Long-term principles for meaningful teaching and learning of mathematics[1]

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Preface: The evolution of mathematical teaching and learning at university level

It is a privilege to be invited to write the preface for this ground-breaking book, bringing together the reflections of university mathematicians and mathematics educators on the teaching and learning of their students.

The last half century has seen phenomenal change. In 1957, Russia launched Sputnik and the teaching of mathematics and science was revolutionized in the US and other western countries to seek to remain competitive and gain an advantage by introducing 'new math'.

When I first began to contemplate the mathematical thinking of my own undergraduates as a young lecturer in mathematics some fifty or so years ago, there was little theoretical basis available to address the issues. Educational research in mathematics included studies of young children's arithmetic and more general aspects of school mathematics, but there was no mathematical education research at university level.

The first significant consideration of education in the professional mathematics community took place at the 1900 International Congress of Mathematicians (ICM) in Paris in a section entitled 'Teaching and History of Mathematics'. This included the famous lecture in which David Hilbert listed the 23 'Mathematical Problems' that shaped much of twentieth-century research mathematics. Major countries in Europe and North America were seeking to introduce significant reforms in school mathematics. In the fourth ICM in Rome 1908, it was decided to establish wide-ranging cooperation between countries through an 'International Commission on the Teaching of Mathematics' under its first president Felix Klein.

The commission held four international meetings before the next ICM in 1912, on topics such as 'What mathematics should be taught to students studying sciences?', 'What is the place of rigour in mathematics teaching?' and 'How can the teaching of the different branches of mathematics best be integrated?' It continued with a vast survey of teaching practices in over 300 reports from eighteen member countries. Then the first world war intervened, Klein died in 1923, and it was only in 1928 that the commission was re-established under its modern name, the 'International Commission of Mathematical Instruction' (ICMI).

Its first task was to collect data on teacher training methods to be presented at the next ICM in 1932 in Zurich. This was in the great depression between the wars and there was little interest in educational innovation. The second world war caused the ICM to be cancelled and ICMI ceased activity until it was re-constituted in 1952, now with a permanent secretariat under the auspices of the International Mathematics Union, but still reporting every four years to the ICM.

Apart from the two world wars, ICM meetings continued every four years with a subsection devoted to mathematics education. As the range of interests proliferated, this became

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inadequate and Hans Freudenthal, as President of ICMI from 1967 to 1970, proposed that a full conference be held every four years, between the mathematical ICMs and organized the first International Congress of Mathematics Education (ICME) at Utrecht in August 1967.

Since then, the last half century has seen phenomenal change. When I first began to contemplate the mathematical thinking of my own undergraduates as a young lecturer in mathematics some fifty or so years ago, educational theory of mathematics teaching and learning focused mainly at the school level.

As a lecturer in mathematics at Sussex University, I saw it as my duty to present mathematical ideas to undergraduates in ways that made sense to them. I was also integrated into the undergraduate community through musical activities in which I participated as a choral and orchestral conductor. Motivated by my increasing pleasure in working with students in both mathematics and music, I saw the link between the two in terms of the joy that arises as an individual working in a group for a common purpose. This contrasted with a growing sense of alienation in mathematical research in K-theory which I felt arid and meaningless. This conflict was addressed when I made the transition from a 'Lecturer in Mathematics' at Sussex University to a 'Lecturer in Mathematics with Special Interests in Education' at Warwick University in 1969.

My first experience of an international conference in mathematics education was at the second ICME conference at Southampton in 1972. Here I encountered the parlous state of mathematics education at the time. I attended two working groups, one on 'the teaching of calculus and analysis', the other on 'history and mathematics teaching'. The first included mathematics professors debating as to whether it was proper to calculate the derivative of sine x using a visual diagram in a unit circle or whether it required the formal definition of the limit of the function as a power series. The second working group had no theoretical content relating to teaching and learning. One professor suggested that many students found mathematics options in the final year too difficult and suggested that history of mathematics could provide them with an alternative where they could gain sufficient credit to be awarded a degree. Another showed his collection of mathematicians on postage stamps, and a third showed his photographs of Euler's birthplace. I left the conference early and returned home to my family. I remained a mathematician giving mathematical support to the local teacher training college as part of my job specification.

