need of greater clarity since researchers looked at different age groups, mathematical topics, used different methodologies or defined visualisation and visualisers differently.

Cognitive styles and Dynamic Geometry

The use of dynamic geometry as a medium of learning has been attracting much research in the field of mathematics education. The development of dynamic geometry provides learners with many opportunities to explore and discover

according to their own individual needs. The basic rationale behind dynamic geometry is that information can be presented in different forms and mainly in a visual dynamic format. However, the visual form of information may come at a price because some learners may find the dynamic and visual reasoning as a complex process (Ellis & Kurniawan, 2000). Therefore, it is necessary to see how different learners perceive the features of dynamic geometry.

In the past decade, many studies have shown evidence of individual differences and their significance in mathematics learning using appropriate software (Lee, Cheng, Rai & Depickere, 2005; Parkinson & Redmond, 2002). Among these differences, cognitive styles are especially related to the manner in which information is acquired and processed. It is well documented in the literature of cognitive style that field-dependence and field-independence are strong predictors of student analytical approaches to problem-solving (Nasser & Abou-Zour, 1997). Whether a student is field-dependent or independent is an important predictor of how the student will approach mathematics problems, analyse embedded context and restructure information. Evidence from research on the effect of cognitive styles on learning suggests that cognitive style characteristics such as perception and processing of information enhance learning outcomes (Riding & Sadler-Smith, 1992). The above studies argue that optimum learning outcomes are obtained when the instructional material can be transferred readily to learners' personal modes of representation. More specifically, studies from the United Kingdom, Canada and Kuwait found that between the ages of 11 and 14 years the wholistic-verbalisers have the lowest attainment in mathematics and imagers almost double their learning performance if they are presented with the same information as text-plus-illustration compared to text alone, while verbalisers are not affected (Riding & Watss, 1997).

THE PRESENT STUDY

The purpose of the study

Since the use of technology in mathematics classrooms has increased dramatically during the past two decades, critical issues such as the role of students' cognitive styles need to be addressed. Connell (1998) reported that a technological environment can enhance construction of knowledge and influence learning. Computers are able to aid in visualizing abstract concepts and to create new environments that extend beyond students' physical capabilities. Dynamic software is often employed as a fertile learning environment in which students can be actively engaged in constructing and exploring mathematical ideas (Cuoco & Goldenberg, 1996). The present study is a part of a larger research project which examines the effects of cognitive styles on students' mathematical performance. Our purpose is to investigate cognitive styles along the Verbaliser-Imager (VI) and Wholistic-Analytic (WA) dimensions. However, due to space limitations, in this paper we only refer to the effects of the VI dimension on students' performance on geometry problems through the use of a dynamic geometry software. Since we know from research that a

"preferred cognitive style" exists, then matching the style with the instructional format may enhance learning (Riding & Sadler-Smith, 1992). If students can access information in a format that matches their cognitive style, then the need to reorganize in accordance with their preferred style prior to learning is not necessary. The elimination of this step in information processing presumably reduces the cognitive load imposed by the task and enhances performance (Halford, 1993). Thus, the first hypothesis of the study is that imagers will outperform verbalisers in geometry tasks. The second hypothesis of the study is that imagers may benefit more from the teaching of geometry with the use of dynamic geometry software, since their cognitive style matches the format of representations in this media.

Participants and procedure

Forty nine sixth graders (27 boys and 22 girls) from two intact classes of an urban school in Cyprus participated in the study. The research consisted of a cognitive style test, a pre-test, the instruction with dynamic geometry, and the post-test. The computerised VICS test (Peterson, 2005) was administered to the students during a 30 minute school period. This test was used as an independent variable to split the sample into verbal (textual) and imager (graphics) groups. After a week, students were given the geometry test (pre-test) which it was again administered to the students the day after the teaching intervention with the dynamic geometry was completed (post-test). The pre-test was used to indicate participants' prior knowledge of complex-programming concepts to facilitate the measurement of cognitive performance on the post-test.

During the intervention program the students were taught the area of triangles and parallelograms with the use of the dynamic geometry software Euclidraw Jr. The intervention program lasted for two weeks (eight 45-minute periods). The students had some experience with the software from previous lessons.

Tasks of the study

All students were assessed for their preferred cognitive style using the VICS test (Peterson, 2005). The test works on the basis of response times to a battery of statements which are categorized into subsets, and a ratio for each subset is calculated. The VICS test measures the VI dimension by asking which of the two shapes is bigger in real life or whether two items are man-made, natural or mixed. Both these questions are presented either with the use of icons or words. A detailed discussion of the rationale for the design of the two tests can be found in Peterson, Austin & Deary (2005). For the purposes of this study on the VI continuum, verbalizers were those with scores equal to or less than 1 and imagers were those who scored more than 1.

The geometry test

The geometry test that was used for the purpose of this study included 24 tasks, 12 on the area of triangle and 12 on the area of parallelogram. In six of these tasks students

were asked to recognise triangles or parallelograms with the same area, or specific elements of these shapes such as their height (Recognition Tasks). There were 8 construction tasks which called upon students' ability to construct triangles or parallelograms if given area, or draw the height and the base of these two shapes (Construction Tasks). Finally, there were 10 tasks which required students to compute the area or the height or the base of triangles and parallelograms (Computation Tasks).

Data Analysis

To examine the hypotheses of the study, descriptive statistics were used and multivariate analysis of variance was applied. The total geometry scores in pre and post-tests as well as in the sub-scales of the tasks (recognition, construction and computational tasks) were used as dependent variables, while the cognitive styles of students on the VI continuum served as independent variable. In addition, the progress of students in the geometry test was calculated using the difference in the mean scores of the pre-test and post-test.

RESULTS

The results of the study are presented according to the hypotheses of the study. We first focused on comparing student performance in geometry tasks based on their cognitive styles. Through the administration of the VICS test, we split the sample in two groups, the verbalizers and the imagers. We found that on the VI continuum 29 students tended towards an imagery preference and 20 towards a verbal preference. At the same time students' scores on geometry tasks in the pre-test for imagers and verbalisers were compared.

Table 1 shows the means of students' performance in the geometry tasks (pre and post tests) and in the subscales of the test. Table 1 also shows the gain scores in the total test and the subscales as resulted from the calculation of the difference between the initial performance of students and their performance after the intervention with the use of the dynamic geometry. Finally, the last two columns of Table 1 show the results of the multivariate analysis. As can be deduced from the table, there was no significant difference between imagers and verbalisers on the pre-test total geometry scores ($F_{(1,48)}= 0.24$, $p= 0.63$), indicating that the cognitive style of students does not affect in a significant way students' performance in geometry. Furthermore, comparing scores for imagers and verbalizers on the sub-tasks of the pre-test (recognition, construction, and computation tasks) also revealed no significant difference. However, the mean score of the verbalisers on the total test and on all sub-tasks was consistently better than that of the imagers (see the means on Table 1). Hence, it may be plausible to suggest that there appears to be a pattern (although not significant) in favour of the verbalisers (Table 1). These results do not seem to confirm the first hypothesis of the study that the imagers would have a better performance in geometry than students' with other cognitive styles and contradict the findings of much of the research conducted by Riding and Watts (1997).

	Verbalizers Means	Imagers Means	F	p
Pre-test total	7.6	6.9	0.24	.63
Post-test total	12.5	9.8	0.05	.82
Recognition pre-test	0.28	0.25	0.54	.47
Construction pre-test	0.32	0.31	0.01	.92
Computation pre-test	0.38	0.35	0.19	.66
Recognition post-test	0.49	0.41	1.61	.21
Construction post-test	0.49	0.37	2.88	.10
Computation post-test	0.62	0.50	2.34	.13
Progress in total score	4.97	2.9	9.97	.00*
Progress in recognition	0.21	0.16	0.50	.48
Progress in construction	0.17	0.06	4.51	.04*
Progress in computation	0.24	0.15	3.04	.09

* Indicates statistical significance at a=0.05

Table 1: Means of students' performance and multivariate analysis of variance with cognitive styles as independent variable

Table 1 also shows that the cognitive styles under investigation do not seem to affect students' performance in the overall geometry test, neither in the subcategories of the test, even after the intervention with the use of the dynamic geometry software (see post-test results). This result was somehow expected since the duration of the intervention course was a short one. Of most importance is the progress of students' performance after the intervention. Thus, a multivariate analysis was conducted with dependent variables the progress of the students from the pre-test to the post-test.

This analysis showed that verbalizers improved significantly their performance in the total score (F=9.97, p=0.00). Specifically, verbalizers gained significantly more than imagers, indicating that the progress of the verbalisers was higher than the progress of the imagers in the total test. This improvement was mainly due to the rise in verbalisers' performance in the test's construction tasks (F=4.51, p=0.04). This finding is in contrast to the second hypothesis of the study as well as to previous research indicating that optimum learning is achieved when individuals are taught in a way that matches their cognitive styles. Finally, no differences were found in computational and recognition tasks between verbalizers and imagers.

DISCUSSION

The findings of previous studies provided evidence that computer learning programs may not be suitable for all learners as a learning tool, and teachers should be aware of individual differences such as cognitive styles possessed. Teachers also should not assume that every student would equally benefit from the use of computers in educational settings (Connell, 1998). In this study, we investigated the effect of cognitive styles on geometry learning, based on the notion that verbal (text-based) instruction is best suited to a verbal cognitive style while pictures (graphical representation) suit an imagery-based cognitive style best (Riding & Rayner, 1998).

First, the findings of the present study do not suggest that cognitive style is a decisive factor in explaining the differences in geometry performance between students with different cognitive styles. Second, we focused on the hypothesis that the use of dynamic geometry, which mostly provides experiences in pictorial forms, may enhance the performance of imagers in geometry. However, we found that the effect between cognitive style and dynamic geometry is at variance with the above hypothesis.

Specifically, the results of the study provide evidence that verbalizers appeared to gain more than imagers from the graphical setting of dynamic geometry. This result may suggest that verbalizers perform best when given an instructional format enhanced with graphical features. In addition, it seems that the improvement of verbalizers was mainly due to the development of their abilities in constructing geometrical shapes. This prompts research into the nature of geometrical abilities required for the improvement of students' performance in specific subcategories of problems referred to as recognition, construction and computation of geometrical concepts. Further investigations of what makes a subcategory more or less suited to certain cognitive styles are currently being considered.

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COMPUTER ALGEBRA THROUGH A PROCEPTUAL LENS

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Computer algebra systems (CAS) have become powerful and flexible tools to support a variety of mathematical activities. This includes learning and teaching, research and assessment. This paper concentrates on the use of mainstream CAS packages to support the learning of mathematics at the undergraduate level. We shall examine the notion of procept and undertake a review of computer algebra systems in the light of this concept. That is to say, we shall examine the extent to which proceptual thinking is evident in the design of CAS, and consider some pedagogic and design implications.

INTRODUCTION

The work of Gray and Tall (1994) develops the notion of a *procept* to capture the duality between process and concept in mathematics. For example, basic arithmetic operations make use of the same symbolism to represent the product of the process: one half as 1/2, and as the process itself: divide one into two equal parts. They comment on the ambiguities in using the same symbol for both as follows.

> By using the notation ambiguously to represent either process or product, whichever is convenient at the time, the mathematician manages to encompass both - neatly side-stepping a possible object/process dichotomy. Gray and Tall (1994)

In this paper we shall be most concerned with the *elementary procept* which is

> the amalgam of three components: a *process* which produces a mathematical *object*, and a *symbol* which is used to represent either process or object. Gray and Tall (1994)

Computer algebra systems operate very much within the proceptual world, Tall (2004). Their purpose is to automate the manipulation of symbolic expressions according to specific rules. The saving of time and reduction of human error is clearly advantageous to the mathematical or scientific researcher. As a result of this automation, CAS are extremely compressive. That is to say, in response to a command from a user the system usually produces output without any intermediate results or details of methods by which such results were obtained. This feature has been harnessed for learning and teaching, allowing students' focus of attention to reside other than in the details of the symbolic manipulations themselves. See for example Guin et al. (2005) as just on example from the extensive literature on this subject.

A third application of CAS is to establish the mathematical properties of expressions provided by students as answers to assessment questions in online computer aided assessments (CAA). Perhaps the first system to make a mainstream CAS a central feature was the AiM System, first described by Klai et al. (2000), with subsequent technical developments described in Strickland (2002). This system operates using

Maple, as does the Wallis system of Mavrikis and Maciocia (2003) and the proprietary system MapleTA. Other systems have access to a different CAS, such as CABLE (Naismith and Sangwin (2004)) which uses AXIOM and the STACK system which uses the CAS Maxima, see Sangwin and Grove (2006). From private correspondence, the author is also aware of systems which use Derive and Mathematica in a similar way.

Many mathematical CAA systems do not incorporate a mainstream CAS, but rather make use of bespoke libraries of mathematical functions. Examples, of CAA systems which take this approach are the CALM system of Ashton et al. (2005), the Metric system of Ramsden (2004) and the Aplusix system of Nicaud et al. (2004), to cite but three examples from a very extensive field. We consider such libraries of functions to be `computer algebra', even if they are not traditional general purpose systems or explicitly conceptualized as such by their authors. The discussion in this paper is relevant to these systems also.

As a simple illustration, imagine a student interacting with a CAA system and in some way entering a mathematical expression as an answer to a question. The CAA system uses a CAS to algebraically subtract the student's response from that of the teacher and to simplify the resulting expression. If the result of this algebraic simplification is zero, then the system has established an algebraic equivalence between the student's answer and the teacher's answer. The key point is that the student's answer contains mathematical content, and is not a multiple choice question or similar. A CAS is used to establish a variety of other properties beyond the prototype of algebraic equivalence. For example, whether an expression entered by a student is expressed using a particular algebraic *form*, see Sangwin and Grove (2006). Having established properties of an expression, feedback can then be given automatically.

This application is particularly relevant to the examination of CAS through a proceptual lens, since manipulating student's expressions requires a subtlety and variety of approaches not encountered in the original application of CAS as a sophisticated calculator.

METHODOLOGY

The methodology adopted in this paper is to review the existing computer algebra systems Maxima (`http://maxima.sourceforge.net/`), Maple and Mathematica to find evidence of proceptual thinking in their design and implementation.

To focus this, the discussion is limited to (i) elementary algebraic operations, (ii) representation and manipulation of functions both in-built and user defined, and (iii) calculus operations including the representation and manipulation of differential equations.

RESULTS

The default behavior of all the CAS considered here is to automatically perform all elementary arithmetic calculations and to simplifying expressions containing these where possible. For example, all rational numbers are simplified to their lowest terms whenever they appear in expressions. Hence, to a CAS and their designers a symbol for an elementary arithmetic operation, such as +, is always interpreted as *something to do*, ie the process. This is regardless as to whether the symbol is entered by a user, or generated internally as the product of another process. Only when the process cannot completed is the symbol used by the CAS, eg 1+x, to represent to the object. Even with such algebraic expressions, the CAS is assertive in performing manipulations to "simplify" such an expression to some canonical form. For algebraic expressions this includes gathering like terms, and sorting them into either ascending or descending powers of terms. There is quite a surprising variety between the CAS reviewed as to the extent to which such "simplifications" take place, although space precludes a more detailed discussion, see instead Wester (1999).

These simplifications combine the *algebraic* properties, such as commutativity, with the *functional* properties of the arithmetic operations. Indeed in all systems but Maxima there is no mechanism by which a finer degree of control over such simplifications can be achieved. For example, when comparing a student's answer in CAA there are circumstances when it is necessary to make comparisons in a context where addition and multiplication are commutative and associative but in which we do not want to actually perform any additions or multiplications. For example, let us assume that a student has been asked to simplify an expression such as (x+1)+(x+2). A CAA system should accept both 2x+3 and 3+2x. In general a simple string match is not sufficient: we need to work in an environment where addition is commutative and associative. However, we would want to draw a distinction between these and an expression such as x+x+3, in which terms have not been fully gathered. A traditional CAS does not permit such fine distinctions to be made, since all three examples are immediately simplified internally by the CAS to the canonical form 2x+3, making comparison effectively impossible. And yet this task is one to which computer algebra should be applicable. In this example, we need a finer degree of control over the behaviour of the CAS to provide an appropriate environment in which the comparisons can take place.

Let us examine this example from the point of view of a student stepping through a calculation to better understand elementary algebra. Currently the CAS would compress all stages into a single step, and in Maple and Maxima it will remove the (mathematically unnecessary) parentheses automatically. Since this is the purpose of the question, it is difficult to understand how the CAS is at all helpful beyond giving the student the final answer against which they can check their intermediate working. By providing the opportunities to obtain a finer degree of control over what is done automatically at each step, the teacher can provide an environment in which the student can work through the process at a level of compression appropriate for their

stage of learning. It is interesting to note how many algebra text books use the phrase "simplify" as an instruction which is synonymous for "do what I have just shown you in the worked example on the following exercise". A general lack of consensus on a differentiated nomenclature for the various algebraic processes, eg "gather/collect", "sort" hampers both the debate on this issue, the ability to implement such commands in a technology rich environment such as a CAS and almost certainly teacher's and student's ability to make and describe these distinctions.

Maxima is unique amongst the CAS considered here in its ability to suppress automatic simplification completely. This is achieved using the command `simp:false`, where in Maxima's language the colon is used to signify the assignment of a value to a variable. When automatic simplification is off, specific and selective simplification is possible allowing the needed finer degree of control. This, in principal at least, allows one to step through a calculation, or to switch on and off algebraic properties of the operators. However, Maxima lacks a user interface, documentation, or a sufficiently developed set of tools which would allow students to *easily and intuitively* perform such manipulations for themselves. Currently the interface results in giving the user an "all or nothing" choice over the extent of simplification. Nevertheless, the fundamental mechanism is in place, which is something the other systems lack. Maple, for example, has an add-on Student package which allows some decompression and finer control, although this is separate from the core system.

