The Articulation Principle: Making long-term sense of mathematical expressions by how they are spoken and heard

Kin Eng Chin, Fui Fong Jiew, David Tall
Edith Cowan University, Australia; Queensland University of Technology, Australia; University of Warwick, UK
<k.chin@ecu.edu.au>, <fuifong.jiew@hdr.qut.edu.au>, <davidtall@mac.com>

Abstract The Articulation Principle offers a theory that explains how mathematical expressions can be given clear and unambiguous meaning by the way in which they are spoken and heard. By leaving short pauses in speech, sub-expressions can be interpreted as operations to be carried out in time or as mental objects that can be manipulated at a more sophisticated level. This offers a fundamental foundation for the growth of meaningful mathematical thinking at all levels from young children to the wide array of adult mathematics. It is part of an evolving comprehensive theory of 'how humans learn to think mathematically' that addresses the challenges of 'math wars' between differing communities of practice with differing beliefs and needs in a complex society. This paper sets the Articulation Principle in a wider long-term learning framework and provides empirical evidence for its use in meaningful interpretation of mathematical expressions.

Keywords: articulation principle, mathematical expressions, sense making, supportive and problematic conceptions, three worlds of mathematics

1. Introduction

The interpretation of mathematical expressions is traditionally handled by introducing conventions such as 'multiplication takes precedence over addition' and mnemonics such as 'Please Excuse My Dear Aunty Sally' (PEMDAS) to specify the order of precedence 'Parenthesis, Exponent, Multiplication, Division, Addition, Subtraction', or its English equivalent BIDMAS (Brackets, Index, Division, Multiplication, Addition, Subtraction).

Over time the increasing complication of rules is accompanied by an increasing array of misconceptions. A practical approach to give unambiguous meaning to mathematical expressions by focusing on how they are interpreted when spoken and heard is formulated as

> **The Articulation Principle**: the meaning of an expression depends on the manner in which it is articulated. (Tall, 2019, p. 7).

For example, the phrase 'ten take away four plus two' can be articulated as 'ten take away four [pause] plus two', leaving a small gap in speech to allow it to be interpreted as 'six … plus two', giving 'eight'. It may also be spoken as 'ten take away … four plus two' and interpreted as 'ten … take away six', which is 'four'.

This principle is already widely grasped in an intuitive sense, but it is rarely made explicit in teaching and learning. Tall (2020) proposes that it offers a foundation for a meaningful long-term learning theory that has practical use to enable teachers as mentors to encourage learners to take control of their own learning. It begins by revealing that articulation gives precise meaning to simple mathematical expressions and gives a reason to use brackets as a meaningful indication of order of operations rather than as an arbitrary convention. It continues by allowing symbols representing operations to be imagined as mental objects that can be manipulated more easily. This applies to successive transitions throughout the mathematics curriculum (Gray & Tall, 1994) and on to the ever broadening horizons of mathematical research (Tall, 2013).

The study of the long-term development of mathematical symbolism took a major step forward in Piaget's study of the development of whole number concepts in young children, published in French in 1941 and translated into English in 1952 as *The Child's Conception of Number*, as part of a much wider theory of *The Psychology of Intelligence* (English translation: Piaget, 1950).

However, in the following years, educational psychology offered a range of different theories. Skemp (1962) reviewed all the papers in major educational psychology journals since 1940 and concluded that they were inadequate for dealing with mathematics:

> … theories relating to conditioning, reinforcement learning, sign learning, perceptual learning, etc., are not adequate for the classroom. A theory is required which takes account (among other things) of the systematic development of an organised body of knowledge, which not only integrates what has been learnt, but is a major factor in new learning: as when a knowledge of arithmetic makes possible the learning of algebra, and when this knowledge of algebra is subsequently used for the understanding of analytical geometry.

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> The only theory yet available which does this has been put forward by Piaget (1950). He calls such a body of knowledge a 'schema'. The incorporation of new knowledge into an existing schema is called 'assimilation'; and the enlargement of a schema, which may be necessary if it is not adequate for the above purpose in its existing form, is called 'accommodation'. These three related concepts would seem to offer a basis for the kind of learning theory which is needed. (Skemp, 1962, p. 133).

This paper outlines the latest stage in the evolution of a broader overall theory evolving from the foundations of Piaget and Skemp, to elaborate *How Humans Learn to Think Mathematically* (Tall, 2013). It evolves through three interrelated forms of mathematical thinking:

- *conceptual embodiment* (broadly based on increasingly sophisticated forms of human perception and operation from physical to mental thought experiment),
- *operational symbolism* (based on symbolising operations and manipulating symbols),
- *formalism* (beginning with the *theoretical* reorganisation of experience based on carefully selected definitions and deduction and, at a higher *axiomatic formal* level, based on set-theoretic definition and logical proof).

The initial link between conceptual embodiment operating on objects and operational symbolism is made meaningful by the Articulation Principle.

This proves to have relevance at all levels of development, from the difficulties young children encounter with word problems, through the transitions between number systems and notations, from counting numbers, to fractions, to signed numbers, decimal representation, rational and irrational numbers, real numbers, complex numbers, from arithmetic to algebra and more advanced forms of symbolic calculus, vector algebra and abstract algebra.

Since the publication of *How Humans Learn to Think Mathematically*, new developments have occurred that broaden the theory to include information about the physical structure of the human brain, how it interprets text and continuous motion, how perception and action are linked to emotional and physical reactions. There are also other essential cultural aspects relating to conflicting approaches and beliefs of differing communities of practice. The interpretation of one community (e.g. pure mathematics) may involve aspects that are inappropriate for others (e.g. engineering, economics, biology, teachers of young children).