I attended ICME in Karlsruhe,1976, still in my role as a mathematician, but did not choose to go to the new working group on 'Psychology of Mathematics Education' which featured at that conference. Travelling back with Richard Skemp, I learnt that the group had proposed an annual conference of PME as a working group of the ICME, with the first arranged by Freudenthal in Utrecht in 1978, which I was fortunate to attend. Most of the topics related to school mathematics with a handful of participants interested in the transition from high school to university and on to college and undergraduate levels. Gontran Ervynck, from Belgium, organized a working group to study this transition into undergraduate mathematics which produced the first multi-author book on undergraduate mathematics teaching and learning, which I was fortunate to edit (Tall, 1991).

In 1990, President H. W. Bush declared 1990–1999 to be 'the decade of the brain' providing huge resources to enhance public awareness of the benefits to arise from brain research, leading to greater insight into the structure and operation of the brain.

Around this time, the mathematical community began to reflect on the issues. In a 1990 article on 'Mathematics Education', Fields Medalist William Thurston observed:

> Mathematics is amazingly compressible: you may struggle a long time, step by step, to work through some process or idea from several approaches. But once you really understand it and have the mental perspective to see it as a whole, there is often a tremendous mental compression. You can file it away, recall it quickly and completely when you need it, and use it as just one step in some other mental process. The insight that goes with this compression is one of the real joys of mathematics.
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> After mastering mathematical concepts, even after great effort, it becomes very hard to put oneself back in the frame of mind of someone to whom they are mysterious. (Thurston, 1990, p. 1)

In a second article 'On Proof and Progress in Mathematics' in 1994, he reflected on how a sub-community of mathematicians (say, analysts) may easily share ideas within their specialism which are opaque to another (such as topologists) and vice versa. He offered a detailed analysis of how communities of mathematicians operate and related this to the difficulties encountered by students:

> The transfer of understanding from one person to another is not automatic. It is hard and tricky. Therefore, to analyze human understanding of mathematics, it is important to consider **who** understands **what**, and **when**. Mathematicians have developed habits of communication that are often dysfunctional. Organizers of colloquium talks everywhere exhort speakers to explain things in elementary terms. Nonetheless, most of the audience at an average colloquium talk gets little of value from it. Perhaps they are lost within the first 5 minutes, yet sit silently through the remaining 55 minutes. Or perhaps they quickly lose interest because the speaker plunges into technical details without presenting any reason to investigate them. At the end of the talk, the few mathematicians who are close to the field of the speaker ask a question or two to avoid embarrassment.
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> This pattern is similar to what often holds in classrooms, where we go through the motions of saying for the record what we think the student "ought" to learn, while the students are trying to grapple with the more fundamental issues of learning our language and guessing at our mental models. Books compensate by giving samples of how to solve every type of homework problem. Professors compensate by giving homework and tests that are much easier than the material "covered" in the course, and then grading the homework and tests on a scale that requires little understanding. We assume that the problem is with the students rather than with the communication: that the students either just don't have what it takes, or else just don't care. Outsiders are amazed at this phenomenon, but within the mathematical community we dismiss it with shrugs (Thurston, 1994, pp. 5,6).

In North America in 1992, Ed Dubinsky encouraged the Mathematical Association of America (MAA) to form a joint *Committee on Research in Undergraduate Mathematics Education* (CRUME) with the American Mathematical Society. One of its first projects of CRUME was the CBMS series of 'occasional volumes of papers' on RUME, called Research in Collegiate Mathematics Education (RCME), producing seven volumes in the next decade.

In England in 1992, the London Mathematical Society produced a report (Neumann, 1992) to respond to the increasing number of students studying a university degree with a far wider range of achievement and the explosion of differing mathematical needs in society. The plan was to reform the current 3-year degree with a 3-year bachelor's degree covering a wider range of material than the first half of the current degree and a 4-year master's degree going beyond the current curriculum. Both degrees were to be 'taught in such a way that students achieve a markedly fuller understanding than they do at present.' However, the term 'understanding' was interpreted in very different ways by mathematicians and mathematics educators.

At an LMS conference to celebrate the life of my colleague the late Rolph Schwarzenberger, who embraced both mathematics and mathematics education, the presentations covered both aspects and I was invited to give a lecture on 'Mathematicians thinking about students thinking about mathematics' (summarised in Tall, 1993). At the same time, members of the LMS were invited to update their areas of research interest and I replied 'Advanced Mathematical Thinking'. The committee reluctantly refused to accept it because it was not an accepted heading in the American Mathematical Society's mathematical subject classification. A formal request to the AMS from CRUME was also rejected.