The conclusion which we draw from this is that designers of CAS have ignored the proceptual view of elementary arithmetic operations entirely, as far as the core system is concerned. For them, arithmetic and elementary algebra is something to be done, immediately and to the maximum extent possible. However the designers have needed to address exceptional situations, for example when a user asks the CAS to factor an integer as prime powers. Here the result must be represented using an internal data structure which is different from a product of powers of integers - otherwise the automatic simplification routines would work at cross purposes.

We now turn our attention to how the notion of function is manifested. It is interesting and apparent to note that the general notion of function is different from the elementary arithmetic operations. That is to say, the elementary operations are not implemented as "functions" in an internally consistent way. While this may be for reasons of computational efficiency, it creates a distinction which is not present in the underlying mathematics. Maxima comes closest to providing a consistent approach by allowing the user to define arbitrary infix, prefix and postfix operators. In other computer algebra systems the user is only provided with mechanisms for defining prefix functions. As specific examples, a prefix function is the cosine, eg cos(x), a postfix function is the factorial, eg n! and examples of infix operators are + or logical operators such as `AND`.

The basic object in all CAS is the "expression", which is strictly speaking a more general notion than that of algebraic expression. From a pedagogic standpoint there are many notions of function, although within a CAS it is that of function defined as an algebraic expression which predominates. Such a function definition can be made in two ways. The first is as an explicit declaration, eg

`f := x -> x^2` (Maple) or `f(x):=x^2` (Maxima)

The second is to take, or generate, an expression in one or more explicit unknown variables and then "un-apply" the function. This turns the explicit variables into bound variables within the function definition effectively localizing them. This draws precisely the distinction between the function and the result of applying the function to a particular letter, eg x. Essentially un-applying a function is the reverse of applying a function to its arguments. As concrete example, we have

`f := unapply(x^2,x)` (Maple) or `define( f(x), x^2)` (Maxima)

While the square function can be defined in both ways, there are occasions on which this second form is necessary, and examples will be given in a moment. Notice that in Maple the symbol `f` is now an unevaluated function although it is treated as an object and may be manipulated as such formally. This allows the user to refer to the process, as well as the product of the process, `f(x)` say, which is usually simplified to an expression which results from the evaluated definition. This distinction provides the first evidence for proceptual thinking in the design of CAS. However the flexible ambiguity of using a single symbol for both, which is the hallmark of proceptual thinking, is absent.

Most of the mainstream CAS have *different* symbolic forms for these two aspects of procept. Sometimes these distinctions are subtle. For example, in Maple an initial capital letter is *sometimes* used to denote the result of the process, rather than as a command to undertake the process. For example, in Maple `Cos(Pi)` represents the result of applying the cosine to the argument π, and remains unsimplified. Without a capital letter, ie `cos(Pi)`, the function is applied and simplification undertaken. However the use of capital letters for this purpose is patchy, confused and confusing. For example `matrix` and `Matrix` are different *data types* and have corresponding functions `add` and `Add` (etc). Here the initial capital letter refers to the datatype to which it is legitimate to apply the function, rather than to draw a distinction between a command `to do' and a representation of the command. Furthermore, while `Pi` is the numerical constant and `pi` is an arbitrary variable, `gamma` is the numerical constant and `Gamma` is an arbitrary variable.

The situation in Mathematics and Maxima are better, and in particular Mathematica's use of square brackets to denote function application disambiguates potential functional application when symbols are juxtaposed with parentheses.

Only Maxima uses the *same symbol* for both process and concept, coming closest to a true proceptual view of function. The designers of Maxima talk of `noun' and `verb'

forms of the symbol representing the function, and these map very closely to concept and process respectively. Both in-built and user defined functions always have parallel noun and verb forms. The distinction is made within an *input syntax* by prefixing any function with a `'` symbol. So, for example, `'f(x)` represents the noun form of the function f, applied to the argument x, rather than a verb form which would result in applying the function definition itself. Once an input string has been parsed and an internal representation of the object generated, noun forms can be selectively activated using a single mechanism as needed. This allows a consistent approach on internally defined functions, user defined functions and operators which accept functions as arguments. If a different notation is adopted for such operators, as in Maple's occasional use of initial capital letters, this conceptual and notational consistency is much harder to achieve.

The ability to explicitly encapsulate a function into an object to which other operators can be applied provides one particularly powerful feature of any CAS, specifically the ability to define functions which take functions as arguments, often referred to as either operators or functionals. The first example is differentiation. When a CAS is first encountered symbolic calculus operations are most often explicitly applied to algebraic expressions. So, for example, we might issue the command `diff(x^2,x)`. Such commands may also be used on evaluated functions, for example `diff(f(x),x)` applies the function f to its argument x, and then differentiates the result. The result of this is *not* a function, but rather is an expression. Hence, to obtain the *derived function* it would be necessary to un-apply the arguments. More important conceptually is the ability of a CAS to operate on a function as a single object. For example, if f and g are both defined as functions, then a command `h:=f+g` would define the function `h` to be the sum of `f` and `g`. Furthermore, operators can be defined on functions as objects. For example, in Maple, `D(cos)` is the input syntax to differentiate the cos function, and is evaluated as –sin, which is an unevaluated function. Here we are genuinely operating on whole functions.

An interesting problem occurs when differential equations are entered into a CAS. For example, to enter a differential equation one might type

```
ODE := diff(y,x,x)-3*diff(y,x)+2*y=0;
```

While the user intends the derivative command `diff` to represent the concept, ie the noun form, the default behaviour of the system is to automatically carry out this procedure and to differentiate y with respect to x. It is necessary to indicate otherwise, since we certainly do not want to differentiate y with respect to x when representing the equation at this stage. Of course, if we are testing a particular value of y and substituting this into the `ODE` we would want the differentiation to become the process. As a result it is at this point at which the designers of all CAS acknowledge the process object duality and fully acknowledge the need for noun forms of calculus operations here. Each CAS system has solved this problem in a unique way by providing a (necessarily) differentiated input syntax.

CONCLUSION

All interactions with a CAS are explicit instructions. Hence the opportunity to exploit the ambiguity which characterizes proceptual thinking is diminished. Nevertheless, the lack of a consistent approach to the internal representation of mathematical objects as procepts has resulted in *different symbols* for processes and concept representations. Unfortunately, the provision of and choices for these does not appear to the author to have been implemented with a consistency which would allow them to be extended to the elementary arithmetic operations in a natural way. Indeed, their presence seems to be the result of evolution to allow, for example, differential equations to be expressed, rather than conscious planning from the outset with a proceptual point of view.

The fragmented way in which this notation is used must have implications for the way students learn to use the CAS, and the mathematics which the CAS helps to reveal. This is not the place to speculate in detail on the effects the current situation may be having. However it seems as well to recall again Babbage (1827) pg 326 who's comments on *"a profusion of notations (when we regard the whole science) which threaten, if not duly corrected, to multiply our difficulties instead of promoting our progress"*. This appears to apply equally well to the fragmented way in which computer algebra systems accommodation the notion of procept through their differentiated notation for expressing the related process and concept.

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DEVELOPING VERSATILITY IN MATHEMATICAL THINKING

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BACKGROUND

Figure 1: A young Dr. Tall

Way back in 1969, as a first year Warwick University undergraduate, I was taught a course called Foundations of Mathematics by a young lecturer called David Tall (see Figure 1). I never imagined then that our paths would later become so intertwined. Two years later I took a course on Mathematics Education that he was teaching, and this was to mark my first encounter with Piaget, problem solving (e.g., in two dimensions what shape has the greatest area that can be pushed around a right-angled corner of width 1 unit?), and knots, which have been an interest ever since. Following graduation I was very happy teaching mathematics in secondary schools for the next 10 years or so, until I felt a need to engage with ideas that would again challenge me. During a chance meeting at Warwick University with my undergraduate tutor Professor Rolph Schwarzenberger, he suggested that I talk with David about some mathematics education research he was engaged in. Thus in 1983 I was accepted by David as a part-time master's student, while continuing to teach mathematics at Bablake School, Coventry. This heralded the beginning of my foray into mathematics education research, learning from someone who has displayed tremendous insight into many of the major issues involved in mathematical thinking and learning.

HISTORICAL OBSERVATIONS—AN EMERGING CONCEPT

While studying at Warwick I was fortunate to be involved at the start of a period of great activity, including the construction of concepts that were to have an international impact. One of these, of course, is the idea of a *procept* (Gray & Tall, 1991; 1994). In many ways I am uniquely placed to chart some historical details of this emerging concept. It must have been around 1986 that I stumbled across a response from one of the 14 year-old students in my doctoral research who had made what I thought to be an interesting comment about a fraction. I followed this up in a questionnaire by asking students to explain whether $\frac{6}{7}$ is the same as 6÷7, or not. The actual question was:

1. A girl wrote the following in a Mathematics test at her school. Write underneath each part in the space provided whether she was right or not and explain why you so answer. …

b) $\frac{6}{7}$ is the same as 6÷7

The results were quite surprising to me at the time, with a number of students, such as those whose work is shown in Figure 2, not seeing these as the same, because one is a ‘sum’ and the other a ‘fraction’.

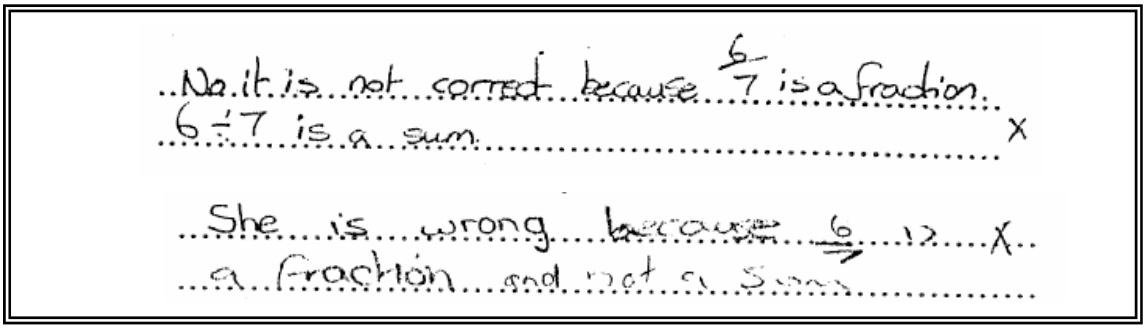


Figure 2: Student work showing differentiation between process ‘sum’ and object ‘fraction’.

Hence, commenting on this differentiated perspective on the symbols, I wrote in my PhD thesis (Thomas, 1988, p. 467)

> Of the 147 pupils given the questionnaire, 22.4% of them did not consider them to be equivalent, because, as they explained it, the former is 'a fraction' or '6 sevenths' but the latter is 'a sum' or 'a divided by' (see Figure 10.12). Thus a high proportion of children at this age see $\frac{6}{7}$ as an indivisible whole, an entity, propagated by the holistic mode of processing which very strongly evokes the concept of fraction for this image. However, for the imagery 6÷7, many pupils process this sequentially, since in their schemas the image is strongly linked to the concept of a process, or a sum rather than an entity, and they view the symbolism in three distinct parts, as 6 (value), divided by (process or operation) and 7 (second value). What these pupils lack is the versatility, described in Chapter 3, which would enable them to view either or both of these symbolisms or images in a global/holistic or a serialistic/sequential way.

As early as 1989 David and I were writing about this idea in a PME paper (Thomas & Tall, 1989) that also referred to the encapsulation of this division process as an entity (or an object as it is now usually described, e.g., Dubinsky, 1991; Dubinsky & McDonald, 2001).

> An interesting example of this, although arithmetic rather than algebraic, is the first question in table 1, where many of the controls did not consider the two notations as the [sic] equivalent because
>
> “$\frac{6}{7}$ is a fraction, 6÷7 is a sum”.
>
> This is a good example of a response which is based on sound conceptual reasoning, but one that is limited because it implies the inability to encapsulate the *process* 6÷7, as a single conceptual entity. The encapsulation occurred far more often amongst the computer group, again underlying what we believe is a more flexible global view.
>
> (Thomas & Tall, 1989, p. 218)

In our paper in *Educational Studies in Mathematics*, written shortly afterwards, but not appearing until two years later, in 1991, we refer to the *process-product* obstacle, where students are unable to see a symbolisation as standing for both the process and the object, which "requires the encapsulation of the process as an object". Here can be seen the genesis of the procept, applied in algebraic thinking, but without the semiotic label.

> Another closely related dilemma is the *process-product obstacle*, caused by the fact that an algebraic expression such as $2+3a$ represents both the *process* by which the computation is carried out and also the *product* of that process. To a child who thinks only in terms of process, the symbols $3(a+b)$ and $3a+3b$ (even if they are understood) are quite different, because the first requires the addition of a and b before multiplication of the result by 3, but the second requires each of a and b to be multiplied by 3 and then the results added. Yet such a child is asked to understand that the two expressions are essentially the same, because they always give the same product. Such a child must face the problem of realizing that the symbol $3a+6$ represents the implied product of any process whereby one takes a number, multiplies it by 3 and then adds 6 to the result. This requires the encapsulation of the process as an object so that one can talk about it without the need to carry out the process with particular values for the variable. When the encapsulation has been performed, two different encapsulated objects must then be coordinated and regarded as the "same" object if they always give the same product – a task of considerable complexity.
>
> (Tall & Thomas, 1991, p. 126)

Later in the same paper (*ibid*, p. 144) we applied these ideas to the $\frac{6}{7}$ versus $6\div7$ dilemma, saying "This reveals the perception of $6\div7$ as a *process* involving value-operation-value rather than as a global entity – the single number – produced by this process." It turns out that many students have not encapsulated the division of integers as fractions, but instead their fraction object is what I have called a *pseudo-encapsulation* (Thomas, 2002—after Vinner's, 1997, *pseudo-conceptual*), based on a sharing conception.

Finally, it was the addition of strong evidence from Eddie Gray's PhD on process-object occurences in arithmetic learning that led to the generalisation of the idea, and cemented in place the now familiar concept of *procept* (Gray & Tall, 1994). More recently I have had the opportunity to write two papers with Eddie Gray too (Tall *et al.*, 2000; Tall, Thomas, Davis, Gray, & Simpson, 2000). The latter considers the nature of mathematical objects, and in particular how encapsulation leads to a mathematical object. In recent years, for me, as may be seen from the quotation above from page 467 of my thesis, it has been an examination of the differences in thinking the examples highlighted, and analysing the versatility of thought necessary to cope with them, that has occupied much of my research over ten or more years. These two areas will be considered below.

DEVELOPING VERSATILITY

Examining the role of proceptual thinking

In a recent paper (Thomas, in print) I have described *versatile mathematical thinking* as comprising three aspects, and I would like to consider the development of my thinking about each of these, in turn, below. The first flows from the work I did with David, described above, and is what I describe as

> *process/object versatility*—the ability to switch at will in any given representational system between a perception of particular mathematical entities that may be seen as a process or an object

This could also be called proceptual versatility. One of the areas of the curriculum where the idea of *process/object versatility* was examined was the concept of integral. Working with a PhD student Ye Yoon Hong (Thomas & Hong, 1996; Hong & Thomas, 1997), we found that students generally had a process view of the integral symbol, and hence could not deal with an integral where the process could not be carried out. Two of the questions we used to examine this thinking were:

> If $\int_1^3 f(t)dt = 8.6$, then write down the value of $\int_2^4 f(t-1)dt$.
>
> If $\int_1^5 f(x)dx = 10$, then write down the value of $\int_1^5 (f(x)+2)dx$.

Our results showed that only 17.0% of 47 Form 7 (age 18 years) high school students were able to answer the first of these questions correctly, and 12.8% the second, while 59.6% were unable to make any response at all. The situation was similar at university, with 49.1% of 161 first year university calculus class students (age 17~22 years) able to use a standard procedure to find $\int_2^4 (x-1)^2 dx$ (surprisingly low) but only 27.3% and 22.4% respectively could answer the above questions. Some students were rather resourceful in their efforts to circumvent the problem of considering the integral as representing an (area) object. For example, to introduce procedures, one student wrote

> Let $f(t) = ax + b$ [interestingly, but not surprisingly, a function in x not t]
>
> Then $\int_1^3 f(t)dt = 3a + b - a - b = 8.6$ so $2a = 8.6$ and $a = 4.3$. [no integration done]
>
> Thus $\int_2^4 f(t-1)dt = 4a + b - 1 - 2a - b - 1 = 2a - 2 = 8.6 - 2 = 6.6$. [using $f(t-1)=f(t)-1$]

A second example of this, a favourite of mine, is seen in Figure 3. Here the *process-oriented* student (Thomas, 1994) is so focussed on carrying out a process that he identifies the t as something he can integrate to $\frac{t^2}{2}$, and so he does, taking the rather troublesome f as a constant whose value is to be found (f=2.15). Combining this with the expansion of $f(t-1)$ as $f(t)-f$ and integrating as before we get…the correct answer!

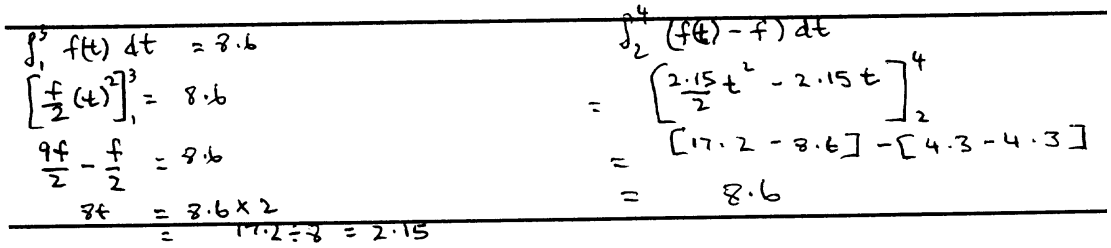


Figure 3: The influence of a strong process-oriented view of integral.

These two examples demonstrate how far students with a procedural perspective will go to introduce known algorithms into a question where they lack the necessary conceptual understanding. Our belief was that investigation of the processes lying behind the concepts of integration would enable improvement of conceptual understanding, assisting with encapsulation. The research showed that it was possible to design curriculum materials using technology to do this and give an improved cognitive base for a flexible proceptual understanding of integral and other concepts (Hong & Thomas, 1998).