Here we focus on the beginning of the development by offering empirical evidence to support the Articulation Principle linking embodiment (through speaking and hearing) to meaningful symbolism (Chin & Jiew, 2018; Tall, 2013) as part of a broader framework for making mathematics meaningful (Tall, 2019, 2020).

2. A Long-term Theoretical Framework

> There is nothing so practical as a good theory. (Skemp, 1989, p. 27)

In the book *Mathematics in the primary school*, Richard Skemp (1989) sought a long-term learning curriculum in which learners build on their previous experience, expanding their knowledge where it fits with what they already know and reconstructing ideas to take account of new situations. His colleague and research student, David Tall—who had completed a doctoral thesis (Tall, 1967) in pure mathematics with Field's medallist Michael Atiyah—completed a second doctorate in mathematics education with Skemp in 1986. Subsequently, this led to building theoretical constructs from practical experience, with others, including the notion of *concept image*, based on a construct of Shlomo Vinner (Tall & Vinner, 1981), the dual use of symbolism as process and concept as *procept* with Eddie Gray (Gray & Tall, 1994), *Advanced Mathematical Thinking* (with the working group of PME, ed. Tall, 1991) and textbooks on university mathematics with Ian Stewart (e.g. *Foundations of Mathematics*, 2nd edition, Stewart & Tall, 2014). He had the privilege of supervising international PhD students who researched the learning of children and students from early childhood to post-graduate level where each thesis added a new aspect to the fuller picture, detailed in Tall (2008). The framework as summarised in *How Humans Learn to Think Mathematically* (Tall, 2013) applies not only to different individuals learning over a lifetime but also to the historical development over generations in different cultures.

Chin (2013) focused on successive stages of development in trigonometry, from triangle trigonometry taught to teenagers aged 14-16, through circle trigonometry with arbitrary angles and functions visualised graphically at ages 16-19, and on to formal trigonometry at university in real or complex analysis. It reveals how teachers and learners make sense of the various ideas within and between stages. For example, triangle trigonometry interprets the visual concept imagery of a right-angled triangle to move flexibly between the symbolic process of calculating the ratio of sides to the symbolic concept of the ratio as a number. Circle trigonometry focuses on varying the angle to study trigonometric functions visually as graphs of general angles measured in radians and to develop their symbolic properties. Formal trigonometry moves on to more general ideas such as series as infinite processes relating to limit concepts and real and complex relationships between trigonometric and exponential functions. At each stage, the Articulation Principle plays a meaningful role in giving precise meaning to expressions consistent with accepted mathematical conventions.

Making sense of spoken expressions involves working in a particular context for a time, say counting and performing simple whole number arithmetic, then shifting to new contexts introducing new ideas which may involve expansion (assimilation) or reconstruction (accommodation). Chin formulated the notion of supportive and problematic conceptions to illustrate how personal conceptions may support or impede learning as the context changes. Supportive conceptions refer to conceptions that work in an old context and continue to work in the new whereas problematic conceptions work in an old context but do not work in the new.

For instance, in introducing fractions, new concepts and procedures come into play, such as the notion of equivalent fractions and new procedures for addition and multiplication. Old experiences with whole numbers, such as 'multiplication gives a bigger number' no longer work and rote learning of the procedures without making sense can alienate the learner. Yet there are other properties that continue to be supportive. For instance, Piaget's Principle of Conservation of Number states that the number of elements in a collection of objects is independent of the way it is counted. This continues to be supportive through successive number contexts, including fractions, signed numbers, finite and infinite decimal representations, real numbers and even complex numbers.

Tall (2020) proposed a supportive approach to deal explicitly with problematic changes of meaning. This entails the curriculum designer and teacher being aware of changes in meaning and to explicitly encourage the learner to be aware of ideas that remain supportive through several changes of context to give them confidence in building on them. Then, when problematic ideas arise, they may be explicitly addressed to encourage the learner to reconstruct their ideas to fit the new context. This led to a major principle formulated to enable learners to take control of their own learning, guided by teachers acting as mentors. It was named in honour of his grandson Simon, then aged 11, who was responsible for giving examples of different ways of speaking mathematical expressions:

> **The Simon Principle**: The teacher should be aware of those ideas that remain supportive through several changes of context, to give confidence to the learner, and to make explicit those ideas that are problematic so that they can be addressed meaningfully. (Tall, 2020, p. 2).

We propose that teachers apply the Articulation Principle in their classrooms so that students can make better sense of the meaning of operations in arithmetic and algebra. It can be introduced at any stage to improve understanding and to build more coherent relationships as mathematics becomes more sophisticated. Spoken articulation offers a way to help students begin to make sense of the sequence of operations in a mathematical expression, mentally and spontaneously.

3. Literature Review

Our purpose in this paper is to formulate and test a simple way of making sense of mathematical expressions. Therefore we limit our review to a small number of relevant sources.

When learners first encounter mathematical concepts, there is a huge difference between their natural language and mathematical terminology of the teacher that Kotsopoulos (2007, p. 301) formulates in terms of 'It's like hearing a foreign language.' Language is an essential means of communication, both between individuals and within the mind (Lakoff & Johnson, 1980; Lakoff & Núñez, 2000; Pimm,1987; Sfard et al., 1998). It is essential not only for communication, it also influences human thought (Freudenthal, 1991). Our interest is in the increasing sophistication of symbolism to support the long-term learning and meaning throughout the curriculum.

