Significant Changes in University Mathematics Education

David Tall
University of Warwick, UK
david.tall@warwick.ac.uk

Request from the ICME-14 Survey Team: Research on University Mathematics Education

What do you see as the most significant advances, changes, and/or gaps in the field of research in university mathematics education? These advances, changes, or gaps might relate to theory, methodology, classroom practices, curricular changes, digital environments, purposes and roles of universities, social policies, preparation of university teachers, etc. Please elaborate on just one or two advances, changes, or gaps most relevant to your experience and expertise. If possible, please include a few key references.

1. Response

The question, as posed, already puts a structure on the discussion, requesting a wide range of possibilities and then encouraging a response to focus on one or two aspects. My response is to look at the bigger picture to develop an overall balanced view of positive and negative developments and, within this overall framework, to offer specific examples.

University Mathematics Education should not be seen in isolation. Insights and difficulties that students encounter at university level are already affected by their earlier experiences. The same is true for university professors in different disciplines (such as pure, applied, engineering, biology, economics, computing, etc) who may have radically different approaches that may be appropriate in one context yet entirely inappropriate in another. For example, a pure mathematician may deal with mathematical analysis based on the epsilon-delta definition of limit and the completeness of the real numbers which do not contain infinitesimals, while an engineer may consider infinitesimals as 'arbitrarily small quantities.'

Those present in the ICME-14 working group will probably be mainly concerned with teaching and learning of undergraduate and graduate mathematics. However, it is essential for the participants to be aware of aspects that affect the ways in which differing experts and learners interpret mathematical ideas.

Recent developments in technology have offered us radically new tools to support our mathematical thinking. For example, the introduction of interactive retinal screens on smart phones offers new ways of interpreting what we see. Less obviously for mathematicians and mathematics educators, research into brain structure and brain activity has subtle implications for how we make sense of mathematics. This causes a variety of emotional reactions, including positive willingness to address problems and negative mathematical anxiety that can inhibit coherent thinking. Then there are cultural aspects in which different communities of practice interpret mathematics according to their own shared views that may cause them to be completely unaware of the root causes of positive and negative aspects of mathematical thinking in their own minds and in the minds of their students.

I will report evidence that cultural views of mathematicians and mathematics educators have led to reforms that are not meaningful for many students and will propose broad principles that may be implemented in current approaches to lead to long-term meaningful growth.

The cultural differences between different communities of mathematics is analysed in Tall (2019b) in terms of 'long-term principles for meaningful teaching and learning of mathematics' at university level. Essential aspects of the differences as applied to calculus and analysis are considered in the video Tall (2019c). The theory I present here applies throughout the whole of mathematics, including the full range of mathematics at university level.

The long-term growth of sophistication in mathematical thinking can usefully be seen in terms of three successive levels that I term *practical*, *theoretical* and *formal*. This analysis applies both to the long-term learning of individuals from child to adult and to the evolution of mathematical thinking in different cultures in history.

Pure mathematicians tend to see mathematics in terms of axiomatic definitions and formal proof, while being concerned that intuitive ideas so often fail and are in need of formally defined structure. I offer an analysis based on structure theorems in formal mathematics that lead to more sophisticated forms of visual and symbolic thinking that involve refined forms of intuition. This is supported by research in the structure and operation of the human brain related to mathematical thinking and cultural relationships between different communities of practice.

2. Practical, Theoretical and Formal Mathematics

Practical mathematics involves recognising properties and patterns, performing operations and noting relationships that link together *coherently*, occurring at the same time. Theoretical mathematics involves explicitly formulating definitions that can be used to reason about properties of familiar situations where other properties can be deduced as a *consequence* of the definitions. Formal mathematics involves quantified set-theoretic definition and proof where all properties are deduced from axioms and definitions using formal proof.

Practical mathematics involves sensing general properties. For example, shopping in a supermarket and collecting together a basket of items, the order in which the items are selected, or the prices are added at the checkout, always gives the same total cost. Practical mathematics observes general principles, such as the principle of conservation, that the number of elements in a collection is always the same, no matter how it is counted. This principle extends successively to sums in the context of whole numbers, fractions, signed numbers, decimal expansions, real and complex numbers and to more general properties in algebra.

Theoretical mathematics notices properties, such as the commutative, associative and distributive properties that can be used as a basis for a theoretical approach to arithmetic and algebra. For instance, the property $x^{m+n} = x^m x^n$ which has a practical meaning for whole numbers m and n can be used to derive theoretical properties for fractional and negative powers.