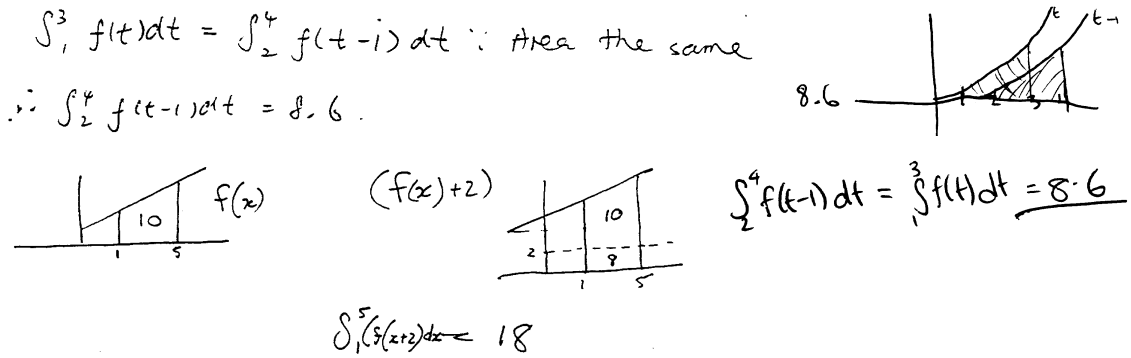


Figure 4: Understanding an integral procept as an object.

Figure 4 shows some of the work of students on the questions after such a programme. Here we see that the students were able to perceive the relationship between the first integration symbolism and the area object and to then operate on this area according to the change in the integration symbols. This is much different from process-oriented students, who need to carry out a procedure.

I have also spent some time investigating student understanding of differentiation procepts such as $\frac{dy}{dx}$ (Delos Santos & Thomas, 2001, 2003, 2005). Among our results, we found (Delos Santos & Thomas, 2001) that only 45% of 22 final year school students could make any interpretation of $\frac{dy}{dx}$ in $z = \frac{d(\frac{dy}{dx})}{dx}$, and only 1 thought that it had anything to do with rate of change or gradient of a tangent. A possible explanation of this problem (Thomas, 2002) can be expressed in terms of the differences in the

manner students perceive $\frac{d^2y}{dx^2}$ and $\frac{d(\frac{dy}{dx})}{dx}$. The first is often seen as a repeated application of the differentiation process, but there are problems interpreting $\frac{d(\frac{dy}{dx})}{dx}$ in this way because it requires one to operate on $\frac{dy}{dx}$ as an object (in the differentiation process), and hence students with only a process view of $\frac{dy}{dx}$ meet a cognitive obstacle. One of the students in the study, Steven, epitomized the lack of an object perspective; when asked about $\frac{dy}{dx}$ he immediately responded in a process-oriented way, saying "I must differentiate." (Delos Santos & Thomas, 2003).

We probed our students' process/object versatility further by taking them into uncharted waters, presenting them with unfamiliar function constructions that required object perspectives. While the students were familiar with the composite function form $f(g(x))$, we employed the unusual notations $f(f'(x))$ and $f'(f'(x))$ to access their thinking. For the first of these Steven responded "the original function times the differential of the original function", and proceeded to illustrate this by multiplying the function $f(x) = 2x^2 + 1$ by its derivative $4x$. He followed through consistently for the second describing it as "the differential times the differential" Thus when faced with an unfamiliar representational form Steven's recourse was to interpret the juxtaposition of f and f' (and later of f' and f') as a known operation or process, namely multiplication. Hence he operated *with* the result of a process $f'(x)$, but not *on* it as an object, as the composite function requires. James on the other hand displayed the ability to think of the symbol $f'(x)$ as an object, what he called the derivative function. He tried to interpret the symbols using specific functions, of the form $f(x) = x^n$, and wrote:

$$\begin{array}{ll} f(x) = x^2 & f'(x) = 2x \\ f(f'(x)) = (2x)^2 = 4x^2 & \end{array} \quad \text{and} \quad \begin{array}{ll} f(x) = x^3 & f'(x) = 3x^2 \\ f(f'(x)) = (3x^2)^3 = 27x^3 & \end{array}$$

Using a graphic calculator he was able to generalise this, getting $n^n x^{(n-1)n}$, and recognise that the power would always be even. Moreover, he provided a graphical interpretation saying that "it's always gonna be steeper than this original function …it's also gonna be concave up". Interestingly, when asked to describe $f'(f'(x))$ he responded "that does imply second derivative". Hence instead of applying the same composite function thinking he had used seconds before, he saw this as the second derivative $f''(x)$. This could be the result of a strictly linguistic interpretation of the symbolism. Reading $f'(x)$ as *f*-dashed of *x*, may cause one to read $f'(f'(x))$ as *f*-dashed of *f*-dashed of *x*. This in turn leads to James' statement that "It's the derived function of the first derived function.", and a parallel with $\frac{d(\frac{dy}{dx})}{dx}$, the second derivative, takes over. Whatever the reason this appears to be a common initial reaction to this unfamiliar proceptual symbolism.

The ideas surrounding process/object versatility, that started with my conversations with David years ago, continue to occupy my time, and more recently I have been considering their application to the learning of linear algebra (Stewart & Thomas, 2006a, b). One problem we have identified here is in the definition of eigenvalue and eigenvector, often a form of:

> A non-zero vector x is called an eigenvector of a square matrix A if and only if there exists a scalar λ such that $Ax = \lambda x$.

Analysing this equation from a proceptual perspective we see that the right hand side involves a process in which a vector is multiplied by a scalar, resulting in a vector object. However, the left hand side of the equation has a quite different process, with a vector multiplied by a square matrix. The potential difficulty for students in understanding the definition is to see that both processes can be encapsulated as the same vector object. We have found that students with a process perspective of the equation's procepts do not see this. This lack has implications for understanding of the standard algorithm for finding the eigenvalues. Figure 5 shows a version of how this is often described in coursebooks or textbooks.

Definition 4.1. *Given a square $n \times n$ matrix A, we can sometimes find a non-zero vector $\mathbf{v} \in \mathbb{R}^n$ and a corresponding scalar λ, such that*

$$\boxed{A\mathbf{v} = \lambda\mathbf{v}.} \tag{4.1}$$

We call any non-zero vector $\mathbf{v}$ which satisfies (4.1) an **eigenvector** *of A, and the corresponding scalar λ an* **eigenvalue.**

The matrix equation (4.1) can be rewritten

$$(A - \lambda I)\mathbf{v} = \mathbf{0}. \tag{4.2}$$

Figure 5: An 'explanation' of the move from $Ax = \lambda x$ to $(A - \lambda I)x = 0$.

Asked to reproduce and explain the missing steps in this transformation students had problems bringing the two processes together. Rather than multiplying either just the x, or both sides by the matrix identity I_n, as the student whose work is shown in Figure 6 did, others ran into process difficulties such as needing to subtract a scalar from a matrix, as seen in Figure 7. We concluded that this apparently simple step is not straightforward for many students, and the one-step jump tends to cause a focus of attention on λI rather than on Ix.

The research description above gives a brief indication of the fruitfulness of an investigation of the role of process/object versatility in mathematical thinking. No doubt there are still many other areas of mathematical thinking where a similar analysis of will repay further dividends.

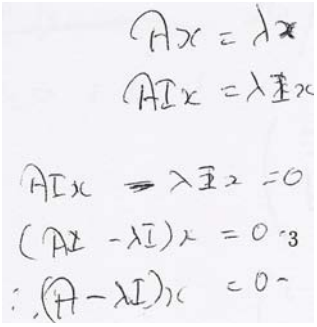


Figure 6: One way of understanding the transition from $Ax = \lambda x$ to $(A - \lambda I)x = 0$.

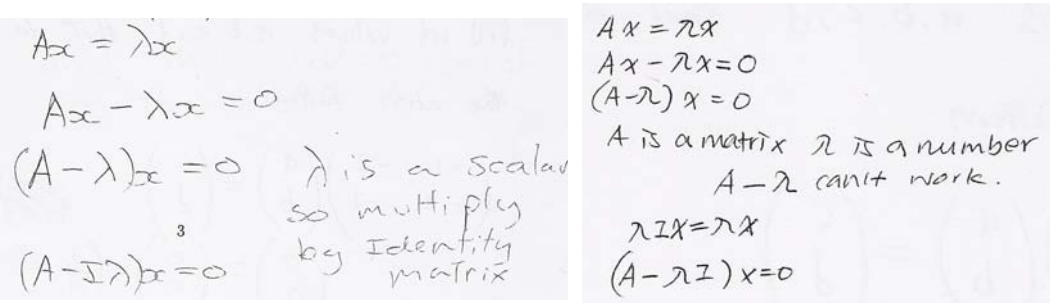


Figure 7: Failure to understand the transition from $Ax = \lambda x$ to $(A - \lambda I)x = 0$.

The role of visualization in versatile mathematical thinking

The versatility of thinking required to switch from a process view of symbol to an object view is only one example of the considerable flexibility needed in mathematical thinking. A second, which I have been thinking about since I wrote my PhD thesis (Thomas, 1988), is also described in a recent paper (Thomas, in print):

> *visuo/analytic versatility*—the ability to exploit the power of visual schemas by linking them to relevant logico/analytic schemas

In my thesis, and in papers since (e.g., Tall & Thomas, 1991; Thomas, 1995; Thomas, 2002; Booth & Thomas, 2000) I have described a model of *cognitive integration* that seeks to incorporate visual thinking into a description of versatility. This model (see Figure 8) comprises a *first-degree* knowledge structure, the elements of which are primarily mental images of various forms, 'existing in' and processed by the brain's minor hemisphere.

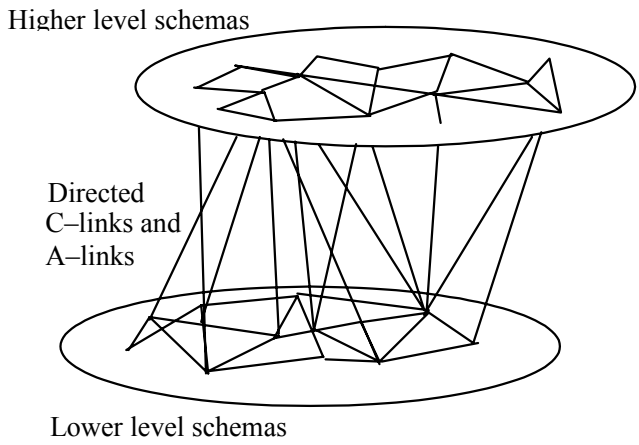


Figure 8: A simplified model of cognitive integration.

These elements or states are connected to form schemas in a way similar to those of the largely language based, serialist/analytic *second-degree* structures of the left hemisphere, and are also constructed by Piagetian

assimilation and accommodation. It is these *first-degree knowledge structures,* I contend, that give rise to global/holistic mental abilities. Piaget and Inhelder (1971, p. 366) have described evidence for the schematic nature of the mental imagery generated by these abilities, and say that

> If we use the word 'scheme' (*scheme*) to designate a generalization instrument …there are perceptual schemes, sensori-motor schemes, operational schemes, and so on. And in this senses there also exist imaginal schemes enabling the subject to construct analagous schemes in comparable situations…Imaginal figuration, on the other hand, is 'schematized' precisely in the 'schema' sense, though at the same time it may entail 'schemes'.

We may glimpse this ourselves by thinking of the name of a town that we are very familiar with. Mentally naming the town and then concentrating on the rapid sequence of images which may appear and disappear we may conclude that those connected to the initial image of the chosen town are linked in a structure such that each image evokes others. With such a structured sequence of images we may even take a tour around the town, though it may be half a world away. Such a structure constitutes, I maintain, an example of the schemas of mental imagery. However, if these *first-degree knowledge structures* are to have value in the goal-directed mental activity of the individual, and their processing is to be acted upon, then a means of accessing the results of the goal-directed activity of this schematic thinking from the qualitatively different, conscious, logical/analytic schemas of the second degree must be included in a psychological model. This is achieved by introducing links between the concepts in each level of the model, facilitated by the flow of data between each hemisphere, across the corpus callosum. Hence it includes, not two separate knowledge structures, but two connected, distinct modes of operation, working in parallel, within an integrated whole. It is by means of the 'vertical' links between the levels, I contend, that the mental imagery schemas influence the higher-level cognitive functions of the mind. In the context of this model, *visuo/analytic versatility* may be defined as attaining the construction of meaningful schemas at both the higher and lower cognitive levels, as well as appropriate two-way, inter-level—that is, inter-hemispheric—links. Thus, such a learner is able to use the conscious higher-level relational schemas (with their serialist/analytic processing) in parallel with the unconscious lower level relational schemas (with their global/holistic processing), and, most importantly, to switch easily, although often unconsciously, between the two as and when appropriate.

I have applied the principles behind this theory of the power of visualization in a number of projects arising from my PhD work. For example, we tried to promote a *versatile* view of equation using a *Dynamic Algebra* computer environment (see Figure 9), which encourages students to construct equations in terms of variable and expression objects that can be simultaneously evaluated (Thomas & Hall, 1998). This approach emphasised the visual aspects of variable as a location or store and an accompanying label, and equation (as two equal expression boxes) in an environment

where a number of processes for equation solving, including trial and error substitution and balancing can be investigated. After the visual module of work the 11 and 12 year-old students improved significantly in their ability to solve linear algebraic equations, including using a method of solution where most of them applied the same operation to both sides of the equation and 'cancelled' terms, for equations such as $5n + 12 = 3n + 24$. We felt able to conclude that they had developed a more versatile view of equation.

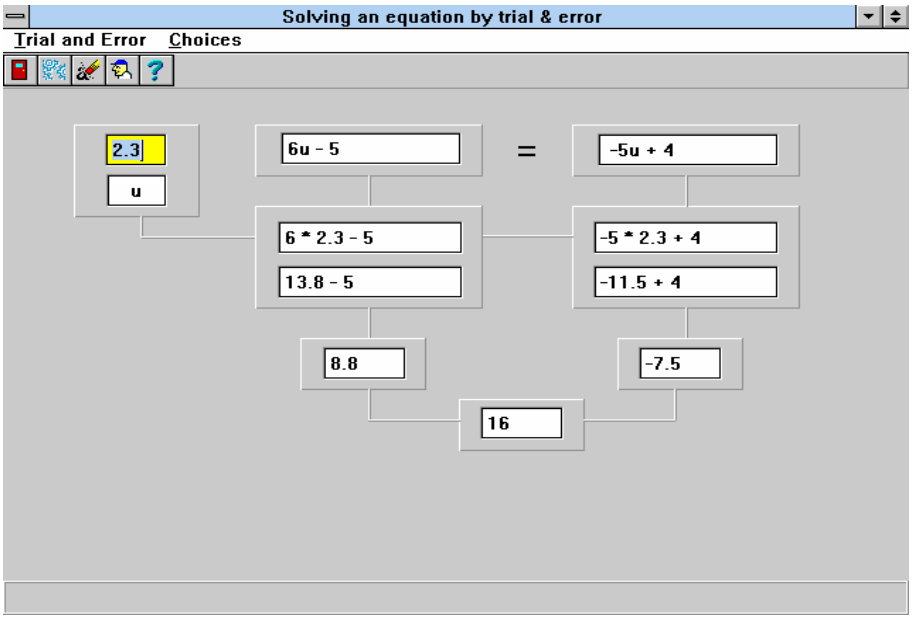


Figure 9: A screen from the Dynamic Algebra program.

A similar approach was employed (Graham & Thomas, 2000) to give students an improved understanding of generalised number or variable, using graphic calculators. The controlled experimental study showed that the visual mental model, using a graphic calculator, assisted in significantly improving student understanding of symbolic literals, regardless of their ability level, although the gains were particularly noticeable for the weakest students. The value of visualisation can be seen in the remarks of one student:

> I think the STORE button really helped, when we stored the numbers in the calculator. I think it helped and made me understand how to do it and the way the screen showed all the numbers coming up I found it much easier than all the other calculators which don't even show the numbers.

It appears that the way the graphic calculator screen preserves several computations on variables in view, along with the mental model, had assisted this student to think mathematically and make powerful connections. The flexibility to link visual and embodied thinking with analytical reasoning is an important part of versatile thinking.

The role of representation in versatile mathematical thinking

Versatility of mathematical thinking involves more than the two types of flexibility described above. Engaging in further discussion of these ideas with David Tall led me to consider further the role of representation in mathematical thinking. In my recent paper (Thomas, in print) I have described this third aspect of *versatile mathematical thinking* as:

> *representational versatility*—the ability to work seamlessly within and between representations, and to engage in procedural and conceptual interactions with representations.

Following the field of semiotics introduced by Peirce (1902) and others, signs (icons, indexes, or symbols) are objects of thought that bring to mind their referent. While there are icons and indexes in mathematics, it predominantly comprises symbols, signs that have become associated with their meaning by usage (Peirce, MS404, 1894); becoming significant simply by virtue of the fact that it will be so interpreted. However, the signs are not usually isolated but are grouped together into families (Saussure, 1966), such as logic signs, algebraic symbols, matrices, or graphs. These groupings can be called *representation systems*, and the individual signs representations, making a representation a sign associated with a given system of signs.

Kaput (1989) drew a distinction between the way some *representation systems* (he also called them *notation systems*) are used mainly to display information and relationships (*display notations*) while others support a variety of transformations and other actions on their objects (*action notations*). An important class of mathematical activity involves manipulation of mathematical concepts both within and between these different representations, or "translations between notation systems, including the coordination of action across notation systems." (Kaput, 1992, p. 524). While the ability to establish meaningful links between and among representational forms and to translate meaning from one representation to another has been recognised, and referred to as *representational fluency* (Lesh, 1999), I introduced the concept of *representational versatility* to include both this fluency of translation between representations, and the ability to interact procedurally and conceptually with individual representations (more details below).