In Australia, the National Numeracy Review Report (COAG, 2008) stated that mathematical symbols and expressions hinder children from understanding concepts in mathematics. For example, in the expression $4 + 3 \times 2$, the multiplication 3×2 needs to be performed first because 'multiplication takes precedence over addition' as a convention. However, reading $4 + 3 \times 2$ as 'four plus three times two' in standard left to right order gives a different result. This introduces a conflict between natural reading from left to right and the conventions for interpreting mathematical expressions.

Mathematics teachers may tend to write mathematical expressions on a whiteboard without focusing on the ways to speak or read them. This may fail to encourage students to develop the capacity to access mathematical meaning through spoken language. Ellerton and Clements (1991) suggested that teachers should encourage students to speak and read mathematics.

Different individuals may interpret various spoken mathematical expressions in different ways. For instance, the spoken expression, 'one divided by two plus three' may be interpreted as either $\frac{1}{2+3}$ or $\frac{1}{2} + 3$. Gellenbeck and Stefik (2009) performed a study to examine if insertion of pauses in spoken mathematical expressions could reduce ambiguity between similar algebraic expressions. Their study involved 16 students who were around 22 years old majoring in computer science. The participants were given two, side-by-side, algebraic expressions and were required to rate each expression on a Likert scale from zero to ten on how well they conceived the expression corresponded to the audio. The spoken expressions were either with or without pauses. Their findings revealed that the use of pauses for spoken expressions dramatically improved participants' ability to distinguish between similar algebraic expressions. However, they claimed that no empirical study had focused on the use of pauses as a causal mechanism for interpreting mathematical expressions and they did not offer a theoretical framework to explain or predict this phenomenon. Our purpose here is to provide a theory and offer empirical evidence in support.

Realising the importance of verbalising mathematics and reading mathematical expressions, this paper aims to answer three research questions:

1. Is there any evidence which shows that our conventional way of reading, speaking and hearing mathematical expressions is problematic?
2. Do the participants agree that we need to have a different way of reading, speaking and hearing different mathematical expressions?
3. How does the Articulation Principle help the participants to make sense of the sequence of operations?

4. Methodology

We report data collected from two Malaysian mathematics teachers (Eric and Fabian – pseudonyms) from two different schools, who participated in the study on a voluntary basis. Both were familiar with the conventions for mathematical expressions such as 'multiplication takes precedence over addition' which they had been taught in the Malaysian primary school curriculum.

Each participant took part in a clinical interview lasting approximately half an hour. Although the data will be interpreted using the Articulation Principle, the term itself was not used in the interviews, which were carried out using the familiar language of the mathematics classroom.

Each interview was conducted in English comprised of three parts. In the first part, the interviewer spoke the expressions 'the square root of four times four' and 'one over two plus one' in a simple even tone and asked the participants to give the answer verbally. The interviewer asked the participants to explain how they got the answers and to write them down. The interviewer then wrote a second symbolic expression to give a pair of expressions that would be spoken in similar ways but had different values. These were $\sqrt{4 \times 4}$, $\sqrt{4} \times 4$ and $\frac{1}{2+1}$, $\frac{1}{2} + 1$.

The second part of the interview briefly addressed the second research question to ask whether the participants agreed on the need for different ways of reading expressions.

The third part researched in greater depth how the participants made sense of a sequence of operations. First the interviewer revisited the spoken expressions 'the square root of four times four' and 'one over two plus one', now spoken with different articulations, corresponding to the written symbolic expressions, $\sqrt{4 \times 4}$, $\sqrt{4} \times 4$ and $\frac{1}{2+1}$, $\frac{1}{2} + 1$. Then two new spoken expressions were investigated: 'negative five squared' and 'two plus three times four'. The main purpose was to seek how the participants gave meaning to expressions through articulation and also how they handled the dual meaning of an expression as a mental process or a mental object.

The interviews were audio-recorded and transcribed for subsequent analysis.

5. Results and Analysis

The results and analysis are here reported to respond to the three research questions. First, we illustrate how Eric and Fabian reacted to mathematical expressions spoken with different articulations. Then we address how the Articulation Principle can potentially impact the participants in getting accurate answers for mathematical expressions. In the following excerpts, "I" refers to interviewer and statements in square brackets indicate the nature of the actions of the person speaking.

5.1 Is there any evidence which shows that our conventional way of reading mathematical expressions is problematic?

5.1.1 The case of Eric

I : What is the square root of four times four?
Eric : The answer is definitely four.
I : Can you please explain how did you get your answer?
Eric : It seems like a straightforward calculation because four times four is four squared, then we can cancel the square root and the square. The square root and square are inverses of each other, if we do the calculation, we get four.
I : Please write down the expression, square root of four times four.
Eric : [Writing on a piece of paper:]

$$\sqrt{4 \times 4}$$

I : What is one over two plus one?
Eric : Half plus one, zero point five plus one, so it's one point five.
I : Can you please explain how did you get your answer?
Eric : I just added half and one together.
I : Please write the expression down.
Eric : [Writing on the paper:]

$$\frac{1}{2} + 1$$

I : Please read your first written expression. [Pointing at Eric's first written expression, indicated above as $\sqrt{4 \times 4}$.]
Eric : Square root of four times four.
I : How would you read this mathematical expression? [Writing the expression, $\sqrt{4} \times 4$].
Eric : It's the same way, I think. Square root of four times four.
I : Are you reading it similar to this one? [Pointing at $\sqrt{4 \times 4}$.]
Eric : Hmm … I think so. Yes, I'll read it the same way, but I might as well write it down to avoid confusion. Now I see the problem.