The deduction of more general properties in arithmetic and algebra require more subtle principles, such as proof by induction. Here, the proof is *potentially infinite*: get started at an initial value, say $n = 1$, then prove the general statement that *if* it is true at $n = k$, *then* it is true at $n = k+1$, then, repeat the process for $k = 1, 2, \ldots$, and so on, potentially reaching any desired value of n, no matter how large.

Formal mathematics involves an enormous *compression* of knowledge. Instead of speaking of the potential infinity of the set of counting numbers, think of it *as a single mental object*: the *set* $\mathbb{N}$ of natural numbers. Now write down just *two* axioms:

1. there is a successor function s: $\mathbb{N} \to \mathbb{N}$ which is one-to-one but not onto (so $\mathbb{N}$ has an element, 1, which is not a successor, $s(n) \neq 1$, for any n in $\mathbb{N}$).
2. If S is any subset of $\mathbb{N}$ where $1 \in \mathbb{N}$ and ($k \in S \Longrightarrow s(k) \in S$), then $S = \mathbb{N}$.

A proof by induction is now finite. It has just *three* steps. Let S be the set of n for which a given statement is true:

i) prove $1 \in \mathbb{N}$,

ii) prove $k \in S \Longrightarrow s(k) \in S$,

iii) then quote axiom 2 to deduce that S is the whole of $\mathbb{N}$.

The three levels of practical, theoretical and formal mathematics are clearly *hierarchical* in that practical mathematics begins before theoretical mathematics and most individuals never reach the formal level of mathematics. They are also *cumulative* in that any level includes interchange with previous levels within an individual and between different individuals. The difference in level of interpretation may exist explicitly or implicitly between expert and student or between different experts or differing cultural communities. Theoretical notions involving potentially infinite processes may co-exist with formal notions of axiomatic mathematical objects.

In *How Humans Learn to Think Mathematically* (Tall, 2013), I analysed the cognitive growth of mathematical thinking throughout the curriculum. This involves highly subtle detail.

Now I stand back to look at mathematical growth, not just in terms of what we teach and what students learn in a curriculum sequence, but in terms of *how the meaning of the mathematics changes as new contexts are encountered*. This involves a new way of balancing positive and negative effects of experiences encountered earlier. Some continue to be *supportive* in the new context (such as the general principle that a sum is independent of the order of calculation) and others become *problematic* in the new context (such as the notion of uniqueness of factorization of prime numbers, which changes meaning when shifting from whole numbers to algebraic numbers). The simple principle I advocate is:

> **The principle of long-term meaningful learning**: It is essential for the teacher to be consciously aware of those ideas that remain supportive through several changes of context, to give confidence to the learner, and to make explicit those ideas that are problematic so that they can be addressed meaningfully.

3. Thinking Mathematically through Conceptual Embodiment, Operational Symbolism, Axiomatic Formalism

Many university mathematicians see formal mathematics as the summit of mathematical thinking. It may be the final stage of proving a particular theorem or building a sequence of theorems into a formal theory, but it is not the end of the creative process of developing new mathematical theory. Fundamentally, we need to be aware of how our thinking becomes more sophisticated through the use of visual dynamic representations and symbolic expressions in advanced forms of mathematics and will offer particular instances in calculus and analysis.

In Tall (2013), I formulated three distinct forms of development of human thinking that I termed 'three worlds of mathematics' that grow in sophistication and are founded on the structure and operation of the human brain. I named these: *conceptual embodiment*, *operational symbolism* and *axiomatic formalism*, abbreviated to *embodiment*, *symbolism* and *formalism* where the context is clear.

Conceptual embodiment began for me in the physical embodiment of Dienes (1960), developing through practical drawing, theoretical definitions and Euclidean proof in geometry, and moving on to visual dynamic representations and enactive gestures in the calculus and to mental thought experiments that may act as a prelude to axiomatic formal proof. Essentially this follows the broad ideas of levels of sophistication inspired by van Hiele (1986).

Operational symbolism focuses on the flexible meaning of mathematical expressions dually representing a process or a mental object (Gray & Tall, 1994), taking account of the precedence of operations. (Initially I called this 'proceptual symbolism' (Tall, 2004) but changed to 'operational symbolism' to include the reality that many individuals learn symbolic algorithms by rote without meaning.)

Axiomatic formalism involves structures defined using set-theoretic axioms and definitions formulated as quantified statements where all other properties of the structures must be deduced from these axioms and definitions using formal proof.