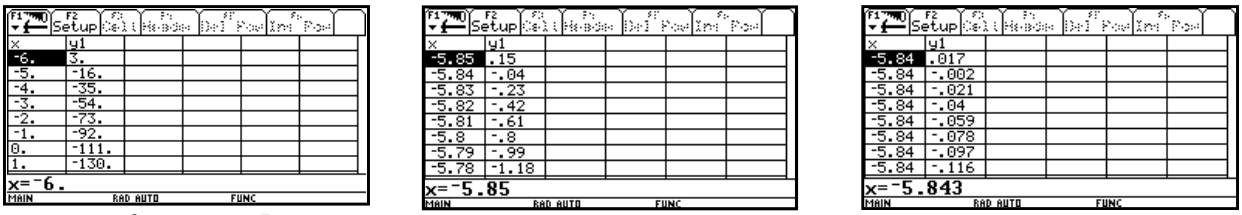
Interacting with mathematical signs or representations can be a complex multi-stage process. One may interact with them by *looking at* the images or *looking through* them (Mason, 1992, 1995) depending on whether the focus of attention is surface or deep. For example, Laborde (1993a, b) described how one may see a geometric icon in two different ways. She talks about how a *Drawing* refers to the material entity while *figure* refers to the theoretical object (Laborde, 1993b, p. 49). Likewise, Fischbein (1993, p. 141) refers to how "successful geometric reasoning can be achieved when we stop considering only two distinct categories of mental entities (images and concepts) and we deal apart from them with a third type of mental

object, the figural concept". In both cases looking through, or deep observation, requires linking of an image with a conceptual base, or schema. This process is part of the *cognitive integration* described above. Thus a surface observation of an icon may lead one to think that it may be a representation of a mathematical object, but in order to move to seeing it as a figure, referring directly to the mathematical object, requires interpretation. The perceived object needs to change role from an icon to a symbol. This interpretation involves the use of a link to an appropriate, existing second-degree mathematical schema to ascertain the properties of a rectangle that may be overlaid in memory on the first-degree drawing or representation. Thus the mathematical concept of rectangle, is a combination of a perceived icon, its object referent and data (properties) from the mathematical 'rectangle' schema. With many signs this process may have more than one stage and in order to produce a mathematical figure from a picture we need to pay attention to the essential property-revealing details of the picture in two steps. First we mentally or physically produce a diagram or figure from the picture, and secondly we need to overlay its conceptual properties in order to see the figure as representing the theoretical object (Booth & Thomas, 2000).

It has struck me as interesting that in all the analysis of process/object versatility the question of how representations other than those of the symbolic algebra representation system relate to the process-object conceptualisation of mathematics had not really been addressed. Since mathematical concepts can clearly be perceived in this dual manner one would expect graphical, tabular, ordered pair, and other representation systems common in secondary mathematics, to be amenable to a corresponding analysis.

When the interaction with a sign or representation progresses from observation to performance of an action on the representation, and learning from it, *doing* and *construing* in the sense of Mason (in print), then I describe the representation as becoming a *cognitive tool* (Thomas & Hong, 2001; under review). I propose (Thomas & Hong, 2001; in print) that a crucial difference between a process and object tool interaction is that the former comprises a discrete approach in terms of its parts (e.g pointwise for a function) while the latter requires a holistic perspective.

Technology, such as graphic calculators, makes available some novel interactions such as the ability to solve equations by numerical processes by zooming in on solutions using tables of values (see Figure 10). This kind of dynamic interaction is not as simple as it appears. It is based on the continuity of the function and the Intermediate Value Theorem, and requires not a discrete, point-by-point view of the table of values, but a holistic focus on at least the part of the function revealed by the representation.



$-6 < x < -5$ $\quad$ $-5.85 < x < -5.84$ $\quad$ $-5.843 < x < -5.842$

Figure 10: Dynamic interactions with a tabular representation using a calculator.

We have also tried to promote representational versatility by using a geometrical approach linked to algebraic representations with CAS calculators for the Newton-Raphson method for approximating zeros of functions (Hong & Thomas, 2002). We noted that local examiners' reports over a number of years, had stated that 'many candidates had no geometrical appreciation of the Newton Raphson method.' and so such students were reduced to calculation processes based on the algebraic formula $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$. We employed a method based on the graphical representation and linked it to the algebraic representation by rearranging the equation of the tangent: $y - f(x_1) = f'(x_1)\,(x - x_1)$, with $y = 0$ when $x = x_2$, to get $f'(x_1) = \frac{f(x_1)}{(x_1 - x_2)}$. The algorithm used was to draw the graph of the function under consideration, choose a suitable first estimate, and then use the CAS to draw the tangent at the point and to find and display its equation. The representation is then acted upon in a conceptual way by using the symbolic manipulator of the CAS to solve $f(x)=0$, and find where the tangent crosses the x-axis. This method can then be repeated until the zero is found to the accuracy required, while zooming in on the graph to see what is happening.

We found that this conceptual interaction with two CAS representations helped students to understand why the Newton-Raphson method works, and to form conceptual links between the graphical and algebraic representations. For example, Figure 11 shows how one student understood how the sign of f and f' (ie the gradient of the tangent) affect whether the second estimate is greater than or less than the first estimate (Hong & Thomas, 2002).

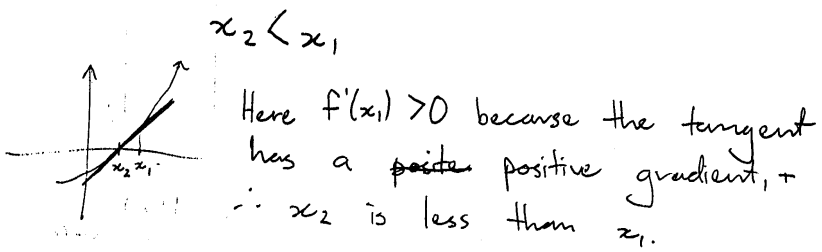


Figure 11: Interacting with a geometric representation in a conceptual way.

Figure 12, shows another student interacting with a graphical representation and appreciating that the first estimate has to be sufficiently close to a for the gradient to be large enough for the tangent to cut between $x=a$ and $x=b$.

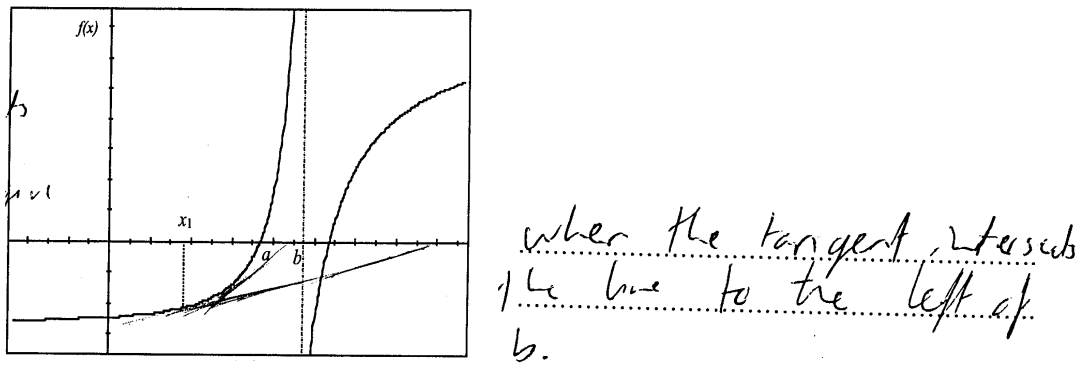


Figure 12: Interacting with a geometric representationin a conceptual way.

Our students also showed an understanding of how the choice of the first approximation to a root influenced the result, especially how the gradient at the point could be too small or be zero, with consequent problems:

Student A: You can't choose a max or min point or else you won't cut the *x*-axis. Also the tangent could go towards the wrong root.

Student B: It must be close to the root so the tangent gives you the nearest value. Also you can't choose a stationary point as a first value.

As part of the study students were asked to construct a graph of a function to put the first and second estimates on opposite sides of the zero. This involves a consideration of the concavity required, and Figure 13 shows the result of the graphical thinking of one student who managed this task.

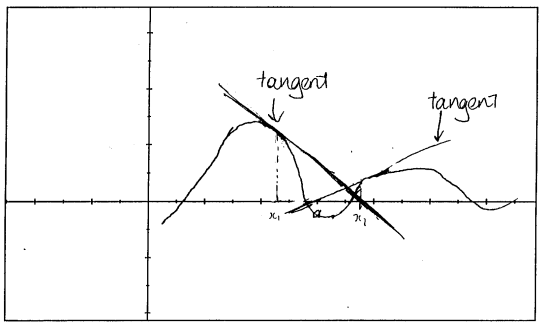


Figure 13: Interacting with a graphical representation in a conceptual way.

The idea of *representational versatility* has also found application in other areas of mathematics. Recently we have applied it to the learning of statistics (Graham & Thomas, 2005). A key part of promoting statistical thinking (Wild & Pfannkuch, 1999) is for students to develop a "curiosity for other ways of examining and thinking about the data and problem at hand" (Chance, 2002, p. 4). It seems that promoting an examination of a number of representations would form a crucial part of such thinking. To illustrate, a student might interact with a statistical data representation in a procedural or a conceptual manner. In the former they might perceive that what is required of them is to carry out a series of steps or actions on their part, while in the latter they would focus, not on a procedural algorithm such as calculating a summary

value, but on an idea or concept. We suggest (Graham and Thomas, 2005, p. 9) that this is "more likely to create in the student an 'object' perspective of the data representation where they can consider the data and its statistical properties as a whole."

For example, when reflecting conceptually on the regression properties of a set of paired data the student may examine the graphical representation to gain an awareness of the overall pattern of the points. The switch of focus from process to object for such a data set is subtle, but in this case students may be encouraged to draw a line of best fit by eye and then consider each point in turn, calculating the vertical deviations from their line along with the sum of the squares of these deviations. This process approach can then progress into a conceptual interaction where the student is able to manipulate a property of the whole of the data as a single object. It is only when thus viewed holistically that questions about minimising the sums of the squares of the deviations and whether a straight line is the best model to fit, can take on meaning for the whole data set. They may then link to a symbolic algebraic representation to consider a suitable regression model, and once this has been found from working within this representation, the data set may no longer perceived as a set of discrete points (ie in a process manner) but rather as a single entity that may be represented by its regression function. It seems important to stress that in statistics the student's interaction is sometimes with a representation of the original data, and sometimes with a mathematical model of the data. They need to appreciate when the representation they are dealing with depicts the original data and when it is one step removed from the data, being a theoretical model fitted to it. This is a key part of representational versatility in statistics.

In the above discussion I have tried to give a flavour of the growth of what I still see very much as a developing conception of versatile mathematical thinking. It is still clearly not complete and is very much a work in progress. The picture may, or may not, become clearer, but the exhilaration of the enquiry is assured.

THE ROAD AHEAD

Over the past 23 years I have been able to work with David Tall in various capacities, writing some 14 papers together, and producing an edited book on Richard Skemp's ideas (Tall & Thomas, 2002). Throughout this whole period I have been conscious of the debt I owe to him for the friendship that has helped refine many of the ideas we have tried to bring to fruition. I am hopeful that our productive collaboration will continue for some time to come, even though David will be 'retired'. At present we are working together on a paper describing what we know, and can know, about mathematical thinking from brain studies. We are also just beginning to look ourselves at fMRI brain scans and what they may (or may not!) tell us about such thinking; so there is much more still to accomplish.

I certainly appreciate the truth of what another of the world's leading mathematics educators, the late Jim Kaput, wrote in a reference for me, when he said "He was fortunate to study with a pioneer in the field, Professor David Tall." While this is most certainly true, to me (and I know Jim would have agreed) David has been much more than just a pioneer in mathematics education; I would suggest that his research has been, like Jim Kaput's, of the highest possible order. His careful and thoughtful elucidation of theoretical concepts, coupled with his innovative uses of technology have often had, and will continue to have, a profound impact on teaching and learning. The advances he has invited us to follow also demonstrate that David Tall is not merely an excellent theoretician but is someone who cares deeply about the practice of mathematics education in the classroom and the lecture theatre. I know I am not the only person who has valued David's keen insights and while I'm sure we all wish him a long and happy retirement, we are eagerly awaiting the publishing of 'the book'.

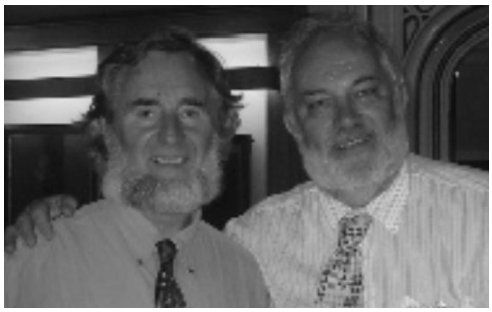


Figure 14: Jim Kaput and David Tall

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LEARNING AND TEACHING INFINITIES: A NEVER-ENDING STORY

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We start this paper by providing a brief account of some of Tall's contribution to the study of the learning and teaching of infinities. Then, we describe an interview that Tall conducted with Nic, his youngest son, when he was seven years and one month. The interview reveals the thinking of a brilliant child about infinity, for instance, his grasp of the idea that aleph plus aleph equals aleph. We then describe another interview that we conducted with a bright seven year old girl, where she expressed her view that while infinity is the largest number, infinity plus one is larger than infinity. We conclude by discussing the contribution of such interviews to the learning and teaching of infinities.

In 2001, a special issue of *Educational Studies in Mathematics* was dedicated to the study of the learning and teaching of infinity. This issue was entitled: *Infinity – The never ending struggle*. In a way, this volume, which was edited by Tall and Tirosh, reflected a long standing shared interest of members of the mathematics educators at the university of Warwick (e.g., Tall, Monaghan) and several members of the mathematics education group at Tel Aviv University (e.g., Fischbein, Tirosh, Tsamir, and later also Dreyfus) in infinity. The volume included papers by Tall, Monaghan, Fischbein and Tsamir, as well as papers by Kleiner, Jahnke and Mamona-Downs.

In his paper Tall coined the terms *natural infinities* and *formal infinities*. He distinguished between "natural concepts of infinity, developed from experiences in the finite world, and formal concepts of infinity, built from formal definitions and deductions" (Tall, 2001a, pp. 235).

Tall explains that

> By beginning with different properties of finite numbers, such as counting, ordering or arithmetic, different formal systems may be developed. Counting and ordering lead to cardinal and ordinal number theory and the properties of arithmetic lead to ordered fields that may contain infinite and infinitesimal quantities… while natural concepts of infinity may contain built-in contradictions, there are several different kinds of formal infinity, each with its own coherent properties, yet each system having properties that differ from the others.
>
> (Tall, 2001a, pp. 199)

Tall further described different types of formal infinites, showing that a formal theory can lead to a *structure theorem*, and concluded:

> The moral of this tale is that there is not a single concept of infinity, but several distinct ones, and that it is not appropriate to assign to any one of these concepts the title of 'the' concept of infinity. In particular, it is not appropriate to consider infinite cardinal number theory (with its lack of subtraction and division) as the *only* mathematical

> infinity. There are other infinite concepts, including the intuitive infinities and infinitesimals in the calculus, that can be given formal theories of their own, consistent in themselves and yet *different* from the notion of infinite cardinal.
>
> (Tall, 2001a, pp. 236)

Indeed, in his extensive work, Tall explored a remarkable spectrum of infinities-related issues (e.g., Tall & Schwarzenberger, 1978; Tall, 1980a; 1980b; 1981a; 1981b; 1990; 1992; 1993; Tall, 2001a; 2001b; 2001c). He addressed, for instance, the learning and teaching of limits, cardinal infinity and ordinal infinity, and the notion of measuring infinity.

Our work addressed mainly the notion of cardinal infinity. We focused on the comparison of infinite sets, being attracted by the deep contradiction between the intuitive and the formal infinity. That is, between our intuitive reasoning, based on practical, real life experiences and the formal theorems. For example, we assumed that propositions like *the whole is equivalent to one of its proper parts* or $\aleph_0 = \aleph_0 + 1$ or $2\times \aleph_0 = \aleph_0$ may contradict our usual mental schemes, and consequently, provide a fruitful setting to examine pivot issues related to the learning and teaching of mathematics.

In the following sections we briefly describe two infinity-related interviews that touched upon the above mentioned surprising characteristics of cardinal infinity. The first interview was conducted by Tall (2001b), the second by Tirosh and Tsamir (Unpublished).

David and Nic Tall: Talking about infinity

Tall opened his paper on *a child thinking about infinity*, by stating that: "young children's thinking about infinity can be fascinating stories of extrapolation and imagination. To capture the development of an individual's thinking requires being in the right place at the right time" (Tall, 2001b, pp. 7).

The paper, then, described several episodes in which David and Nic (his youngest son) talked about infinity. At the age of seven years and one month, Nic came to David with a comment that took him (David) "totally by surprise".

Nic: I've invented a number bigger than infinity.

The result of the comment was an outstanding manuscript published in the *Journal of Mathematical Behavior* where David described the development of Nic's thinking about infinity. Nic started by treating infinity as a very large number.

David: What is infinity?

Nic: A very, very high number.

David: How high is it?

…

David: What about a googol times a googol, is it bigger than that?"

Nic [thinks briefly]: Well, … I think that equals just about infinity.

…

David: I see … Is a billion, billion, billion, billion, infinity?

…

Nic: Well, I think it just about is.

At that point David challenged Nic in an attempt to explore whether Nic believed infinity to be the largest number, discovering that (in the first episode), for Nic, infinity and one was larger than infinity.

David: ... Do you know any numbers bigger than infinity?

Nic [firmly]: Infinity and one.

David: How much bigger than infinity is that?

Nic [firmly, with a questioning tone]: One.

…

David: Tell me another number bigger than infinity.

Nic [firmly]: Infinity and two.

This led David and Nic to discussing operations with infinity. David found out that Nic regarded infinity as a large number that can be added, subtracted and multiplied like any other number. Nic said, for example,

Nic [with great conviction and force]: …if you take away infinity from infinity, you're left up with nothing, but if you take away two infinity from infinity, you get minus infinity.

At the end of the interview, after David challenged Nic's conceptions regarding the arithmetic of infinite numbers, and introducing Nic to א, Nic concluded that 2× א= א.

David: … How many whole numbers are there?