At this point Eric has sensed there is a problem but has yet to verbalise what it is.

I : Do you think that these two expressions have different answers? [Pointing at $\sqrt{4} \times 4$ and $\sqrt{4 \times 4}$.]
Eric : Yes, definitely, because the answer for the second expression is eight. [Pointing at $\sqrt{4} \times 4$.]
I : How did you get your answer, eight?
Eric : Because the square root of four is equal to two, so two times four is eight.

I : How would you read this mathematical expression? [Writing the expression, $\frac{1}{2+1}$.]
Eric : One over two plus one.
I : Do you think that these two expressions have the same answer? [Pointing at $\frac{1}{2+1}$ and $\frac{1}{2} + 1$.]
Eric : No.
I : What is the answer for this expression? [Pointing at $\frac{1}{2+1}$.]
Eric : One over three.

Based on the excerpts above, the conventional way of reading mathematical expressions without articulation is problematic. Eric read the expressions $\sqrt{4} \times 4$ and $\sqrt{4 \times 4}$ with the same sequence of words and he was unable to distinguish them verbally. He also read $\frac{1}{2} + 1$ and $\frac{1}{2+1}$ in the same way. Note that he was aware that there was a problem, but had not yet formulated what the problem was.

5.1.2 The case of Fabian

I : What is the square root of four times four?
Fabian : Four.
I : Can you please explain how you get your answer?
Fabian : Four times four is equal to 16, so the square root of 16 is four.
I : Please write down the expression, square root of four times four.
Fabian : [Writing on a piece of paper:]

$$\sqrt{4 \times 4}$$

I : What is one over two plus one?
Fabian : One and a half.

I : Can you please explain how you get your answer?

Fabian : Half, then plus one, so it equals one and a half.

I : Please write down the expression, one over two plus one.

Fabian : [Writing on the paper:]

$$\frac{1}{2}+1$$

I : Please read your first written expression [Pointing at Fabian's first written expression, $\sqrt{4 \times 4}$.]

Fabian : Square root of four times four.

I : How would you read this mathematical expression? [Writing the expression, $\sqrt{4} \times 4$.]

Fabian : Square root … haha … same thing. Square root of four times four.

Note at this point, Fabian has a sudden 'Aha!' insight.

I : Do you mean that these two expressions have the same answer? [Pointing at $\sqrt{4} \times 4$ and $\sqrt{4 \times 4}$.]

Fabian : No, they have different answers.

I : What is the answer for this expression? [Pointing at $\sqrt{4} \times 4$.]

Fabian : Eight.

I : How did you get your answer?

Fabian : Square root of four is two then two times four is eight.

I : Please read your second written expression. [Pointing at Fabian's second written expression, indicated above as $\frac{1}{2} + 1$.]

Fabian : One over two plus one.

I : How would you read this mathematical expression? [Writing the expression, $\frac{1}{2+1}$.]

Fabian : One over two plus one. Same thing!

Again, Fabian expresses an insight.

I : Do you mean that these two expressions have the same answer? [Pointing at $\frac{1}{2+1}$ and $\frac{1}{2} + 1$.]

Fabian : No, they have different answers.

I : What is the answer for this expression? [Pointing at $\frac{1}{2+1}$.]

Fabian : One third.

Fabian's response was broadly similar to Eric's. However, he expressed an 'Aha!' moment when he realised that he would speak $\sqrt{4 \times 4}$ and $\sqrt{4} \times 4$ using exactly the same words that do not distinguish between their different meanings. In the case of $\frac{1}{2} + 1$ and $\frac{1}{2+1}$, he realised the same phenomenon, saying 'Same thing!'.

5.2 Do the participants agree that we need to have different ways of reading different mathematical expressions?

5.2.1 The case of Eric

I : Do you think that we should have two different ways to read these two mathematical expressions? [Pointing at $\sqrt{4 \times 4}$ and $\sqrt{4} \times 4$.]

Eric : Yes, I think so because they are actually two different things but read the same way.

I : Do you think that the way we read mathematical expressions is problematic?

Eric : Yes, if just reading without any writing.

Now Eric explicitly agrees that it is necessary to have two different ways to read the two expressions and that reading mathematical expressions could cause problems.

5.2.2 The case of Fabian

Fabian also agreed that we should have different ways of reading different mathematical expressions:

I : Do you think that we should have two different ways to read these two mathematical expressions? [Pointing at $\sqrt{4 \times 4}$ and $\sqrt{4} \times 4$.]

Fabian : Yes.

5.3 How does the Articulation Principle help the participants to make sense of the sequence of operations?

Now the interviewer speaks expressions that have different meanings with different articulations, to investigate how the students interpret them.

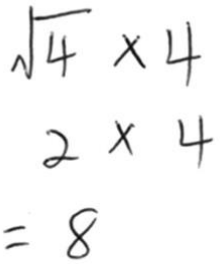
5.3.1 The case of Eric

I : What is the square root of four [pause] times four? I repeat, what is the square root of four [pause] times four?

Eric : Eight.

I : How did you get your answer?

Eric : When you say it like that, for me, instinctively I will square root the four first. Square root of four is two, two times four is eight. [Writing on the paper:]



I : What is the square root of [pause] four times four? I repeat, what is the square root of [pause] four times four?

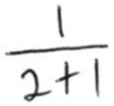
Eric : Four, I think.

I : How did you get your answer?

Eric : I did four times four first then square root the product.

I : What is one over [pause] two plus one?