Pure mathematicians often regard Euclidean geometry as the first stage of formal proof. This is consistent with the categorization of Euclidean geometry as theoretical mathematics and set-theoretic mathematics as axiomatic formal mathematics. The difference is that Euclidean geometry builds on mental imagination of visual figures, which may be termed 'natural mathematics' following the historical interpretation of mathematics as 'natural philosophy' before the change to the formal approach of Hilbert (1900) at the turn of the twentieth century.

All three forms of development are relevant to university mathematics. Axiomatic formalism refers to the final precising stage of axiomatic mathematical theories based on set-theoretic definitions and mathematical proof. These emerge as the result of more informal thinking about possible relationships, formulating hypotheses, bouncing off ideas with other mathematicians and deep, thoughtful reflection on how to organise the formal proof.

4. An Example: The Teaching and Learning of Calculus and Analysis

The framework of embodiment, symbolism and formalism evolving through practical, theoretical and formal mathematics applies directly to the long-term teaching and learning of calculus and analysis. This is particularly relevant in recent years as the development of digital technology offers us new tools to make sense of ideas in new ways. For example, the recent international explosion in the use of smart phones with retinal displays offers new ways to look at the graphs of functions to *see* that, as the graph of a differentiable function is magnified, a small part looks less curved until it looks 'locally straight'. Once an individual realises this, it becomes possible to look along the graph to *see* its changing slope. This *embodies* the meaning of the process of differentiation and the concept of derivative. The derivative is now represented by a graph whose value at any point equals the slope of the original function at that point.

At a practical level, the visual embodiment offers a human meaning for the rate of change of a quantity. In a simple case such as $y = x^2$, the slope over a short distance from x to $x + h$ can be calculated symbolically as $2x + h$, and for small values of h, the slope stabilises on the value $2x$. The same technique generalises to powers of x and polynomials.

At a theoretical level, once the property of powers of a variable is generalised to give the power law $x^{m+n} = x^m x^n$, it becomes possible to deduce new meanings for fractional and negative powers and to draw the graphs of 2^x and 3^x. Both are increasing steadily while the slope for 2^x is below the graph of 2^x and the slope of 3^x is above the graph of 3^x, so it is straightforward to seek a polynomial approximation to the value of e between 2 and 3 where the slope of e^x is again e^x. It is also possible to look at the slope functions of $\sin(x)$ and $\cos(x)$ in radians to *see* the slope of $\sin(x)$ is $\cos(x)$ and the slope of $\cos(x)$ is the same as the graph of $\sin(x)$ upside down, so the derivative of $\sin(x)$ is *minus* $\cos(x)$.

By such methods it is possible to give meaning to the standard derivatives and to experience the process in which the *practical slope function* $(f(x + h) – f(x)/h$ gets close to the derivative as h gets small. The *theoretical slope function* involves a change of focus from the *process* of getting close to the derivative $f'(x)$ to the *object* that the process stabilises upon. In analysis, the formal level involves translating the theoretical definition into its set-theoretic epsilon-delta form.

In the cognitive development of calculus, this framework reveals the increasing sophistication from the embodied notion of local straightness to the symbolic notion of local linearity. There is a huge difference between the two. Local straightness is a simple visual idea, that arises as a natural product of how the eye and brain interpret visual information (Tall, 2019b). This can be introduced meaningfully before introducing the more sophisticated notion of symbolic local linearity. AP calculus, as designed by the US College Board (2019), bases its development on local linearity and makes no mention of local straightness or embodiment. This is a serious issue in the teaching and learning of calculus that needs to be addressed.

5. Structure Theorems: from Formalism to Refined Embodiment and Symbolism

While formal proof is the final precising stage of pure mathematics, it is not the final stage of mathematical thinking. Given a formal theory, it may be possible to prove theorems, called *structure theorems*, that give more sophisticated forms of embodiment and symbolism. In the second edition of *Foundations of Mathematics* (Stewart & Tall, 2014) we focus on the role of structure theorems that give more sophisticated embodied and symbolic forms of various axiomatic systems. These include the interpretation of the natural numbers and the real numbers as unique systems represented visually as points on a number line and symbolically as decimal representations. Visual embodiment and symbolic representation can also be proved in other axiomatic systems such as finite dimensional vector spaces, and more general structures such as group theory.

Structure theorems lift the discourse to a more sophisticated level of embodiment and symbolism and, from there, the process may be repeated with the new embodiment and symbolism, suggesting new possible relationships, hypotheses and the quest for even more sophisticated theorems and proof.