I gave an invited presentation in the Mathematics Education section of ICM in Strasbourg (Tall, 1994a) in which I said:

> I cannot believe that mathematicians can continue to ignore the study of mathematical thinking as part of the totality of the profession, for if it is not done by mathematicians, others surely lack the mathematical knowledge to research it in depth. I suggest that the study of mathematical thinking be given a place in the canons of mathematical activity comparable with other areas of mathematics. Just as a topologist will defend a number-theorist's right to do research within the umbrella of mathematics I hope that specialists in mathematical research will similarly defend the right of mathematicians to do research into mathematical thinking. Respect will have to be earned by mathematics educators. But if opportunities to earn respect are not honoured then mathematics itself can only be the poorer. (Tall, 1994b, p. l6)

The issue was finally resolved after a meeting of RUME in 1996 when Hyman Bass took up the matter and 'mathematics education' was added to the AMS Mathematics Subject Classification as Topic 97. Now it became possible for an individual applying for a post in a college or university to specify their mathematical area of interest as 'mathematics education'. Even so, it still remained necessary for mathematics educators to gain respect within the mathematical community.

The editor of this book, Sepideh Stewart, was fortunate to study for a PhD in mathematics education in a university in New Zealand where the mathematics department took mathematics education seriously and integrated their work within a single community. Even so, when a mathematician and mathematics educator were given equal support in recommendation for promotion, the university promotion committee chose the mathematician over the educator.

Having obtained and held a position in an American university for ten years, Sepideh opens the first chapter of this book reporting the development of her research working with mathematics professors willing to reflect on their teaching and sharing their experience with other members of their department.

In the second chapter, she collaborates with Bharath Sriraman to review theories and models for collaboration between mathematicians and mathematics educators at college and university level.

Around the world there has been a widespread activity in seeking to link activities of university and college mathematicians, mathematics educators and teachers. These include the 20-year-old annual conference of *Research in Undergraduate Mathematics Education*, Special Interest Group of the Mathematical Association of America); the biennial *Delta* conferences since 1997 nurturing exchanges between mathematicians, educators and researchers in the southern hemisphere; the Thematic Working Group on *University Mathematics Education* in the the Congress of European Researchers in Mathematics Education (CERME) evolving into ERME Topic Conferences (Montpellier, 2016; Kristiansand, 2018; Bizerte, 2020), and a range of national activities on teaching university mathematics. Of particular interest is the development of bilingual conferences in English and the language of the hosting country in the International Network for Didactics Research in University Mathematics (INDRUM), first held in France in 2016. These have the advantage of directly linking international research to the local community which has the potential to advance the link between theory and practice nationally and internationally.

The chapters which follow in this book report individual researchers' developments in undergraduate teaching and learning. In chapter three, Elena Nardi traces the relationship between mathematicians and mathematics educators in research on the teaching and learning of mathematics, recounting examples of initiatives that developed over time in research, teaching, professional development and public engagement. She re-imagines this not just as a story of paths crossing, but as paths meeting at a vanishing point in the future where boundaries between mathematics and mathematics education may fade into insignificance to become a joint, multi-faceted enterprise.

In chapter four, Michael Thomas reports his experience in developing collaborative work between mathematicians and mathematics educators. His title 'mind the gap' arose from the 1960s London underground rail warning of the inherent danger in the gap stepping between the train and the platform. Thomas uses Schoenfeld's Framework of 'Resources, Orientations and Goals' (ROG) as part of an extensive collaboration between mathematicians and mathematics educators as equal partners in his university mathematics department to show how interchange can be of mutual benefit. Techniques include the lecturer writing up his experiences after giving a lecture to form the basis for discussion and lecturers choosing a short selection from one of their lectures to illustrate aspects of interest.

In chapter five, Simon Goodchild writes of his experience as the founding director (2013-2020) of the Norwegian National Centre for Research, Innovation and Coordination of University Mathematics Teaching (MatRIC). The Centre was formed to promote the vision of 'students enjoying transformed and improved learning experiences of mathematics in higher education'. He categorises Higher Mathematics Education Teachers (HEMT) to include mathematicians and specialists in service subjects. These adopt a variety of instructional approaches which are not resolved by the current complexity of Mathematical Education Research (MER). He reports initiatives in his own institution to address these problems directed at teaching staff and students. Essentially the staff seek practical strategies that they can use to improve the effectiveness of their teaching in their own terms without the need to translate from technical terms in educational theory.