Eric : If you say it slowly, the two plus one is the denominator. [Writing on the paper:]



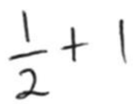
I : What is your answer?

Eric : One over three.

Notice that Eric stated that the difference in meaning could be explained 'if you say it slowly'. He had an implicit awareness of the role of articulation without explicitly saying what it is.

I : What is one over two [pause] plus one?

Eric : This is my expression. [Writing on the paper:]



I : What is your answer?

Eric : One and a half.

The next part of the interview was carried out only in spoken form. There were no written expressions in view and the interviewer read his questions from the interview protocol without showing it to either participant.

I : What is negative [pause] five squared. I repeat, what is negative [pause] five squared?

Eric : Negative twenty-five.

I : Please explain how did you get your answer?

Eric : When you square the number five you will get twenty-five, a positive answer. It depends on whether or not we have a bracket here. [As Eric spoke, he moved his index finger on the table as if he was writing a minus sign followed by a bracket round the number five.] But in this case, I think this is a negative twenty-five.

I : What is negative five [pause] squared? I repeat, what is negative five [pause] squared?
Eric : For me, right now I'm thinking the answer is twenty-five.
I : Please explain how did you get your answer.
Eric : I squared negative five.

The interviewer went on to speak a further question from the interview protocol without showing any written symbolism:

I : What is two plus [pause] three times four?
Eric : This is fourteen. [Writing on the paper:]

$$2+3\times 4=14$$

I : Can you please explain how you get your answer?
Eric : I multiply first then do addition. Three times four is twelve, plus two is fourteen.

The interviewer next sought to explore if Eric noticed the differences between spoken expressions without the insertion of pauses and those spoken expressions with the insertion of pauses.

I : Did you notice any difference when I read the expressions?
Eric : Yes, the way you read the expressions somehow affected my thinking, how I pictured the expressions in my mind. It is actually like a space button in a computer. It separates things.
I : How does it affect your thinking about the expression?
Eric : It's a way to imagine, a way of saying the expression that will help us to convey the message without writing the expression down.

When listening to spoken expressions, such as 'the square root of four times four' without any articulation, as noted in section 5.1, Eric definitely interpreted it as 'four', giving precedence to calculating the product. When the expression was spoken with a pause, he separated out the parts before and after the pause. For instance, when hearing 'the square root of four [pause] times four', he immediately square-rooted the four first to get two then he performed two times four. He now sensed that 'the square root of four' could be processed to get the result, 2.

This exhibits the notion of procept where the symbol can be interpreted flexibly either as a process (an operation) or as a concept (a mental object). He was able to switch fluently between the two as if they are two different ways of thinking of the same thing.

He explicitly said that he 'pictured the expression in his mind' to 'see' imaginary brackets and that certain spaces 'acted like a space button in a computer to separate things'. He used his previous learning to 'see' imaginary brackets, to distinguish the two different meanings of 'negative five squared'.

When the interviewer verbalised 'the square root of four [pause] times four', Eric computed the square root of four first to get two then multiplied it by four to get eight as his final answer. He made sense of 'the square root of four' as a process that he needed to perform first to get the result 4 (as an object) before performing the next process.

When the interviewer spoke a few different mathematical expressions using articulation to separate different sub-expressions, Eric was able to sequence the operations correctly because he was able to identify the sub-expressions that needed to be performed first. For instance, when he heard 'one over [pause] two plus one', he spontaneously sensed 'one' and 'two plus one' as separate sub-expressions. In a similar vein, when he heard 'negative [pause] five squared', Eric sensed 'five squared' as a sub-expression and imagined it having a bracket round it. He performed the operation to square five first before forming the negative.

In the Malaysian primary school curriculum, he will have been taught the convention that multiplication must be performed before addition; he interpreted the expression $2 + 3 \times 4$ by first performing the operation 3×4 to get 12, then adding $2 + 12$ to get 14.

5.3.2 The case of Fabian

I : What is the square root of four [pause] times four? I repeat, what is the square root of four [pause] times four?
Fabian : Hmm … which square root of four times four are you referring to?

Here Fabian is almost certainly referring to the two expressions encountered earlier.

I : Let me repeat the expression once again. What is the square root of four [pause] times four?
Fabian : Hmm …
I : Maybe we shall try another expression. Please listen carefully. What is the square root of [pause] four times four? I repeat, what is the square root of [pause] four times four?
Fabian : Oh! I got it! Ask me again the first one!

Here Fabian suddenly has an Aha! moment when he senses the required link.

I : What is the square root of four [pause] times four?
Fabian : Eight!
I : How did you get your answer?
Fabian : Square root of four is two, times four, then is eight.
I : Please write down the expression, square root of four [pause] times four.
Fabian : [Writing on the paper:]

$$\sqrt{4} \times 4$$

I : What is the square root of [pause] four times four?
Fabian : Four.
I : How did you get your answer?
Fabian : Four times four is 16, the square root of 16 is four.
I : Please write down the expression, square root of [pause] four times four.
Fabian : [Writing on the paper:]

$$\sqrt{4 \times 4}$$

I : Let’s try another one. What is one over [pause] two plus one?
Fabian : One over three.
I : How did you get the answer?
Fabian : Two plus one is three, so one over three.

I : Please write down the expression, one over [pause] two plus one.
Fabian : [Writing on the paper:]

$$\frac{1}{2+1}$$

I : What is one over two [pause] plus one?
Fabian : One and a half.