Nic: Aleph.

David: Aleph. That's right! Well, how many even numbers are there then?

Nic: Aleph?

David: … and how many odd numbers are there?

Nic: Aleph.

[These questions were repeated to confirm the ideas.]

David: So what happens if we add aleph plus aleph, what's the answer?

Nic [immediately]: Aleph!

David [Feigning amazement]: Aleph plus aleph equals aleph? Why is that?

Nic: Well it's just … like so! [With sudden conviction]: Like nothing add nothing equals nothing!

…

David: So … what happens if I have two aleph, is that the same as aleph?

Nic: Yes!

This interview was conducted by a special interviewer (mathematician, math educator, and father) with a brilliant child. In his reflection on these interviews, Tall raised the question: “Did I do right to press on with infinite correspondences with a seven-year old?” He admitted: "I don’t know". This issue is still open to research and discussion.

Dina Tirosh, Pessia Tsamir and Noa: Comparing infinite sets

Our conversation with Noa, a seven year old Israeli girl, illustrates the fascinating nature of young children's ideas about infinity. Noa was identified as highly talented in mathematics. She was also very talkative and extremely communicative, volunteering her thoughts, hesitations and ideas in a very open manner. Noa knew that we were interested in infinity, and one day as we were working together and Noa was around, she approached us saying enthusiastically:

Noa: I can show you infinity.

Before we had a chance to react to her statement, she took a piece of paper and wrote: 100000000000000 (one and 14 zeroes, so that the entire row was filled with the numerals and the last zero was somewhat squeezed to the edge. No room was left for another digit). Then, Noa looked at us with great satisfaction and said:

Noa: That's infinity.

We realized that:

(1) The number “infinity” that Noa wrote was one, followed by 14 zeroes;

(2) The 14th zero was squeezed into the row, touching the edge of the page. There was no room on that row, on the piece of paper, for another digit. It seemed that Noa attempted to completely fill an entire row.

This led to an unexpected conversation.

Dina: Yesterday, another child was here and wrote: 1000000000000000 (one followed by 15 zeroes, to complete an entire row). He said that this is infinity…

Noa [counting the zeroes and interrupting]: NO! This is not infinity! Infinity is one and fourteen zeroes.

Dina: So… what is this (one and fifteen zeroes)?

Noa: This is not a number… one and fourteen zeroes is infinity… the greatest number… the greatest! So…this (points to the one and fifteen zeroes) can not be a number…

At that point Noa went to her room. It seemed that for Noa, infinity was a number that could be presented in a finite manner with a fixed number of digits. She grasped infinity as the largest number, and so any representation that included more digits could not be a number.

After some time Noa returned to us, asked for her paper, and wrote below her previous item (100000000000000, i.e. one with fourteen zeroes), another number: 1000000000000000000, i.e., one with eighteen zeroes, in smaller digits, still completing the row.

Noa: You see? One and (counts) eighteen zeroes is also the number infinity…

Pessia: Why?

Noa [writes 100000000000000 and again another 100000000000000 below it, and then below both numbers she writes 200000000000000.] She said:

Look, if I write it like that, I get 2 infinities. But infinity is the biggest number…. So it's (pointing at 200000000000000)… infinity…

Pessia: So what is infinity plus infinity?

Noa [laughing]: Infinity!

Pessia: ...and… infinity plus infinity plus infinity plus infinity plus infinity?

Noa [laughing]: Also infinity!

Dina: And this (writing one with twenty seven zeroes)?

Noa [confidently]: Also infinity! From 1 and fourteen zeroes all are infinity… infinity is the biggest number… nothing is bigger than infinity…

At that point Noa seemed to believe that:

(1) Infinity is a number;

(2) The number 100000000000000 (one with fourteen zeroes) is infinity;

(3) Any number beyond the number 100000000000000 (infinity) is infinity;

(4) Two times infinity and five times infinity are infinity (perhaps any number times infinity is infinity).

(5) There is nothing larger than infinity;

In an attempt to further explore Noa's ideas of infinity, we asked:

Dina: What about infinity plus one?

Noa [hesitating]: Infinity and one?...

Pessia: What about it? Is it larger than (writing): 1000000000000000 (one and fourteen zeroes)?

Noa: Yes. It's infinity and one.

Dina: So???

Noa: It's one more than infinity…

Interestingly, Noa kept thinking that infinity is *a number*, and that it can be presented by one with fourteen zeroes. These two ideas resisted various manipulations, suggested by us or by her. However, while "infinity times two" and "infinity plus infinity plus infinity plus infinity plus infinity" was grasped as "the same infinity", a seemingly simple manipulation (adding one) challenged her belief that this "infinity number" (100000000000000), is *the largest* number, and that no number is larger

than the “infinity number”. She regarded infinity plus one as a number, larger than the “infinity number”.

Some Concluding Remarks

What can be learnt from such conversations? What is the relevance of conversations with young, bright children to the learning and teaching of mathematics?

In both cases, at certain points of the interview the interviewees regarded (1) two times infinity as infinity, and (2) infinity and one as larger than infinity. Yet, these conclusions might reflect different reasoning. These seemingly small, yet interesting data guided us in our attempts to explore students’, prospective teachers’ and teachers’ conceptions of cardinal infinity, and to assist them in constructing related secondary intuitions (e.g., Fischbein, 1987; Tsamir & Tirosh, 1999; Tsamir, 2001).

For instance a task that we used in our attempts to present ideas related to cardinal infinity, to prospective teachers and teachers is:

☆ + ☆ = ☆
2 × ☆ ☆
☆ + 1 = ☆

Till now, all participants, in various groups responded that there is a solution to the first two items (i.e., zero), but no solution for the last item.

This leads us to a discussion of extensions of number systems (e.g., from positive natural numbers to non-negative natural numbers). The discussion of cardinal infinite numbers is then presented as an additional extension of the number system (à la Cantor).

Clearly, more research on the learning and teaching of infinities is needed. We are sure that David Tall will continue to play a major role in advancing this domain of research as well as other domains of advanced mathematics thinking.

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THE RETIREMENT ERA - THE END OF MATHEMATICS EDUCATION: A TALK IN HONOR OF DAVID TALL AND EDDIE GRAY ON THE OCCASION OF THEIR RETIREMENT

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If life is a journey, the metaphor can be used in order to distinguish between two types of people. Those who look back from time to time and reflect on the events and landscapes they have gone through during this journey, and those who always look forward. I myself belong to the first type. I like to reflect on my own development and also on the development of other people. We, the retired generation of this group (which can roughly be described as PME veterans), started our career in mathematics education as a research discipline, when this notion (mathematics education) was quite clear to us. We believed that the goal of this discipline was to investigate the processes of learning and teaching mathematics, to describe them, to analyze them, to understand them and, as a result of all these, to make some suggestions how to improve the practice of mathematics education.

As a consequence of this belief, several mathematics concepts and topics were chosen and investigated. Here is a partial list: Fractions, equations, geometrical figures, limits, functions, derivatives, problem solving, definitions and proofs. After a while, for some reasons, this direction of research became unsatisfactory for some people, veterans as well as new comers. That is a fact. The explanation can be that these people found the above direction too narrow. They wanted to theorize on wider domains and to relate to more general questions than those which were investigated in the original domain. Thus, to the primary direction of mathematics education as a research discipline, some new aspects were added: sociological aspects, anthropological aspects, historical aspects, and philosophical aspects. Epistemological and ontological questions were raised. All of a sudden we were involved with questions like: what are the origins of mathematics? What is it? What are mathematical objects? How were they formed in our mind? And so on.

Recalling a line from Julius Caesar, some of these questions were always Greek for me. However, many people were fascinated by these questions, and, even more, by the answers. Were Tall and Gray part of this movement? I think it is better to ask them to elaborate on this. An answer to such a question depends also on interpretations to some given texts. Interpretations, as we all know, depend very often on the desires of the interpreter. My desire is not to consider Tall and Gray among those who chose the Greek direction. I prefer to believe that they have stayed in the famous English path of common sense. Therefore I do not interpret their PROCEPT theory as an attempt to explain how mathematical objects came into

being. In my view, it is an attempt to point at the fact that some central mathematical symbols are ambiguous. They denote both an operation (a process) and the product of that operation (a concept). In fact, it is already claimed by the title of their paper: *Duality, Ambiguity, and Flexibility* (Tall & Gray, 1994). Thus, the comprehension of mathematical situations requires a flexible mind. Human beings with such flexibility form a minority in our society and therefore only few people are capable of studying mathematics. Fortunately, we are, of course, among the few who are capable, so we believe. Unfortunately, because we are so few, like all minorities, we do not have a simple life.

Since this paper is not about Tall and Gray, but is in honour of Tall and Gray, I would not discuss any further their contribution to mathematics education (that is dealing with thousands of printed pages, not to mention teaching, supervision and other services to the community of mathematics education). I would rather use this occasion to confess that I myself, few years ago, like other people whom I mentioned earlier, started to move in a direction different from the original direction of mathematics education. I have done this not because the original direction became too narrow for me. I did it because, as a result of accumulated research in the original direction, I started to have serious doubts about the necessity and importance of teaching mathematics the way it is taught, to the extent it is taught, to the entire population in almost the entire world. The result of the common teaching of mathematics is that the decisive majority of the world population does not know mathematics and very often also hates mathematics. On the other hand, this lack of knowledge has no negative impact on the life of the mathematically ignorant, or even on the society in which they live. The mathematical needs of this society are supplied by few people who have studied mathematics properly and can use it professionally whenever it is needed. A reasonable conclusion of this state of affairs is to give up mathematics as a compulsory subject in school. However, even expressing this view is dangerous to any speaker who dares to express it in any educational forum. Why is it so? First of all, it contradicts the common rhetoric about the importance of knowing mathematics. People hate to give up rhetoric. The rhetoric helps us to make our life meaningful. It is quite frustrating to realize that we are involved in meaningless activities. In addition to that, mathematics has a very important role in the educational system. It is an excellent selection tool. By means of mathematics scores, the society can select those who will be accepted to a higher stage of education, whether this is a junior high school, high school, college or special prestigious disciplines as law, medicine, economics or business administration. The claim that in order to study some disciplines one should know mathematics is only partly true. Medicine does not require mathematics. If you have any doubts about that, examine your doctors' mathematical knowledge. I have done it and, frankly, I could not care less about the poor results. I was concerned about their medical knowledge and this was, fortunately, quite satisfactory. As I said, one can argue about the wisdom of the current practice, but since any attempt to change it will raise brutal objection, there is no point in even mentioning it.

So, if this is the case, what are we supposed to do? Well, retirement is a reasonable option if you are old enough. But before elaborating on retirement, I would like to share with you more of my reflections. They relate to my attempt to reach a global view on the overall disciplines which are taught in school and the special role mathematics plays there. In fact, the achievement aspect of other school discipline is not more impressive than the achievement aspect of mathematics. It is only less traumatic. But if this is the case, why do we have schools at all? Of course, we need them as a giant baby-sitting system. We need them, people remind me, for the sake of socialization. We need them, I eventually recalled, in order to educate people. Quite funny - to spend almost my entire life on mathematics education without thinking about its relation to general education! But better late than never, and since in the near future mathematics will remain a compulsory subject in schools, and since we have a huge population of mathematics teachers and mathematics educators, why won't we use this constellation and try to improve education?

The title of my talk is a modification of a famous book by Neil Postman (1996), the title of which is: The End of Education. It is an ambiguous title since "end" can be understood as a goal, purpose, etc., and it can also be understood as termination. The bottom line of Postman's book is that if we do not set goals for education it will die. Some people claim it is already dead. For me, Postman's title is a warning on a wall (I am referring here to a biblical event described in the Book of Daniel). However, as in the biblical story, people do not take seriously such warnings. They prefer to go on with the party. It is more convenient to follow inertia, old habits and current routines than to initiate change. And what are the goals of education anyway? An academic attempt to answer this question will imply infinitely long discussions, will raise bitter controversies and will last for ever. In order to avoid this, I would like to suggest a non-academic and simple approach. The goal of education, according to this approach, is an educated person. Since this looks circular I will add: an educated person is a thoughtful person. "Thoughtful" in English, like the above mentioned "end," is ambiguous. It means contemplative as well as considerate. I suggest that to be considerate in behavioral terms means to follow the rule: *what you hate - do not do to other people.* This was the answer which was given in ancient Jewish text to somebody who asked one of the sages to summarize the entire Jewish moral theory in one sentence. A few years ago, I saw in London underground a poster saying: To be considerate means not to carry your backpack on your shoulders. This is a small example of the above rule and it is quite typical. Each day, in case we have interaction with other people, we face dozens of situations with similar characteristics. If we have decided to be considerate, we should be aware of the factors involved in the in the situation, foresee whether an action of ours can disturb another person and avoid such an action, in case we understand that it disturbs somebody. This requires analytical thinking and control. Can this be related to mathematics education? Can it be related to any other school discipline? Very often we speak in mathematics education about analytical thinking and control. We certainly do it in mathematics education conferences. Some of us may speak about it

even in our mathematics class, if we teach mathematics. It is considered as a meta cognitive activity or a reflective activity. In teacher training we encourage our student to include it in their future teaching. I assume that very few of us and very few mathematics teachers, when they speak about it in their mathematics classes, point at the fact that these are also relevant to our every day life. If the notions "moral behavior" or "moral values" were not so emotionally loaded, I would have said that the issues at stake at this point are moral behavior and moral values. Unfortunately, when people hear these notions they think immediately about issues like abortions or homosexual marriages and so on. In order to avoid these connotations, I will use the terms "educated behavior" and "educational values." Unfortunately, in most cases, these are not part of any discipline taught in schools. So, on what ground do we expect our graduates to behave in a civilized manner?

My claim is that educational values should be integrated in school disciplines by means of the principles and the nature of these disciplines. Usually, it does not happen. Moreover, even reflection on thinking in general does not occur. Just a small illustration: One of the courses I taught for many years was 'topics in mathematics education'. It was a graduate course the students of which were middle school and high school teachers. In the beginning of each year I asked them about the goals of mathematics education. Many of them claimed that one of the main goals of teaching mathematics is to develop the students' thinking. However, when I ask them to elaborate on it or to demonstrate it by means of some examples, the decisive majority practices its legal right to remain silent. I realized that *developing the students' thinking* has become a common rhetoric associated with mathematics teaching. But for the majority of mathematics teachers it is not clear at all what it really means and how their actions in class can help their students to develop their thinking. At this point I start a class discussion. I try to draw my students' attention to the fact that there are many kinds of thinking: religious thinking, scientific thinking, intuitive thinking, dogmatic thinking, mystical thinking, analytical thinking and more. - *Which kind of thinking do we want to develop in our students?* I ask. - *Mathematical thinking* (the answer does not hang around too long). With the previous frustration and being afraid of some more frustrating answers, I do not ask the student who gave the answer to elaborate on it this time. *Well,* I say, *isn't mathematical thinking a too narrow goal? How will it help our students in the rest of their lives outside the mathematics classes?* Silence again. *Well*, I say after a while, *it is unfair to expect you to read my thoughts. What I have in mind is* ***rational thinking***. At this point, twenty pairs of cold eyes are staring at me. - *All right,* I am trying a new direction. - *What is the teacher's main activity?* - *Teaching* (another answer which does not hang around too long). - *This is correct, but can you be more specific?* - *Making silence in the class.* -*Again,* I say, *it is unfair to expect you to read my thoughts. The main activity of the teacher is explaining.* The twenty pairs of cold eyes become hostile at this point. - *Consider a certain event,* I suggest, *like somebody becomes sick. Assume you want to explain it. Can you think about two explanations to it, a religious one and a scientific one and tell the difference between*

them? - Yes. The religious explanation will use arguments of punishment and reward, whereas the scientific explanation will use medical arguments. Something starts to move in the class. - *And which one do you prefer?* I ask and add immediately, *please, don't tell me, because if you prefer the religious explanation you have a problem as science teachers. - But we are not science teachers, we are mathematics teachers. Didn't you tell us that mathematics is not about the real world, it is about an abstract world which exists only in our mind? - I am glad you remember this,* I reply*, but didn't you tell me that one of the main reasons for teaching mathematics is that science desperately needs mathematics in order to develop its theories? Are you trying to tell me that you work so hard serving science without believing in it? - Is this discussion related to mathematical thinking?* (a skeptic reaction which comes directly from the seats of the opposition). - *Well,* I conclude with my own question, *isn't it related to rational thinking? And isn't rational thinking related to mathematical thinking?*

So far with this illustration. I would like to return now to my original line of thoughts. Mathematical thinking and scientific thinking are parts of rational thinking. However, rational thinking is broader. Rational thinking is the kind of thinking which is needed to maintain our society. By "our society" I mean the liberal democratic society with its values, its various institutions, its science, technology and medicine. Other kinds of societies may need, perhaps, other kinds of thinking.

It seems, at this point, that if I recommend rational thinking as the main goal of teaching mathematics I have to define it. Well, definitions in the mathematical sense of the word exist mainly in mathematics. (This is another lesson that we should teach our mathematics students; namely, when they are outside the domain of mathematics and they want to clarify some notions, then looking for definitions is not necessarily an appropriate activity.) This does not necessarily have to be a problem. There are many notions for which we do not have strict definitions and yet we use them in academic discussions. But rationality is enormously broad, ambiguous and vague notion which has become a research topic for several academic domains like philosophy, psychology, economics, game theory, etc., and it will be extremely hard to uniquely characterize such a notion without arousing objection from scholars of different disciplines or of different personalities. It is quite typical to the academic community that controversies about central concepts of scholarly research never end. On the other hand, it seems that for ordinary people, using ordinary language, the concept of rationality is quite clear. In everyday situations, people recommend to each other to behave rationally. When they do it, it is quite clear what they mean. It is true that very often we fail to behave rationally, but this happens not because we do not know what it means to be rational. It happens because we are driven by strong impulses that we fail to control. This reality does not imply that we should give up rationality as an educational goal. On the contrary, education is about overcoming and controlling negative tendencies. The claim that human beings are irrational is irrelevant to education exactly as the claim that human beings are evil (or in the

biblical formulation: *The desire of man's heart is evil from his youth* (Genesis 8, 21)). The end of education is to teach us how to control our evil desires. Moreover, even the claim that human beings are irrational is quite inaccurate. Some people use the collective work of Tversky and Kahneman to establish this claim. As a matter of fact, what Tversky and Kahneman show is that under certain circumstances, when people are given some intellectual tasks, they fail to reach the correct answer. It happens because they make all kinds of mistakes. It does not happen because they do not care about rationality. To be irrational, in my opinion, means that we know what should be a rational decision in a given situation and in spite of that, we decide not to follow it. Such examples are gambling, smoking, eating frequently at McDonald's and so on and so forth. Therefore, I would rather claim that what Tversky and Kahneman show us is that people even try to be rational but they fail because they do not have suitable tools to deal with the problems posed to them. An indication to this claim is the title of Kahneman's Nobel Prize lecture (2002): Maps of bounded rationality. (More about this issue can be found in Leron & Hazzan (2006) and in Stanovich (1999)). Therefore, if this is the case about people's attempt and failure to be rational, why won't we teach them?