University and college lecturers often find that they are constrained in how they can teach. They are part of a system which is subject to a range of differing demands, from preparing students of varying abilities for future employment to encouraging highly able students to become research mathematicians of the future.

In chapter six, Paul Christian Dawkins and Keith Weber observe that, although some mathematics educators have developed radically new approaches for students to take an active part through some form of 'inquiry-based learning', most mathematicians still see the lecture as a central form of teaching. They are unlikely to take on new approaches that are unfamiliar and do not guarantee success in their own teaching. Nevertheless, based on a synthesis of the research literature, they suggest that mathematicians and mathematics educators would agree that

> Students need opportunities to reflect on central ideas and understandings in their advanced mathematics courses.

They propose that lectures can be enhanced by including activities that encourage students to reflect on their understanding. As an example, clickers can be used to allow students to choose from contrasting multi-choice options that can then be displayed to form a basis for discussion of their meaning. Or lecture notes can be printed with gaps so that the lecturer can speak about ideas and the students can fill in the details. This involves a general principle to develop 'minimally invasive classroom activities' in partnership between mathematicians and educators.

In chapter seven, Carl Winsløw begins by outlining his personal development. He was taught the 'New Mathematics' in school, based on the structural approach of the French Bourbaki group. After completing his undergraduate degree, he became a mathematics researcher in an algebraic area of functional analysis, and then an associate professor in mathematics teaching undergraduates, using this experience to develop an integrated perspective encompassing mathematics education research (which he terms the *Didactics of Mathematics*) and the mathematical sciences.

He bases his approach on the Anthropological Theory of the Didactic (Chevellard, 1992) which encompasses two major components in historical and personal development of knowledge in mathematical communities– praxis (practical knowledge) and logos (theoretical knowledge). Praxis involves recognising a problem and knowing the technique to solve it. Logos uses words, pictures and diagrams to build logical relationships in increasingly sophisticated forms of deduction and proof. A praxeology is a theory of the relationship between the two.

This leads to an analysis of the evolution of mathematical knowledge in history and in mathematical communities where individuals play different roles. There is a particular focus on Felix Klein's distinction between a curriculum that separates different topics into self-contained units and his preference for mathematical science as a great connected whole. (Klein, 1908/2016, vol. I, pp. 82-83)

Winsløw proposes the need for 'mathematics teacher educators who can effectively pursue Klein's vision, we need to prepare mathematicians-didacticians who are both acquainted with contemporary mathematics, with creative mathematical work, and with modern methods and results from the Didactics of Mathematics.'

In chapter eight, Paola Iannone, a mathematician who developed a deep interest in teaching and learning, describes her shift to mathematics education and subsequent collaboration with mathematicians who wish to design and evaluate new approaches to their own teaching and assessment. She notes growing evidence in the literature that written examinations. as they are currently structured, fail to assess types of reasoning valued by the mathematics community such as conceptual understanding and problem solving. This is investigated in a summative question and answer session with the student writing on the board (or using pen and paper) answering questions which can be theoretical (stating known definitions, theorems or proofs) or applied (working out examples, tackling unseen problems or proofs, or using algorithms appropriately). A second study considers students using theorem-proving software to investigate how this changes students' understanding of proof. In both cases the mathematicians and educators involved learnt a great deal about the subtleties of their own perceptions of mathematics and the understandings of the students taking part but questioned how these specific experiences could be generalised for wider dissemination. This was related to the chapter by Dawkins and Weber (reference to page number) that questioned why teaching and curriculum innovations proposed by mathematics educators have had little impact on how university mathematics is taught and echoed the need for mathematicians to be aware of tools to study the impact of the transfer of interventions into their own context.

Chapter nine by Barbara Jaworski reviews the development of inquiry-based mathematics and learning, starting from her own experiences which have influenced the development of several other authors in this collection. After a first career as a schoolteacher in a comprehensive school for students aged 13-18, she took an active interest in the Open University where she was introduced to the ideas of mathematical investigations as a means of developing mathematical thinking and learning. In subsequent university posts, she developed local communities of practice in which researchers and teachers shared their expertise to their mutual benefit. Several authors in this book have taken part in these developments, including Elena Nardi, Simon Goodchild, and Carl Winsløw. A common thread is the development of Inquiry Based Learning, which also features in chapter six written by Paul Christian Dawkins and Keith Weber.