I : Please write down the expression, one over two [pause] plus one.
Fabian : [Writing on the paper:]

$$\frac{1}{2}+1$$

I : Please explain how did you get two different answers and expressions.
Fabian : You read the expressions differently.
I : What is the difference?
Fabian : The way you speak differentiates the operations. You separate the operations obviously. For example, you read ‘one over two … plus one,’ you are separating the half and one. Just like one object … plus … one object.

Notice here how Fabian actually speaks of the expression as an *object*.

I : How would you interpret this way of separation?
Fabian : When there is an operation, you will slow down to indicate the separation. Whenever you slow down, I will instantly separate what you have read with what you are going to read.

Here, he talks about ‘whenever you slow down’ to indicate a gap in speaking, in a similar manner to Eric.

I : What is negative five [pause] squared? I repeat, what is negative five [pause] squared?
Fabian : Twenty-five.
I : Please explain how did you get your answer?
Fabian : Actually, I think of a bracket for the negative five. Let me show you. [Writing on paper:]

$$(-5)^2$$

I : Please explain why you have included a bracket.
Fabian : Because in this case, negative five squared does not make sense without a bracket. This one is like personal familiarity, I straight away think of a bracket.

I : What is two plus [pause] three times four? I repeat, what is ‘two plus [pause] three times four’?
Fabian : Fourteen.
I : Please write down the expression, two plus [pause] three times four.
Fabian : [Writing on the paper:]

$$2 + \underline{\underline{3 \times 4}}$$

Notice how he double underlines the expression 3×4 to emphasise it as a single item.

I : Can you please explain how did you get your answer?
Fabian : Do multiplication before the addition.
I : Why?
Fabian : My primary teacher told me to do so.
I : Is my way of reading helping you to get the answer in this case?
Fabian : Hmm … If you read the expression without slowing down, I might just add two and three to get five, then multiply four to get twenty. But you read “two plus … three times four”, so I know that I need to do three times four first. But if I were given this as a written expression, it is easy to notice that multiplication needs to be done before addition because we used to recite the rules to simplify expressions.

Fabian distinguished the different ways of speaking an expression in terms of ‘slowing down’, as had Eric. He went a step further by referring to sub-expressions as ‘objects’. For example, he read ‘one over two … plus one’ saying ‘you are separating the half and one. Just like one object … plus … one object’. He also explained how he imagined brackets in his mind to interpret the articulated expressions in a precise manner, enabling him to notice each sub-expression as process or object to flexibly interpret the expressions correctly according to the standard conventions.

5.4 Summary of data and analysis

The research presented here has addressed the three research questions. The findings confirm

- that our conventional ways of reading, speaking and hearing mathematical expressions is problematic,
- that the participants agree that it is helpful to have different ways of reading, speaking and hearing mathematical expressions,
- that the Articulation Principle helps the participants make sense of the sequence of operations.

This confirms the insight available in clarifying mathematical expressions by speaking them in such a way that sub-objects are separated by articulating brief pauses.

Previous studies had identified the ambiguity in spoken mathematical expressions (Gellenbeck and Stefik, 2009) and had showed that the insertion of pauses could dramatically improve participants' ability to disambiguate two similar algebraic expressions.

Looking more closely at the responses of the participants in this study illustrates another more fundamental idea. Articulating the expression 'the square root of four times four' as 'the square root of four … times four' breaks the expression into two parts, 'the square root of four' times 'four'. The sub-expression 'square root of four' is here seen fluently as an operation to find the square root of four and also as the number resulting from that operation, which is 2. The 'square root of four' is thus dually an operation (or process) and a mental object (or concept), which Gray & Tall (1994) named a procept.

The same happens with all the other expressions, for example 'negative five squared' is either 'negative … five squared' (the negative of five squared), which is –25, or 'negative five … squared' (the square of negative five) which is +25.

Not only do both participants read sub-expressions dually as concepts in their own right that can be calculated as arithmetic processes, they also speak of mentally processing them in their mind to 'see' brackets to clarify the meanings.

The Articulation Principle offers a spoken way to give unambiguous meaning to expressions by translating precise spoken meaning into written symbols, putting brackets around the sub-expressions, indicating which should be evaluated first.

6. Looking to The Future

As the number of operations in an expression increases, the use of the Articulation Principle becomes increasingly complicated. It can give a meaning to relatively simple expressions, such as the two different meanings of $2 + 3 \times 4$, one reading left to right as $2 + 3 \ldots \times 4$ to give 5×4, which is 20, the other as $2 \ldots + 3 \times 4$ to give $2 + 12$, which is 14. But it soon becomes complicated as it deals with more complex expressions.

For this reason it is sensible to adopt conventions to make the symbolism simpler. For example, the convention that multiplication takes precedence over addition allows $2 + (3 \times 4)$ to be written as $2 + 3 \times 4$, while it is still necessary to keep the brackets in $(2 + 3) \times 4$.

Tall (2019) takes these ideas through the whole of symbolic mathematics, in which the duality of symbolism as process or concept plays a fundamental role. Essentially, an expression such as 2 + 3 may be seen as the operation of adding 2 and 3 or as the single object which is the sum 2 + 3. The aim is for the individual to flexibly imagine the two possibilities. It can be illustrated visually by putting boxes round the objects. Then

$\boxed{2}+\boxed{3}$ is the operation of adding the objects 2 and 3

while

$\boxed{2+3}$ is the object which is the sum of 2 and 3.