When discussing rationality, some researchers prefer to speak about rational behavior rather than rational people. This is quite reasonable, because a person can behave rationally under certain circumstances and irrational under different circumstances. Robert Auman, in his Nobel Prize lecture (December 2005) suggested the following definition for rational behavior*: A person's behavior is rational if, when given his information, his behavior is in his best interest*. This definition was given within the framework of game theory and we see that rationality is related here only to means and not to goals. It can be applied to moral as well as amoral goals. A criminal planning a perfect robbery, having all the necessary information about the place he is going to rob, is demonstrating rational behavior. Even if he is caught later on, because he was not aware of some alarms in the place of the crime, his behavior can still be considered as rational because Auman's definition is relating to the information the person has. What I am saying here only demonstrates what I claimed earlier that the moment a definition is suggested it arouses infinite discussions which never reach a conclusion. In addition, Auman's approach is not suitable for us because we want to speak of rationality in the context of education. Therefore I suggest to use a non technical approach to the notion of rationality, namely, to use it as it is used in colloquial language. I am aware of the possibility that even here it might be hard to obtain a consensus. I have no illusions about that, and if somebody does not agree we can just agree to disagree.

One way to clarify the meaning of a notion in a colloquial language is to look it up in a dictionary. The Merriam-Webster dictionary suggests that to be rational is to be reasonable. Rationality is the quality or state of being rational. Rationality is applied to opinions, beliefs and practices. About being reasonable, the dictionary adds that 'reasonable' is not extreme or excessive and it is moderate and fair.

If we wish to do bring a broader support for the dictionary suggestion, we can collect various excerpts from newspapers, magazines, everyday discourse, and political statements and examine the meaning of rationality in these contexts. I have done this in my own language but this is not necessarily a support for the use of "rationality" in English. A third resource can be looking it up in an encyclopedia. Generally, encyclopedias do not deal with colloquial terms. And indeed, the Encyclopedia Britannica (the 1974 edition) does not have "rationality" as an entry. It discusses "rationalism" in its scholarly style. However, "rationalism" is not the technical equivalence for "rationality" in the colloquial language. On the other hand, Wikipedia, the free encyclopedia of the INTERNET has "rationality" as an entry. In addition to its explication for "rationality" it also elaborates about the use of the term rational. It suggests that *in a number of kinds of speech, "rational" may also denote a hodge-podge of generally positive attributes, including: reasonable, not foolish, sane and good* (Wikipedia, 2006). I hope that this reference can be used as a support to my claim about the meaning of rationality, at least, in colloquial English. I would like to add that the interpretation of rationally which I am suggesting includes also one characterization of rationalism that *regards reason as the chief source and test of knowledge* (Encyclopedia Britannica, 1974; rationalism). On the other hand, I am not considering "rationality" as a rival to empiricism (which is true about "rationalism"). On the contrary, I claim that **to be rational implies taking into account science, medicine and technology.** A person behaves rationally if he or she takes into account all the scientific information which is relevant to their decision making. Thus, rationality is a relative notion. A rational behavior in Newton's era is not necessarily rational in our era since science has been changed enormously.

Now I hope that I have clarified my interpretation to the notion of rationality and I want to assume that it is quite close to its interpretation among people who have at least a college degree in Western society (especially, our mathematics and science teachers). I expect them to distinguish between rational thinking and other kinds of thinking (dogmatic, mystical, divine, etc.). However, I take into consideration that the gap between my expectation and the practice might be huge and therefore I suggest that rationality in the above sense should be discussed in pre-service and in in-service teacher training. What I want to avoid is a university requirement of another dry academic highly sophisticated course in philosophy, which deals with rationality in a technical way. Such a course will, probably, discuss rationalism as a philosophical movement rather than rationality in the above sense.

I feel that after the above discussion about rationality I can return to mathematics teaching and suggest for it an educational goal. As I claimed earlier, education is primarily about values. I suggested that an educated person is a thoughtful person where the emphasis is on being considerate. My claim was that in order to be considerate we should use rational thinking. Please, note that my only concern here is about being considerate. This is not because I am unaware of other potential values. It is because I am aware of potential controversies over additional values that

I can suggest. Even about being considerate some people may claim that it is inappropriate as an educational goal. They believe that selfish behavior is more useful for their interests (and note that it is quite coherent with Auman's definition of rational behavior and with the view that the human being is *homo economicus* (economic man) and as such he or she is logically consistent but immoral). However, within the educational domain, I hope, nobody will dare to recommend selfish behavior as an educational goal. Now, values and rationality are not part of any mathematics curriculum. They are not part of any curriculum at all. My suggestion is that teachers will address them in their classes at suitable moments. This requires a different state of mind from the teachers. It requires that while dealing with a mathematical issue, the teacher is expected be aware of the possibility to add a reflective dimension to the lesson and integrate in it a discussion about rationality and its importance to values. It requires from the teacher some improvisation skills (of course, one can plan it or partly plan it, but improvisation is better). What I am suggesting here is clearly in contradiction with the current conception of mathematics lessons which are supposed to be strictly task oriented, structured and planned to the last detail.

For example: when dealing with fractions one can discuss sharing as a value. Sharing is one of the principle educational values mentioned by Postman quoting Robert Fulghum's book: *All I ever really needed to know I learned in Kindergarten*. Fulghum's list is the following: "**sharing everything**, playing fair, don't hit people, put things back where you found them, clean up your own mess, wash your hands before you eat, and flush" (Postman, 1996, p.46). Space restrictions, again, prevent me from presenting more examples in which particular mathematical topics can be used in order to discuss educational values.

So, my slogan can be: From covering the syllabus as a main goal to a value oriented meaningful learning. I am aware of the difficulties of such a move. The official educational systems will reject it strongly. They developed a system in which learning achievements have the first priority. The common belief is that they also developed tools to measure these learning achievements. Unfortunately, measuring has become the ultimate criterion for any educational project. *I measure therefore I exist* can replace Descartes' *cogito ergo sum.* As long as the authorities of the educational system will not change their *covering the syllabus* policy teachers won't change their teaching style. The question is whether in one school or several schools, which decide to have an autonomous approach to education, will the teachers be able to teach in the above recommended style? I believe that this is quite feasible. As I said earlier, some training is necessary, but teachers are sensitive human beings who care about educational values (if they are not, they chose a wrong profession). The majority prefers meaningful teaching to meaningless teaching. Many of them are quite frustrated with the current style of teaching and learning. However, they feel they can do nothing about it. But even in the current state of affairs, if they take only five minutes from each lesson to deal with values in the way recommended above,

this will be an important change. Thus, this is not a huge change in the system. It needs only some amount of approval and encouragement from the educational authorities.

As to teacher training, I recommend that it will include discussion of rationality and values in the above mention form. Since I am recommending spontaneous discussions on these issues as a tool in school teaching and not lectures, I also recommend that these issues will be dealt in teaching training by means of spontaneous discussions and not in lectures. It is quite important that a teaching style which we recommend in teacher training will be presented to future teachers in that recommended style. I remember a student in teacher training who, in a private conversation, expressed criticism of one of her teachers who recommended to his students to use many teaching styles in their classes while the only style he used himself was lecturing.

My recommendations to deal with values in mathematics classes can certainly be applied to other school disciplines. It can clearly be applied to all natural science disciplines where rational thinking is an essential issue. However, it can also be applied to social sciences and humanities. Rationality there is perhaps not a first priority. However, since they all deal with human beings they can easily be tied to human values. A mathematics education conference is perhaps not the place to advocate this claim, but I would like to tell you one short story from my own biography which is relevant to my recommendation.

When I was a high school, I had a history teacher who was considered by the class as a lazy person. When I analyzed his behavior years later I understood that the reason for this view was that very often he avoided covering the syllabus. We were not worried by it because history was not one of our matriculation disciplines. Very often this teacher discussed with us football games and westerns (which he called "historical films"). Football games and westerns were not my cup of tea, even as a teenager. However, I enjoyed these discussions because they saved me writing down academic comments in my history notebook. Now, in historical thinking there is an extremely important principle. It is the distinction between a fact and an opinion. A fact is a fact, I am trying to be short here, but an opinion can be true or false. This abstract principle requires some concrete examples. My teacher's generic example was the following (and I am making a necessary adaptation of it for my current audience): *That Arsenal lost the game against Barcelona is a fact. However, that their game was weak is an opinion.* Many years have passed since then and when I compare now my historical knowledge to the historical knowledge of my friends who studied history with teachers who covered the syllabus I see no difference. All of us know almost nothing. However, some of them still cannot tell the difference between facts and opinion. I consider it as a thought defect which can be labeled, in the terminology of Tversky and Kahneman as bounded rationality. Moreover, every week I face situations, in politics, economics, psychology and more, where the people involved do not distinguish between facts and their own opinions (in fact, sometimes,

the distinction is not so easy). I am grateful to my history teacher for the distinction he taught me in his special humorous way. It serves me in my everyday life not less than in my academic life. It helped me also resisting post-modernism in which the notion of fact vanished in order to promote the ultimate notion of narrative.

Another example is taken from my own teaching experience. In a research methodology course which I used to teach, I very often reminded my students that *a correlation is not necessarily a causal relation.* This is quite a subtle idea, since it does not exclude the possibility of a causal relation. It only calls us to examine very carefully the situation before deciding about a causal relation. It also tells us that in many cases we cannot be sure and that in many cases we even won't be able to know. This is clearly part of rational thinking. It is not part of mathematical thinking. It is undoubtedly part of scientific thinking (whether it is in natural sciences or in social sciences). I believe that this idea, if internalized, can lead to a moderate behavior and to more balanced analysis of various situations, and therefore to a more desirable behavior which is the end of education. Nevertheless, every week, as in the case of fact and opinion, I face situations in which people do not take into account this idea. They are politicians, economists, police investigators, lawyers and medical doctors. All of them can do excellent jobs in their field of expertise, but sometimes, when ignoring the above idea, they can make fatal mistakes when drawing conclusions.

At this point I would like to call my diverging recommendations and analyses to converge. My suggestion was to set rational thinking as a main goal of mathematics education by means of which it can be also tied to educational values. It is not an entirely new idea. Processes such as control, analysis, planning, meaning negotiation, reasoning and more have been recommended long ago by mathematics educators. They are all part of rational thinking. What is new is that all these should be presented to the students as part of rational thinking together with additional principles which I mentioned earlier. Teachers are recommended to discuss all these in their classes in appropriate contexts which can come out by reflection from mathematical situations. It should be emphasized that rational thinking is the best way of thinking for our liberal democratic society. We should remember that rational thinking has produced science, medicine and technology, and where would we be without them nowadays. Rational thinking is also the best way of thinking for solving conflicts, whether these are conflicts between individuals, groups or nations. Rational thinking is the best way of thinking for our students. It can lead them to worthy destinies and help them choose desirable goals for their lives. Although game theory researchers, as well as some other researchers, are reluctant to relate rationality to goals, we do not have to accept it in the domain of education. Here, I believe, we are allowed to speak also about rational goals. If somebody needs a philosophical support for this approach, we can send them to Kant and Aristotle.

Ingmar Bergman, in his television series *Scenes from a Marriage,* where the hero's behavior is extremely irrational, complains that schools taught him all kinds of things like the Pythagorean Theorem or the capital of Brazil, but they did not teach him how

to behave to other people. Thus he finds himself at a certain moment in a terrible conflict with his wife, which turns into a physical violence. If Ingmar Bergman were an educator he would recommend schools which teach students how to behave to each other. Since Bergman is an artist he avoids making a formal recommendation. However, many great artists have implicit educational recommendations in their art.

A school which teaches its students how to behave is a great vision. It is a huge project. Certainly, it is not a project for some old retired mathematics educators. However, it can be recommended as their will.

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AND FINALLY …

A festschrift normally consists of essays to honour people, written by others. However, as we have gleaned so much from their writing over the years, it is entirely appropriate that Eddie gets to reflect on his academic life and David, of course, gets to have the last word!

LESSONS FROM THE PAST AND CONCERNS FOR THE FUTURE

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This paper places the theory that I have shared in developing at Warwick into the broader spectrum of the search for an approach to the teaching and learning of mathematics that would support high levels of achievement. There is no apology that much of the discussion is contextualized within the UK, seen as a typical country that continually questions the general levels of mathematical achievement, implements a variety of initiatives to improve and raise standards but fails to recognize within any of these initiatives that learners do things differently. Without implementing some understanding of the way in which mathematics is learned even the most recent initiatives will need reconsidering.

INTRODUCTION

As I began to think about this paper I became aware of two UK reports that made me wonder about our ability as mathematics educators to communicate and influence the way that mathematics is taught and thought about.

The first, a publication from the Schools Inspectorate for England and Wales reporting upon the mathematical achievement of 14 to 19 year-olds (OFSTED, 2006), illustrated the distinction between the emphases and outcomes from two forms of teaching. The "best" teaching focused on developing understanding in mathematical concepts to develop critical thinking and reasoning whilst that which focused on teaching a set of arbitrary rules did not motivate students and limited their achievement — "teaching to the test" might ensure that students pass examinations but it would not ensure mathematical flexibility.

The second report was a discussion paper published by the Advisory Committee for Mathematics Education (ACME, 2006). It suggested that top-down pressure on pupil test results imposed by Government, Local Authorities, governing bodies and headteachers has caused primary school teachers to passively and uncritically implement curriculum and pedagogic styles recommended within documents published through the Primary National Strategy, (see for example DfEE, 1999) without due consideration to the needs of their pupils.

The first of these items received considerable attention within the daily press and the BBC. It made me wonder why something that, because of our research interest is so obvious, should become an news item expressed in such a way that it appeared to be suddenly discovered. It is almost thirty years since Skemp (1976), in slightly different terms, pre-empted the conclusions of the OfSTED evaluation, whilst almost 70 years earlier the Mathematical Association within the UK had been expressing concern that the educational value of arithmetic and algebra could be seriously impaired through a

tendency to sacrifice clear understanding to “mere mechanical skills” (Mathematical Association, 1905, p.7). One hundred years separates expression of concerns on the same theme — teaching the rules of mathematics does not lead to true understanding of mathematics.

GOING FULL CIRCLE: SOME CONSIDERATIONS ON THE CURRICULUM

Hardly a decade seems to pass before concern is being expressed about the generally poor level of children's attainment, the nature of the mathematics curriculum or the quality of children's learning. In response, a variety of interest groups, mathematicians, general educators, politicians and mathematics educators have recommended change to either the teaching methods, to the curriculum or to the ways in which we assess children’s achievement. Frequently there has been strong contradiction, even discord, in the way each group has perceived the mathematics curriculum and the emphasis that should be placed upon its content. Should the emphasis be on the acquisition of skill or the development of understanding that will provide conceptual flexibility? Should teaching focus on classes or individuals. The answers to such questions can only be meaningful if we have reflected upon the ways that children learn mathematics and the influences that may support this learning.

One hundred years ago, the mathematics to be taught within the UK was codified within ‘Codes of Regulations’ (a sort of National Curriculum) that specified the criteria children had to meet to fulfill the knowledge requirements of annual inspections. A successful outcome implied that the school had met one of the necessary conditions under which it could be funded. The mathematics specified within the codes was seen as a mathematics for the majority and it was essentially utilitarian — a thoroughly good arithmetic that was taught to satisfy the demands of parents “whatever else had to give way to it” (Ministry of Education, 1958, 4).

It was this utilitarian aspect of mathematics that came to dominate elementary schools, those schools that taught children aged 4 to 14 and its justification was confirmed by the Mathematical Association (1932) because it was:

> only through examples in arithmetic that all children will learn anything of correct mathematical reasoning… and it is perhaps the only subject where mastery is attained.
> (Maths Assoc, 1932, 5).

However, even this report did not appear to have taken notice of the caution expressed 10 years earlier

> Arithmetic differs from most subjects in that the hardest step has to be taken right at the beginning. At the very start the child has to leave the world of concrete fascinating realities and concentrate on abstraction, on the creation of human intellect.
> (Drummond, 1922. p10)

(It is interesting to note one of the Discussion Groups within PME this year is on abstraction and mathematical learning.)

When I started teaching 45 years ago in a Secondary Modern School —the type of school to which the greater proportion of children aged 11-15, soon to rise to 16, attended if they had not succeeded in the selection examinations at 11 — the words of the 1932 report formed the basis for the mathematics teaching philosophy. It was assumed the children were being taught to think by doing arithmetic, whilst evidence of their mastery or otherwise could be easily identified by the frequency of ticks or crosses. The children left school with no formal recognition of their achievement and were not expected to take part in any mechanism that would give them any. There was no algebra and no geometry—these areas of mathematics were to be within the realm of those who had been successful in the selection process.

However, unknown to me at the time, and apparently to the school I taught in, things were about to change. Recognition of changing world conditions and an increasing appreciation that mathematics was a language suggested that adaptability and understanding were the qualities that required emphasis more that ever before. It was also recognised too few children were attaining reasonable and sufficient skills in the computations that they were being taught. Centralisation of the mathematics curriculum and a focus upon teacher as teacher which had variously dominated the previous 50 years was receding. The curriculum was undergoing rationalization in content, delivery and in its perception of the child as learner.