Jaworski formulates three layers of inquiry represented diagrammatically as an inner layer where students engage in inquiry with a teacher in the classroom, a middle layer of teachers engaging in professional inquiry and an outer layer of didacticians inquiring with teachers to research processes, practices and issues in developing mathematics teachers and learning. The chapter includes the story of development of international organisations in which she has taken an active part.

In chapter ten, John Mason focuses on how individuals can keep themselves mathematically alive by being attentive to their own thinking, to sensitise themselves to the struggles faced by learners and to improve their own pedagogy. Based on this attitude, he recommends readers to think through mathematical questions he poses before reading his account which follows so that they have personal experience as a foundation to consider his observations.

Chapter eleven sees Günter Törner incorporating Shulman's notion of a *'signature of teaching mathematics'* that characterise each teacher's approach to teaching and learning. He offers a range of examples in terms of four overarching aspects: the *mathematical content* as such and its structure, the underlying *understanding of teaching and learning*, the characteristics of the partly socially acting *classroom*, and the *immanent philosophies of mathematics*.

My own chapter on 'long-term principles for meaningful teaching and learning of mathematics' is appropriately placed last as it is my own personal attempt to formulate how mathematical thinking evolves in sophistication over time in history and in the individual, taking account of different approaches appropriate for differing specialisms, experts, teachers and learners. In particular, it formulates how different communities of practice may have approaches that are appropriate for some yet be problematic for others and proposes a 'multi-contextual overview' where each community is aware of the values shared between the two, to build confidence based on their communalities while respecting and addressing their differences. Its main purpose is not to conflate a highly complicated theory. Its objective is to find fundamental ideas that can be observed meaningfully by most readers. This includes how we *speak* and *hear* mathematical expressions that can be interpreted both as *operations* and as mental *objects*, how we *see* moving objects as variables – which allows the imagination of an infinitesimal as a quantity that grows arbitrarily small – and how we *read* mathematical proofs to make sense of them. This offers new possibilities for readers to reflect on their own experiences and beliefs, taking into account other chapters in the book.

References

Chevallard, Y. (1992). Fundamental concepts in didactics: Perspectives provided by an anthropological approach. In R. Douady and A. Mercier (Eds.), *Research in Didactique of Mathematics, Selected Papers*. La Pensée Sauvage, Grenoble, pp. 131-167.

Klein, F. (1908/2016). *Fundamental Mathematics from a Higher Standpoint, I-III. Translated by G. Schubring.* Berlin: Springer.

Neumann, P. (1992). The Future for Honours Degree Courses in Mathematics. *Journal of the Royal Statistical Society. Series A (Statistics in Society)*,155 (2), 185-189. doi:10.2307/2982954
Selden A. (2012). A Home for RUME: The Story of the Formation of the Mathematical Association of America's Special Interest Group on Research in Mathematics Education. https://www.tntech.edu/cas/pdf/math/techreports/TR-2012-6.pdf
Tall, D. O. (1993). Mathematicians Thinking about Students Thinking about Mathematics, *Newsletter of the London Mathematical Society*, **202**, 12– 13.
Tall, D. O. (1994a). Understanding the Processes of Advanced Mathematical Thinking, *Abstracts of Invited Talks*, International Congress of Mathematicians, Zurich, August 1994, 182–183.
Tall, D. O. (1994b). Understanding the Processes of Advanced Mathematical Thinking, *International Congress of Mathematicians*, Zurich, August 1994. Full Lecture at: http://homepages.warwick.ac.uk/staff/David.Tall/pdfs/dot1996i-amt-icm.pdf
Thurston, W. P. (1990). Mathematical Education, *Notices of the American Mathematical Society*, 37, 7, 844–850. https://arxiv.org/pdf/math/0503081.pdf (Note: page numbers in this text refer to the available pdf, not the original pagination.)
Thurston, W. P. (1994). On proof and progress in mathematics, *Bulletin of the American Mathematical Society*, 30 2, 161–177. https://arxiv.org/pdf/math/9404236.pdf (page numbers in the text refer to the latter pdf.)