This notation can be used to visually represent the structure of more general mathematical expressions as part of a long-term development in sophistication. It is a general technique which applies to more sophisticated expressions such as

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

which may be seen as a whole as an object, then as a process of division, with sub-objects which are the numerator and denominator, each of which may be further seen as objects, then processes, recursively moving down the hierarchy to see terms, then sub-terms as object and process (Figure 1).

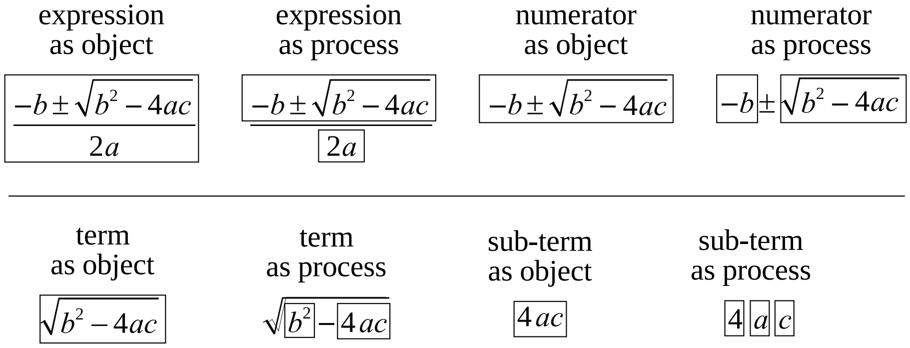


Fig. 1 Sub-expressions as operation or object

This use of boxes to see the duality of process and object is helpful in simple cases. But more complex examples depend on the individual looking at a written expression and imagining sub-expressions being interpreted flexibly as process or object.

This can be seen for a written expression such as $2x^2 + 3x + 7$ by imagining it as the sum of three terms:

$$\boxed{3x^2} + \boxed{2x} + \boxed{5}$$

which can be added together in any order by Piaget's principle of conservation. These terms can be further broken down as

$$3\boxed{x^2} + 2\boxed{x} + \boxed{5}$$

to see it as 3 lots of x^2 plus 2 lots of x plus 5.

It is not necessary to draw the boxes. What is important is to be able to imagine the expression in flexible ways. While successful individuals may do this implicitly, a wider range of students may be helped by explicitly introducing the Articulation principle and encouraging them to focus on long-term supportive principles that give them confidence to address new aspects that may initially be problematic.

In practice, the increasing sophistication of expressions over the long term will be introduced in appropriate stages. The first stage involves counting objects and fingers and building simple arithmetic of whole numbers, to formulate the principle that addition of a list of numbers is independent of the manner in which the sum is calculated. This includes not only different ways of counting, but different layouts, such as 3 lots of 2 giving the same number as 2 lots of 3, extending flexibility from addition of whole numbers to multiplication. Subsequent stages are studied successively in a long-term framework which may be enhanced by curriculum designers and teachers who are aware of long-term supportive concepts that give confidence to the learner and encourages them to explicitly address problematic changes that require reconstruction of knowledge to develop new ways of thinking.

6.1 Implication for teaching and research

The framework offers practical ways to re-think how we as individuals make sense of mathematics and how we can help others to progress in their mathematical thinking. The Principle of Articulation can be introduced at any level to open up discussion about the meaning of mathematical expressions. It can be used with young children or with adults with learning difficulties. It can be helpful to teachers to encourage long-term strategies to grasp the meanings of operations in arithmetic and algebra and it can help experts and curriculum designers to plan for long-term success. By making the Articulation Principle explicit, it can highlight the long-term principles for the operations of arithmetic that lead more naturally to the underlying principles of algebra and offer a supportive basis to build confidence throughout the whole curriculum.

Many of the problems raised in the literature can be linked to the interpretation of symbolic expressions. For example, children first encounter simple operations such as $2 + 3 = 5$ with a process on the left giving a result on the right. In algebra an equation such as $2x + 1 = 7$ with a process on the left and a number on the right can be 'undone' by reversing the process. Meanwhile an equation with expressions on both sides is better understood as having an object on either side expressed in different ways and is solved by 'doing the same thing to both sides' (Tall et al., 2014).

Over the long-term, there are successive transitions as different processes give the same object: counting a set in any way gives the same number, equivalent fractions become a single rational number, algebraic equivalences become the same function, equivalent Cauchy sequences give the same real number. While this is usually interpreted symbolically using the concept of equivalence, it is more meaningfully sensed as different symbolic representations of the same object. When these ideas are embodied on a number line and as graphs in the plane, equivalent fractions are seen as a single point, algebraically equivalent expressions and trigonometric identities give the same graph, and, more generally, infinite limiting processes stabilise visibly on their limit object.

Dienes (1960) advocated working with several different embodiments from which children can construct generalities for themselves. The alternative suggested here is for the curriculum to be planned in such a way that the teacher is aware of the foundational supportive principles to make them explicit for the learners to use them to build confidence to address problematic aspects that impede transition to new contexts.

Embodied ideas of space and shape evolve in sophistication from practical perception and action to theoretical definition and deduction in Euclidean geometry. In the transition to calculus, a problematic conflict arises between embodiment and symbolism. Circle geometry describes a tangent as a line that touches a circle in a single point. This precise definition enables a tangent to a circle to be drawn by constructing a radius from the centre of the circle to the point in question and drawing the tangent at right angles. The definition fails to work for a tangent to a more general curve computed by symbolic differentiation. It can be resolved using embodiment to imagine zooming in on the curve at a point to see the curve as 'locally straight' and give human meaning to the derivative as the slope of the curve itself (Tall, 1985).