After making a move to primary schools the mathematics that I taught in the late 1960's and the 1970's was characteristed by the belief that mathematics should be broadened beyond arithmetic. It should become embedded in the relationships underpinning the processes that the children used in recognition that mathematics was the universal language of communication. Perhaps the most important aspect of these beliefs was nothing more than an effort to spread an acute awareness of the role of symbolism in mathematics, not only in its conceptual form but also in its operational consequences. It was an exciting time but whilst the resulting decentralization evoked widespread professional interest in the teaching and learning of mathematics it also brought about a high degree of inconsistency in the degree of mathematical provision. But I was lucky, the children I taught were bright, motivated and a delight but the full effect of the inconsistency and my growing insight into an awareness of the role of symbolism had to wait until I had been at Warwick for a few years.

My almost 25 years at Warwick has been charactarised by two developments. Firstly my apprenticeship and eventual growth as a mathematics educator and secondly a increasing concern with children's achievement that appeared to be derived from the inconsitency lead eventually to recentralisation of mathematics curricula and a growing emphasis on examinations and testing. Within the UK this centralization was invoked through the introduction the National Curriculum in 1989, and the introduction of Standard Attainment Tasks in 1991. However, even by 1998, only 59% of 11-year-olds achieved level 4 (the expected standard for their age) in mathematics. The consequence was that The National Numeracy Strategy (NNS)

(DfEE, 1999a), developed from a Numeracy Project that appeared to be successful mainly because it addressed and appeared to reduce underachievement in mathematics amongst all groups of pupils, was introduced within English schools from September 1999 and incorporated into the National Curriculum in 2000.

It is immediately apparent when the totality of the core document, *Framework for Teaching Mathematics from Reception to Year 6* (1999) is considered that throughout it is a strong bias towards skills, particularly those that are associated with arithmetic. The perception presented in the document is that mathematics consists of a set of facts and procedures, emphases that seem to reflect the view of the then Secretary of State for Education:

> Numeracy is a vital skill which every youngster must learn properly. Yet for thirty years we have not focused on what we know works. The new daily maths lesson will ensure that children know their tables, can do basic sums in their heads and are taught effectively in whole class settings.
>
> (Blunkett, Press Release 10/99)

It was almost as if the experiences of the previous hundred years had not existed. We returned to the centralization that promoted the failure of so many with the consequences identified within the reports from OFSTED and ACME. Despite almost a century of initiatives from which the focus of attention had changed from the acquisition of a body of mathematical knowledge and the tempering and broadening of that knowledge to take into account the development of individual children, the political reaction was to centralise, examine and evaluate. Teachers were once again being told what to teach, when to teach and how to teach and both they and their children were to be evaluated on the outcome.

More and more I am becoming convinced that until such time as we can begin to consolidate our sense of the way in which children learn mathematics, the factors that may contribute to this learning and then effectively communicate this to curriculum designers and to teachers the recurring cycle of initiatives designed to address underachievement will continue.

INSIGHT INTO THE QUALITY OF LEARNING

The observation that some individuals are more successful than others in mathematics has been evident for generations and could perhaps be explained through Piaget's novel method of interpreting empirical evidence to hypothesise that all individuals pass through the same cognitive stages but at different paces. I am become more and more doubtful that this is the case. The philosophy behind the National Numeracy Strategy suggest that as long as children are taught the subject matter in the correct order and as long as teachers teach properly then all will not only be well but that learning targets for each year of a child's time within school can be identified. Not only the order but the pace is given and, as in the earlier Codes,

children are then examined against specified outcomes. No wonder the two OfSTED and ACME reports demonstrated concern. We had been here before.

I like to think that our work at Warwick refutes the philosophy behind the current curriculum initiative and I believe it has made some contribution to our insight into the ways children learn mathematics.

After arriving at Warwick I was given time. Fresh from being head teacher of a large school, my supposed expertise as administrator and teacher were the things that were used and appeared important to the department. I had no solid research agenda and neither did I possess much more than superficial research training but I used my time to visit schools, observe teaching and talk to children about they ways they thought about mathematics. There was no explicit research philosophy — it was simply an open-ended experience that involved talking to children about their mathematics and how they solved simple arithmetical problems. But how the children talked — it was eye opening. At one extreme the children were very precise, offering clear explanations of their attempts to solve questions and address issues raised in conversation. At the other, they demonstrated some considerable mathematical confusion - but I was taught how to count on my fingers, how to count if I didn't use fingers and how to count in my brain. On one occasion I listened with a sense of sadness as Jonathan (aged 9), attempted with extreme difficulty to use mental images of fingers to add 4 + 3 explaining:

> I like trying to do things in my mind. I like them to be harder because when I grow up I will be able to do harder things. (Jonathan, aged 9)

Jonathan thought that even the best children in his class counted fingers in their heads. They were just quicker than him.

I watched with amusement and incredulity as Tara (9) suddenly took her shoes off to use her toes to support counting back to 2 from 11 and I listened with awe as Maria (4) gave me the answer to a wide range of number combinations to 20 and all of the table facts to 10 x 10. I also watched the delight when Simon, also four, who when asked to use a calculator to give me two numbers that make eight responded:

> Look at that. See. One million take away nine hundred and ninety nine thousand nine hundred and ninety two — that makes eight.

I listened incredulously as children talked about moving the calculators in their head, combining a red 4 with a yellow 5 to get a blue 9, effectively solved three digit subtraction problems by taking largest from smallest (and mentally recorded the difference so that an adjustment could be made later). And I watched as teachers cast aside such novel approaches to teach standard algorithms which, temporarily at least, caused considerable confusion.

The outcome of these almost random observations began to lead to some tentative assumptions:

- Children can develop mathematically at different rates but it was not to be assumed that given enough time all children will reach the same level of achievement.
- Children make qualitatively different interpretations of mathematical activity and thus give different meanings to subsequent activity.
- There are several key points in the construction of mathematical knowledge where the consequences of a divergence in thinking make the next stage of development easy for some but very difficult if not impossible for others.

The key issue that guided later ways of working was to consider these assumptions in the context of "What is it that children are doing differently and why". In establishing some insight into making a response to these questions the seminal work of Richard Skemp and developments with, and by, David Tall together with the studies of an increasingly large group of PhD students contributed towards an increasingly wide range of evidence that provided insights into the mathematical development of learners across the spectrum of learning, from elementary school to university.

Many of the key outcomes from this work have been reported in a variety of individual papers whilst summaries are presented within Tall et al (2001) and Gray & Tall (submitted) but here I would like to highlight several features relevant to the issues associated with the development of children's thinking within the primary school, that is schools for children with median ages from 4.5 to 10.5.

The divergence reported by Gray (1991) was later conceptualized within the notion of procept (Gray & Tall, 1994). Conceptualising this construct has allowed us to articulate the essence of mathematical symbolism in the way that was attempted by the curriculum reforms of the 1960's and the 1970's and it provided an insight into explaining a fundamental paradox: mathematical symbolism gives the power to think mathematically but can be the source of considerable complexity for those who are trying to learn it.

The different interpretions of the conceptualised notion of procept enabled us to see the operational consequences of the different forms of mathematical thinking that lead to a proceptual divide. An interpretation of the evidence drawn from Demetra Pitta's PhD (Pitta, 1998) would seem to suggest that particular frames of reference may be associated with learners moving along each arm of this divide (Gray & Pitta. 2006). The disposition of low achievers to articulate the descriptive properties of items, without any reference to their more relational characteristics, would seem to be related to their tendency to focus upon procedures that are fundamentally episodic. More recently, the contribution that styles of teaching may contribute to procedural conceptions of mathematics have been indicated by Md Ali (2006) and to children's descriptive and episodic interpretations of representations by Doritou (2006).

Gray & Tall (submitted) suggest the evidence coming from the full range of studies carried out at Warwick fit a theoretical framework that sees abstraction as a natural process of mental compression. As Bills (2002) indicated, not all pupils abstract the

same things from their experiences and neither do all children abstract the same experience from common activities — if existing mental constructions are different then new activities will be construed in different ways.

The indications are that moves towards increasingly sophisticated concepts often fail if pre-requisite concepts, ('thinkable concepts') are not available to make sense of the new materials. Howat's (2006) study of the outcomes of a remediation programme, developed to bridge the gap between low achievers' current knowledge and the knowledge required to participate more fully within designated whole class teaching phases, amply illustrates this. When working individually with the children it became clear that some of the children's cognitive difficulties were so elementary that these children would require much more than a simple remediation programme.

CONCLUSION

One of the features that seems to emerge from an examination of the variety of mathematics curricula proposed for schools over the past 100 years is that they appear to be conceived of by people who are successful. Those who are mathematicians think in sophisticated ways because of their ability to compress knowledge. Their mathematics is seen from a mature viewpoint in which the structures have great richness and interiority and they therefore have a perception of simplicity in which this structural richness plays an implicit fundamental role.

Learners do not yet have this conceptual richness, but the belief is that as long as teachers explain to them how to manipulate sophisticated thoughts they are giving them power and strength.

In seeking to design a contemporary curriculum for the UK the over-riding emphasis has been placed upon the achievement of results that seem to illustrate improvement in mathematical skill. This impetus would appear to be derived from the politicians' need to see success in a way that they believed was apparent in past generations. Perhaps they also have misguided notions of equality since they invoke attempts to teach a whole population increasingly sophisticated mathematics. However, the response from curriculum designers not only meets this desire by identifying 'What' should be taught but also indicates 'When' and 'How'. But as we have seen, short term need in the form of procedural growth, tends to take precedence over long term development.

Within the current mathematical climate there is a tendency to provide practice to develop and confirm "understanding" but we need to provide practice which disconfirms and requires the search for alternative approaches. The more we work at remembering procedures the more we are likely to use them. If all of our effort goes into this solution to our problems it is perhaps the case that the more we will remember procedures but, paradoxically, we may possess less understanding. Failure becomes a more distinct possibility than long term success.

We as members of a mathematics education community are all interested in the development of mathematical knowledge. We recognise some of the strengths and weaknesses of curriculum implementation within our own countries. We also have insight into the difficulties and misconceptions that may be associated with particular areas of mathematics. We support our own subject knowledge with an ever evolving pedagogic content knowledge but how good are we at actually communicating, not simply to ourselves but to those who "set standards" and those who attempt to help children reach them. Without such communication and some acceptance of our message I would suggest that what it will not be long before we undergo another cycle of change and then yet another until it is recognized that what is being attempted is undoable until such time as the nature of the sophistication required to compress complicated ideas is grasped. I believe that our studies at Warwick have made a contribution towards explaining why this may be the case but how do we make it part of mainstream thinking.

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A LIFE-TIME'S JOURNEY FROM DEFINITION AND DEDUCTION TO AMBIGUITY AND INSIGHT

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In this paper I speak of a personal journey in mathematics and mathematical thinking that began in a mathematical world of precision and certainty and found a world of mathematical thinking full of ambiguity and insight. It is a journey on which I have had many fellow travellers from whom I have learnt most of what I know, particularly my colleague Eddie Gray and our research students, and other very special co-authors. They have accompanied me on various quests in a life-time's journey by way of 'concept image', 'measuring number', 'local straightness', 'generic organiser', 'cognitive root', 'procept', 'cognitive unit', 'advanced mathematical thinking', to 'three worlds of mathematics', where the route to the world of formal mathematics is by way of two other mental worlds— conceptual embodiment (thought experiments based on perceptions) and proceptual symbolism (actions, such as counting, symbolised as concepts, such as number). On the journey I also discovered new insights, at least for myself: that the proliferation of mathematics education research is constructing a growing environment for individuals to build theories and careers, leading to a welcome level of productivity, but that this productivity has led to a wealth of complication that needs compression into an insightful simplicity. We know more than we knew thirty years ago, but our knowledge has not produced universal success in teaching mathematics. Looking back, I see my journey as a quest to seek the underlying simplicity that enables us to think in a powerful mathematical way in our increasingly complicated lives.

STARTING FROM WHERE WE ARE

We all begin our journey from where we are at the start. My own journey in mathematics education began from my position as a mathematics lecturer in a university mathematics department which coloured the ways in which I viewed mathematics and mathematical thinking. I sought clear definitions, clear deductions and the construction of a coherent theory of mathematics education.

In the early 1970s, the main cognitive frameworks available were stimulus-response behaviourism (hardly appropriate for the subtleties of mathematical thinking) and Piaget's epistemological approach to child development. The simple but profound book on *Psychology of Learning Mathematics* by Richard Skemp (1971) came as a breath of fresh air. I was so enamoured of his work that I invited him to speak to the Mathematics Department at Warwick and the mathematicians were so impressed that, when the Education Professor resigned that year, Richard was invited to apply for the post and became Professor of Education.

When Richard was asked to review Freudenthal's book *Mathematics as an Educational Task*, having already bought his own copy and not wanting another, he passed the invitation to me. To review a work of the great Freudenthal was a huge task for a young mathematics lecturer and I sought advice from a senior colleague, James Eels, who knew him well. He confirmed that I should say *exactly* what I felt and, emboldened by his advice, I wrote a welcoming but critical essay. I received a post-card from Freudenthal after the review appeared: 'thank you for the review which I enjoyed, *especially* the critical parts.'

My position in the Mathematics Department was as 'a lecturer in mathematics with special interests in education'. Rolph Schwarzenberger, a professor and a friend who gave me guidance, encouraged me to begin research into the learning of undergraduates and I began a study on students' understanding of limit processes (Tall, 1976) where I found that most students believed that '$0.\dot{9}$ is just less than one'. I had data, but where was the theory?

I still saw mathematical thinking from the viewpoint of a mathematician, becoming enamoured of catastrophe theory through the research of my departmental chair Christopher Zeeman (1977). The paper I presented at the very first meeting of PME (Tall, 1977) used catastrophe theory to describe how the brain could leap suddenly from one viewpoint that becomes untenable to another that is more stable.

This paper interested Shlomo Vinner and I travelled to Israel in 1979 to work with him. There I met Efraim Fischbein, the first President of PME, who was at that time working on the concept of infinity. As a mathematician, I suggested to him that the conflicts in the data he had collected related in part to two different ways of looking at infinity: cardinal infinity arising as an infinite extension of counting and the infinities and infinitesimals of the calculus of Leibniz that I suggested were infinite extensions of 'measuring' (Tall, 1980a). He told me he wanted to 'see' an infinitesimal and I suggested that rational functions could be ordered by looking at their graphs and defining $f(x) > g(x)$ if the graph of *f* is above the graph of *g* for sufficiently large values of *x*. Functions such as $y = x$ or $y = x^2$ are, in this sense, greater than zero but ultimately below any positive constant graph so they are 'infinitesimals'. He protested saying that he wanted to 'see' an infinitesimal as a tiny value and these rational functions did not look 'small' to him. He remained unmoved when I suggested we shift our attention to a vertical line far off to the right to see that constant functions $y = c$ met the line in fixed points, but variable functions met the line in variable points where those that were positive but ultimately lower than any given positive constant were infinitesimal. I wrote about this in a paper (Tall 1980b) and was met with the wrath of Israel when Tommy Dreyfus alerted me that Fischbein was concerned that he had more strength in psychology than in mathematics and that, in referring to his rejection of mathematical constructions, I had touched a sensitive spot. I wrote an immediate letter of apology to Efraim and he and his student Dina

Tirosh became long-term friends and continued to keep in contact over the years. When he passed away, I was invited to give his eulogy at PME.

Back in Warwick, at the age of around forty, I started a second PhD in Mathematics Education with Richard Skemp who was about to become the second President of PME. He was such an inspiration with his clear and simple approach to theory, and gave me great insight into the psychological side of mathematical thinking. I was invited to present the eulogy in his memory at PME too.

When Shlomo Vinner visited me at Warwick at Easter in 1980, I had a great deal of data available and no way of making sense of it because the students' answers weren't mathematically coherent. He showed me his paper with Rina Hershkowitz (1980) prepared for PME later that year in which students' interpretations of geometric concepts were analysed in terms of the new notion of 'concept image' and 'concept definition'. It was exactly what I needed to make sense of my data. I began writing with great speed and energy and wrote the first draft of a paper in a day, finishing it off in subsequent discussion with Shlomo (Tall & Vinner, 1981).

This became my second source of embarrassment with my Israeli friends. Following the custom in my experience as a mathematician, authors' names were written in alphabetical order and so it happened that 'Tall and Vinner' became quoted as the origin of 'concept image' and 'concept definition', when it really was the invention of Vinner, shared earlier with Rina Hershkowitz. In my defence, I recalled Richard Skemp's immortalization of 'instrumental and relational understanding' using terms formulated by his friend Stieg Mellin-Olsen in a social setting, which Richard converted to a psychological setting. His paper became a classic. In the case of 'concept image' and 'concept definition', I took Shlomo's definition of the terms in two separate compartments in the philosophical mind and turned them into mental conceptions in the biological brain. This paper became a classic too. Publication is all. The credit goes to those who publish first, even if they acknowledge earlier sources.

As I worked on my doctoral thesis in the early eighties, computers arrived and I began to bring together my mathematical knowledge with my personal version of cognitive psychology. I knew that students had a serious difficulty with limits, which they saw as potential processes that were never finished rather than fixed concepts. I had used infinitesimal concepts in teaching a course on 'Development of Mathematical Concepts' long ago in the early seventies and a young David Pimm—then a mathematics undergraduate—had persuaded me to teach a mathematics course on non-standard analysis. I therefore knew the mathematical theorem that when the graph of a differentiable function is magnified by an infinite quantity, with infinitesimals too small and infinite elements too far away to see, the resulting image is precisely an infinite straight real line. (The technical details are in Tall, 1980c.) So I was on mathematically solid ground when I introduced students to 'locally straight' functions that looked straight under appropriately high magnification. This led to my

programming the software for *Graphic Calculus*, which was the first approach to the calculus using a balanced combination of mathematical correctness and visual conceptual meaning.