Representing such a framework in a two-dimensional picture is necessarily limited. It may be imagined as an upward spiral, moving to successive levels of embodiment, symbolism and formalism, or as a 'folding back' of formalism to more sophisticated forms of embodiment and symbolism (Figure 1).

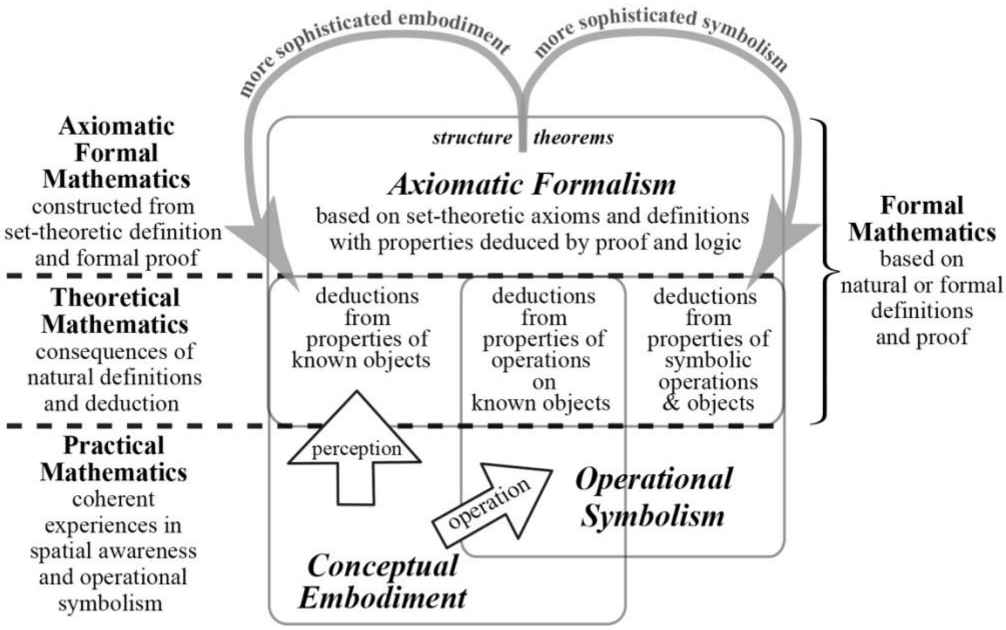

Figure 1: Practical, theoretical and formal mathematics
moving through the worlds of embodiment, symbolism and formalism

As an example, there is a simple structure theorem that takes us beyond the real number system represented visually on the number line. It reveals that the completeness axiom (that every non-empty subset bounded above has a least upper bound) not only tells us that the real number line consists only of infinite decimals and cannot contain infinitesimals that are 'arbitrarily small, but not zero'. The very same axiom proves that any ordered extension field of the real numbers *must* contain infinitesimal quantities.

The rational number system does not contain the irrational number √2, but this lies in the larger ordered field of real numbers which again can be represented on a number line. By the same token, the completeness axiom can be used to prove that *any* ordered field K that contains the real numbers as an ordered subfield *must* contain infinitesimals.

Elements of the real numbers ℝ will be referred to as 'constants' and elements of the larger system K as 'quantities.' A quantity x is said to be *finite* if it lies between two real numbers a, b, so that, in the ordering of K, we have $a < x < b$. A quantity ε is said to be an *infinitesimal* if $\varepsilon \neq 0$ and $-a < \varepsilon < a$ for every positive real number a.

It is then straightforward to prove:

Structure Theorem for any ordered field extension K of the real numbers ℝ.
Every finite quantity is either a real number or a real number plus an infinitesimal.

The proof considers any finite quantity x and the set of real numbers, $L = \{t \in \mathbb{R} \mid t < x\}$. This is non-empty (because it contains a) and bounded above by b, so it has a unique least upper bound $c \in \mathbb{R}$. Let $\varepsilon = x - c$, then, by a contradiction argument, it can be proved that ε is either zero or infinitesimal. The unique real number c is called the *standard part* of x, $c = \text{st}(x)$. Infinitesimal detail for a quantity t near x can then be visualised using the linear map

$$m(t) = (t - c)/\varepsilon$$

called *the ε-microscope pointed at c*. The subset V of quantities where $(t - c)/\varepsilon$ is finite is *the field of view* of the microscope.