In the 1980s, dynamic computer graphics were introduced to give embodied meaning to calculus concepts, together with powerful numeric computation and symbolic manipulation. Now the Articulation Principle and related supportive principles address the interpretation of symbolism to link meaningful conceptual embodiment to meaningful operational symbolism.

A serious problem in the current curriculum is the fragmentation of the whole system dealing with learning by developing expertise in separate parts of the whole: pre-school, early learning, kindergarten, primary, secondary, high school, college, adult learning, university, post-graduate, special needs, gifted and talented, and so on. All of these are essential, but they need to be seen as part of a greater whole, so that different communities of practice are aware of a bigger picture. What happens currently is that learning is broken into stages, with tests to decide who passes on from one stage to another. This can lead to a desire to pass the examination by rote learning, especially when there are problematic aspects involving a change in meaning. Over the longer term, cumulative changes that occur without making meaningful connections are likely to make mathematics more complicated. Long-term success may be enhanced by meaningful connections that compress complex operations into mental objects that can be manipulated in simpler ways in more sophisticated situations.

6.2 Long-term principles for meaningful learning

The Articulation Principle offers an unambiguous meaning to spoken mathematical expressions. Teachers who are aware of it can encourage learners to make long-term strategies explicit to help students to grasp the changing meanings of symbolic operations throughout the whole mathematics curriculum. These include extensions of Piaget's principle of conservation of number and general properties such as the principles that the sum of a list of numbers and the product of a list of numbers are both independent of how they are calculated. They are the foundation of the equal precedence of addition and subtraction and of multiplication and division in the rule P-E-MD-AS or its English equivalent B-I-DM-AS.

Meaningful long-term learning takes account of supportive and problematic aspects of increasingly sophisticated thinking in both embodiment and symbolism. This is enshrined in the earlier-mentioned Simon Principle, which underlies the whole framework as follows:

> The teacher should be aware of those ideas that remain supportive through several changes of context, to give confidence to the learner, and to make explicit those ideas that are problematic so that they can be addressed meaningfully (Tall, 2020, p. 25).

This broader theory uses ideas that are already intuitively familiar, but this does not mean that there is nothing new in their explicit use. Writing about the historical evolution of calculus, which also applies to the evolution of mathematical thinking, Edwards remarked:

> What is involved here is the difference between the mere discovery of an important fact and the recognition that it is important—that is, that it provides the basis for further progress. In mathematics, the recognition of the significance of a concept ordinarily involves its embodiment in new terminology or notation that facilitates its interpretation in investigation (Edwards, 1979, p. 189).

He went on to quote Hadamard, who wrote:

> The creation of a word or notation for a class of ideas may be, and often is, a scientific fact of great importance, because it means connecting these ideas together in our subsequent thought (Hadamard, 1947, p. 38).

The creation of the naming of the *Articulation Principle* opens up the possibility of further progress and subsequent thought in integrating the long-term teaching and learning of mathematics as it grows in sophistication throughout the lifetime of different individuals.

7. Future Research and Development

This specific study involving just two students with similar educational backgrounds clearly has its limitations. However, the reader can test out the practical value of the Articulation Principle. Simply ask a few individuals in different circumstances from young children to adults, 'What is $2 + 2 \times 2$?' You will find that some say '8' because they perform the operations in the given sequence, while some say '6' because they remember the rule that 'multiplication takes precedence over addition.' Now, speak the problem using two different articulations to see whether they make sense of the difference. Further research is possible into the Articulation Principle in different contexts.

Of far greater importance is the wider study of the use of fundamental principles in long-term learning and the transition from one context to another as new concepts are introduced and previously supportive concepts may become problematic.

The Articulation Principle is unusual in that it can be introduced at any stage, in any curriculum once the learners have encountered the operations of counting and simple arithmetic. This may be with young children at the beginning of their encounter with arithmetic, adults who have severe difficulties with mathematics, or individuals at any stage of learning mathematics. If used sensitively, it can offer *meaning* to symbolic expressions in arithmetic and algebra. It offers a precise and accurate interpretation of simple expressions, in particular it offers a *reason* why brackets should be used rather than rote-learning arbitrary conventions.

While many approaches to teaching and learning mathematics focus on positive organisation of the curriculum or research into the errors and misconceptions that arise, the Articulation Principle is part of a much broader long-term approach that balances the positive nature of supportive ideas and the negative possibilities of problematic ideas. This balance shifts the context of learning to wider possibilities, just as the introduction of a balance between positive and negative numbers expands the possibilities of the number system. It offers the possibility of seeing old problems in new ways.

For example, the current US curriculum lacks the meaning of the Articulation Principle in facing the known problems that arise in expressing and solving word problems in arithmetic. The Articulation Principle encourages a focus on the meaning of expressions, giving new ways of interpreting the known problems of transitioning from arithmetic to algebra. Extending the symbolism to calculus, it offers new insights into the transition between the symbolic idea of the process of tending to a limit and the concept of limit as an object. In the wider relationship between embodiment and symbolism, using dynamic interactive software to zoom in on the graph of a function, it opens up the link between the embodied insight of 'local straightness' and the process of differentiation, to see the symbolic derivative as the slope of the graph itself. Meanwhile, the US curriculum for the calculus (College Board, 2016) lists only items that can be tested, with no mention of the *meaning* of the calculus through local straightness. Instead of building up from the student's experience, it builds down from the formal definition by phrasing it in an informal manner that is known to be problematic.

The Articulation Principle opens the door to question the very basis of current teaching practice and mathematical education research, not only in operational symbolism, but in the wider framework of the long-term development of mathematical thinking.

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