As I studied for a PhD in education under Richard Skemp, my earlier PhD in mathematics entitled me to have my own PhD student, John Monaghan. He radically changed my earlier analysis of infinite concepts by showing me that his students thought that $\sqrt{2}$ was an infinite number too. It is infinite *in extent* because its decimal goes on and on forever, 1.414… . He found that students conceived of 'proper' numbers that could be calculated precisely, such as integers, fractions and finite decimals, and 'improper' numbers that could not. (Monaghan, PhD[1]).

The eighties became a period of great excitement as I travelled to show *Graphic Calculus*, and developed more theory relating to the visualisation of concepts. I theorised that a picture can be *specific*, such as the graph of $\sin x$ and its slope function, looking like $\cos x$. It can also be imagined as being *typical* of more general locally straight functions whose slope function could be seen by looking along the changing slope of the graph. So I formulated the notion of 'generic organiser' as an environment that enables the learner to manipulate examples and (if possible) non-examples of a specific mathematical concept or a related system of concepts (Tall 1989). At the time I considered this as a complementary construction to the notion of advance organizer (Ausubel *et al,* 1978), a higher-level structure used to organize future learning from above, whereas a generic organiser builds up generalisations from below. I also proposed the notion of 'cognitive root', which is 'an anchoring concept which the learner finds easy to comprehend, yet forms a basis on which a theory may be built' (Tall, 1989). A good example of a cognitive root is the concept of 'local straightness'.

At this time in the mid-eighties I was becoming considered as an 'expert' in visualization, though I knew little about the wider cognitive aspects of the topic. I worked with the idea of linking visualization and symbolism with my second PhD student Michael Thomas who has continued to work with me ever since. He built up the idea of 'versatile thinking' in which visualization and symbolism were used in complementary ways (Thomas PhD, Tall & Thomas, 1991) and together we began to build a 'principle of selective construction' where the computer could be used to carry part of the burden of internal computation while the learner could focus on the higher level relationships. In doing so, we began to be interested in the relationship between process and concept.

At this point there was a veritable explosion of related ideas. Ed Dubinsky (1991) was developing his APOS theory whereby ACTIONS were routinized into PROCESSES, encapsulated into OBJECTS to become part of a larger SCHEMA of thought, and Anna Sfard was developing a theory with complementary aspects of STRUCTURAL and

[1] In this paper, all PhD theses are given in a supplementary list at the end in chronological order.

OPERATIONAL mathematics (Sfard 1991). I spent time with both, but we never came close enough to write joint papers in process-object theory, although my thinking was greatly influenced by both of them. I misunderstood Anna's notion of 'structural' because she related it to the way Ian Stewart and I had written about complex numbers as ordered pairs of real numbers and I naturally related the term to the structural approach of Bourbaki with axiomatic definitions and formal deduction.

It was some time before the light dawned. Eddie Gray and I had been colleagues for many years at Warwick when he began his own research in the late eighties. His first major work (Gray, 1991) studied the responses of young children to simple arithmetic problems and he agreed to submit his current research for a PhD. One late Thursday afternoon in December 1990 as we looked at his data on the relationship between counting processes and number concepts, it became clear that we did not have a word to cover *both* possibilities. Dubinsky (1991) had his notion of process that could be encapsulated into object and de-encapsulated into process, Sfard had her notion of process that could be reified into object, but she referred to process and object as 'two sides of the same coin' (Sfard 1991), and asked, 'How can anything be a process and an object at the same time?' Suddenly I suggested we call this 'thing' that can be both process *and* concept, a *procept*. It was a moment of supreme revelation, comparable with the time that Shlomo showed me his paper on concept image. All the disparate pieces fell into place. Some children were thinking flexibly, shifting seamlessly from process to concept, others were fixed in the procedures of counting. Immediately we saw applications in algebra (expressions as process or concept) and analysis (limits as process or concept). This was huge!

At that very moment, Rolph Schwarzenberger, currently the head of our department, walked into my office. Before he could speak, I said, 'On your knees, Schwarzenberger! You were not there when Leibniz first said "functio", or when Cantor first said, "set". But you *were* there when Tall said "procept"!' He smiled indulgently and said, 'I am not sure of the etymology of the word,' meaning that the prefix 'pro' was a Greek term with a different usage. I looked back and said, 'If you have a delicate new plant, the last thing you do is to prune it, instead you nurture it and pour manure all over it.' (Except that I did not use the word 'manure'.)

That evening we were delighted with our work and considered all the possibilities that lay before us. But then, as I sat in my car in the car park in the gathering gloom, I suddenly lost heart and said to Eddie: 'I'm not so sure; all we have is a word.' He leaned in through the car window and said, 'We have more than that, we have *duality*, *ambiguity* and *flexibility*!' This most profound insight became the title of our first paper together: 'Duality, Ambiguity and Flexibility in Thinking' (Gray & Tall, 1991). It was sometime later that Eddie's wife Mareea christened it the DAFT paper.

Eddie used the theory to suggest the idea of the 'proceptual divide' in the spectrum of performance between those children who cling to remembered procedures and those who become increasingly flexible through the use of proceptual thinking. The notion of procept was the product of an equal partnership in which we shared different

experiences to produce something new that was genuinely greater than the sum of its parts. It began a productive relationship over a decade and a half, in which we worked with our doctoral students to develop and enrich the theoretical framework.

Meanwhile, a development that had begun back at the beginning of the 1980s and had run in parallel for years began to bear fruit. The papers presented in the first years of PME were mainly about mathematics education at elementary level. I presented a paper in 1981 which sounded the clarion call to build a theory of mathematics education from early beginnings right through to university undergraduate and research level (Tall, 1981). Gontran Ervynck was also working on this idea and coined the phrase 'advanced mathematical thinking', proposing a working group of that title and inviting me to share in its organisation. I had no talent for administration and he did it all.

By 1979 we had agreed to write a book that I edited and it appeared as *Advanced Mathematical Thinking* in 1991. As I wrote the epilogue to that book, I saw the contrast between two distinct strands of thought leading to formal mathematics.

> There are therefore (at least) two different kinds of mathematics. One builds from gestalts, through identification of properties and their coherence, on to definition and deduction at advanced levels of mathematical thinking. The other continually encapsulates processes as concepts, to build up arithmetic, then generalizes these ideas in algebra before formalizing them as definitions and deductive theorems in the advanced mathematics of abstract algebra. David Tall, 1991, p.254

At the same time, PhD student Md Nor Bakar had introduced me to the theory of prototypes (Smith, 1988), which resonated with my thoughts on concept images and generic examples and led to our realisation that students' conceptions of functions were very much influenced—often unconsciously—by their previous experiences (Bakar, PhD; Bakar & Tall, 1991).

Meanwhile, Norman Blackett studied trigonometric ratios as process and concept and researched how students might gain more conceptual insight using interactive visual software to explore the properties of trigonometric ratios in right-angled triangles. As a by-product it gave a significant gender difference, with the girls cooperating while the boys took the activity less seriously so that a previous advantage enjoyed by the boys was turned around (Blackett, PhD; Blackett & Tall, 1991).

In 1993 I was struck down by an illness (sarcoidosis) which left me exhausted and I had fourteen months off work. I continued to see my research students during this time, but everything else stopped. On attempting to return to work, it was clear that I could cope for a day, perhaps two, but then the duties began to fall behind schedule and within a week it had all ground to a halt. So I took early retirement in 1994, while continuing to work at the university on a one-third timetable. The remainder of my working life was the best time of all. I had a stream of willing PhD students, supported by Eddie as second supervisor and we developed a range of new ideas as I shifted from writing a majority of solo papers to joint papers with others.

My three American PhD Students each gave me insight into the nature of American College Mathematics. Phil Demarois studied the relationship between different representations (facets) and different levels (layers) of encapsulation in the function concept (Demarois PhD, Demarois & Tall, 1996). Mercedes McGowen studied the cognitive collages of knowledge construction produced by college students to show how the students concept maps changed with time, as the more successful built new knowledge on old while the less successful had more fragile structure which failed to build on previous knowledge (McGowen PhD, McGowen & Tall 1999). Lillie Crowley revealed how students could be awarded the same grade at one stage, yet diverge in the next stage with the more successful producing coherent knowledge structures while the less successful remained with limited procedures that trapped them in less appropriate ways of working (Crowley PhD, Crowley and Tall, 1999).

Malaysian PhD student Yudariah Yusof took me along a different track by investigating student attitudes to problem-solving (Yusof, PhD; Yusof & Tall, 1995). Amazingly, she found that the qualities that mathematicians desired of their students were encouraged in problem-solving, but suppressed in regular mathematics lectures.

Maselan bin Ali studied how Malaysian students coped with the symbolic routines of differentiation, showing that the more successful tended to have two or more procedures available for a given problem (classified as multi-procedure or process) while the less successful were more likely to have at most one (classified as procedural) (Ali, PhD; Ali & Tall, 1996).

Robin Foster studied young children's solutions of equations of the form $3+4=\square$, $3+\square=7$ and $\square+4=7$. We hypothesised together that the first could be solved by any method of addition, the second solved by 'counting-on' from 3, but the third would be more difficult for children at the 'count-all' stage because it would require them to guess a starting point from which to count-on, to see if the desired objective was reached (Foster, 1994). He also considered the way in which children used Dienes' Multibase Blocks to solve subtraction problems and found that while the most successful would use the materials to illustrate what they knew, a second group might use the materials to see the underlying mathematics, a third group might function satisfactorily with the materials, but not without, and a fourth group might fail entirely. (Foster, PhD). This has widespread importance in using statistics in any controlled experiment where one group are given a special treatment: the statistics only measure the change in *one* sub-group—the second sub-group who have a genuine improvement in long-term learning—while the first, third and fourth have no permanent long-term effect at all.

At University Level, Marcia Pinto from Brazil considered how students at Warwick coped with a first course of university analysis, distinguishing between those who rely on their concept imagery to give meaning to the definition (natural learners) and those who give meaning to the definition through studying the formal definitions and related proofs to derive properties (formal learners) (Pinto, PhD; Pinto & Tall, 1999).

Erh-Tsung Chin (Abe) from Taiwan studied how students made sense of equivalence relations, revealing that underlying embodiments interfered with their understanding of the three properties of an equivalence relation (Chin, PhD; Chin & Tall, 2002).

Soo Duck Chae from Korea studied the use of computer software in investigating the meaning of the bifurcation of solutions of iteration of the equation $f(x) = \lambda x(1-x)$ as λ increases. Initially the iteration successively replacing x by $f(x)$ tends to a limit for $0 < \lambda \leq 3$ then bifurcates to an orbit of period 2 at a value of $\lambda = \lambda_1$, and to orbits of period 4, 8, 16, … at λ_2, λ_3, λ_4, … to give a sequence that converges to the Feigenbaum constant λ_∞. According to Dubinsky's original APOS theory, it is possible to distinguish three stages of encapsulation: from the process of performing the iteration encapsulated as a final orbit, from the varying orbit to the sequence of bifurcation points, and from the sequence of bifurcation points to the Feigenbaum limit. APOS theory as originally formulated implies that encapsulation needs to be performed at each stage so that the objects formed at that stage could be used at the next. Yet, Soo Duck found that for $\lambda \leq 3$ many students were still at the process stage getting closer and closer to the limit, while for $\lambda > 3$, they switched focus to the visual picture of the orbit and used this as an object which bifurcated to reveal the sequence (λ_n) whose approximate numerical values looked as if they would converge geometrically. This confirmed to me that a symbolic APOS theory required complementing with embodied visualisation and human action to explain how mathematical learning occurred. On the other hand, the embodied theory of Lakoff (Lakoff & Nunez, 2000) apart from 'the basic metaphor of infinity' which I had discussed in terms of Leibniz's 'Principle of Continuity' long before in the early eighties) did not focus on process-object encapsulation at all. In 2001, Eddie and I expanded on the relationship between embodied objects and symbolic procepts (Gray and Tall, 2001).

The major step in my journey to link embodiment and the symbolic compression from process to concept occurred when Anna Poynter (previously Anna Watson) revealed the insight of a student Joshua. In talking about the sum of two vectors, he explained that the sum was a single vector 'had the same effect' as the combination of two individual vectors. This key unlocked the door of the relationship between embodiment and process-object encapsulation. Embodied encapsulation involves a 'delicate shift of attention' (in the sense of Mason, 1980) from the *action* being performed to the *effect* of that action (Poynter, PhD; Watson, 2004).

This parallel between embodiment and symbolism led me immediately to the idea of *three distinct worlds of mathematics* (Tall 2004). We begin in *a world of conceptual embodiment* focusing on (real-world) objects and actions and by thought experiments focusing on generic properties, we construct hierarchies of mental objects through to the platonic world of Euclidean geometry and visual representations of algebra and the calculus. *The world of proceptual symbolism* compresses actions from processes

into thinkable concepts (procepts) to lead to a hierarchical development in arithmetic, algebra and symbolic calculus. The two operating in tandem, with a tendency to shift from embodiment to symbolism, lead by natural or formal thinking to *the formal axiomatic world* of concept definition and formal proof.

I had begun thirty years before in the formal world of mathematics and I had backtracked and found a route from the perceptions of a child to the conceptions of mathematicians. And virtually every insight came from someone else! As Richard Skemp taught me, 'pleasure is a signpost, not a destination'. 'The journey is the reward'. The journey still continues, clarifying issues in the theoretical framework, for instance, in the way in which learners build on previous knowledge to produce what I call 'met-befores' in the concept image that can be helpful in some contexts, but act as cognitive obstacles in others. We already have a map of the cognitive growth of procepts through arithmetic, algebra, calculus and on to the formal definitions of analysis and other formal mathematics (Tall, Gray, et al 2000).

The journey uses the natural strengths and limitations of the biological brain to compress complicated detail into the simplicity of thinkable concepts that can be handled by the limited focus of attention (Akkoc & Tall 2004).

The research continued as Amir Asghari produced a highly personal Lakotos-style thesis in which he challenged not only my work in textbooks with Ian Stewart and research with Abe (Erh-Tsung) Chin, but also analysed that of the Greeks, Gauss, Russell to point out subtle flaws in the theories of Dienes and Skemp. Once again, challenge has led to personal enrichment. (Asghari, PhD; Asghari & Tall, 2005).

(Juan) Pablo Meira-Ramos, from Columbia is studying proof. He is showing me how students' attempts to understand proof are affected not only by their earlier experiences of embodiment and symbolism, but also by what they perceive as being required by the mathematical culture into which they are being introduced. It is a new journey combining personal growth and social construction.

From Brazil came Victor Giraldo and Rosana Nogueira de Lima to study for a year towards their Brazilian PhDs. Victor had amazing insight into the cognitive root of local straightness and how the conflict that occurs between imperfect numerical computation of limits on a computer and mental imagery of its ultimate perfection can be used to help students understand the need for the formal theory of analysis.

Rosana brought puzzling algebraic performances of students working under caring teachers who knew of their difficulties and taught them what they felt was necessary to pass the exam. In solving quadratics, they focused on teaching the formula, which will solve all quadratic equations and is more general than factorisation into two linear factors. However, in order to be able to *use* the formula, it may be necessary to first manipulate the symbols in the equation and here the problem arose with showing that the equation $(x-2)(x-3)=0$ had roots 2, 3. Many could not begin and of those that could, none saw that on substituting the values the equation was satisfied, instead

they attempted to multiply out the brackets and solve the equation using the formula. Few succeeded.

Here we have the ultimate problem of mathematical education around the globe. Every country is competing with every other country to be 'better' at teaching mathematics. To 'measure' how much better, various countries set up 'standards' to teach and test by. The result is that students year on year get better at responding to standard questions and the 'measure' rises everywhere. But the *understanding* of mathematics is rarely to be seen. My own work has been corrupted to fit the perceptions of others. In America, everyone speaks of 'local straightness' as synonymous with 'local linearity'. It is not. Local straightness is an embodied thought experiment magnifying a graph more and more to see if it straightens under magnification, to develop the insight to 'look along' a curve and *see* its changing slope as a new function. 'Local linearity' is the finding of a symbolic formula that expresses the best linear approximation at a fixed point. Then the fixed point is varied to give the derivative as the formula arising as the limit of the linear approximation. Thus the embodied meaning of local straightness is rejected in favour of the symbolism of the limit process, which we know causes cognitive problems.

The journey continues. Is it better to seek more and more complicated ways to write more and more papers to secure promotion, or is there still some worth in trying to understand what is going on and express it simply? I see the journey ahead as one of compressing complicated ideas into rich and simple meaningful concepts.

My journey has been enhanced by the companionship and insight of many other companions who have taught me so much. Other than my full PhD students, Tony Barnard shared with me his concept of 'cognitive unit', Dina Tirosh her research on infinity, John Pegg his research in SOLO and Van Hiele Theory and a host of others shared ideas including Gary Davis, Demetra Pitta, Shakar Rasslan, Juan Pablo Meija-Ramos, and Adrian Simpson, the organiser of this volume and the related celebrations. A special mention should also be made of Ian Stewart with whom I had the pleasure of writing three university mathematics books, all still in print after a quarter of a century. But it is to my research students I give the greatest thanks, particularly Eddie Gray, John Monaghan and Michael Thomas, all of whom have distinguished careers supervising doctoral students in a direct line from Richard Skemp through myself and them to succeeding generations.

Christopher Zeeman once said to me that the main test for PhD students to be awarded the degree is that they have taught their supervisor something important. It is an interesting idea.

An individual cannot be the source of limitless power of thought. In the real world there is no such thing as perpetual motion. To develop one needs new sources of energy and that energy comes from other people freely giving of their ideas. In the recent review of the first thirty years of PME (Guttiérrez & Boero, 2006), the ideas discussed in this paper have the largest number of references for a single person in

the whole volume. I take quiet pride in this statistic for someone who retired over a decade ago on ill-health. Of course, it is not the work of a single person, but the contributions of a whole family working for a common purpose. As can be seen from this review, almost everything I have done has been gifted to me willingly by others. My work has only been made possible by the support of all my collaborators, especially the PhD students at Warwick University who all earned the award of a doctorate for teaching their supervisor something important.

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