In Tall (1992), I defined the *optical ε-microscope pointed at c* as $\mu: V \to \mathbb{R}$, given by

$$\mu(t) = \text{st}(m(t)) = \text{st}(t - c)/\varepsilon).$$

For a real number k, $\mu(c - k\varepsilon) = k$, so the optical microscope maps the field of view onto the whole real line. More detailed information can be found in Stewart and Tall (2014).

This can be generalised to multiple dimensions by using an optical microscope on each coordinate, for instance in two dimensions the (ε, δ)-microscope pointed at (c, d) is

$$\mu(s, t) = (\text{st}((s - c)/\varepsilon, \text{st}(t - d)/\delta).$$

For example, if infinitesimals ε, δ are taken to be equal, this gives an infinite magnification of a differentiable function $y = f(x)$. Using the convention that the images of points on a map are denoted by the same name as the original point in K^2, we can name the image as (x, y) and its change as the vector (dx, dy) where $f'(x) = dy/dx$.

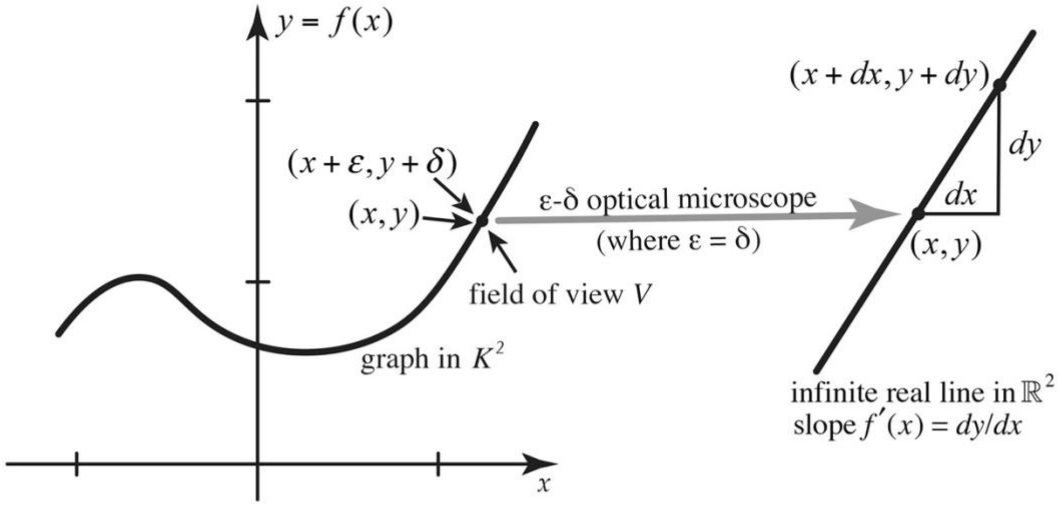

Figure 2: Infinite magnification of a locally straight graph to see finite detail as a real picture

This insight belongs in the formal world of analysis, not in the practical and theoretical worlds of the calculus. It is introduced to show that an 'infinitesimal' arises as a structural extension of formal analysis and should not be disparaged as mathematically unsound. Different cultures have differing modes of operation appropriate for their needs. Legitimate approaches to the calculus can be based on the real numbers with or without infinitesimals.

6. Relationships between Mathematical Cultures

The relevance of different cultural approaches for different cultural communities was formulated by the mathematician Raymond Wilder (1968, 1981). He described the anthropological term 'culture' as:

> A collection of customs rituals, beliefs, tools, mores, and so on, called *cultural elements*, possessed by a group of people who are related by some associative factor (or factors) such as common membership in a primitive tribe, geographical contiguity, or common occupation. (Wilder, 1968, p. 18.)

He speaks of *cultural stress* that occurs when there is a need in the culture to be satisfied, such as making sense of a new mathematical context. Cultures benefit from shared elements that are stable and useful. These may *diffuse* from one culture to another, but this is likely to take time to do so, called *cultural lag*, and may even involve *cultural resistance* if the new element challenges current elements that are considered to operate successfully. To balance elements that resist change, I add the notion of *cultural stability* that seeks to maintain familiar elements that allow the culture to continue to operate in a shared coherent manner. This is essential to my argument as I wish to engage with individuals and cultures that interpret mathematics in different ways. Instead of using the negative idea that some viewpoints are culturally resistant to change, I note that they may be seeking the positive virtue of maintaining cultural stability.

Different cultures evolve different ways of working that are highly relevant to their own needs. That same approach may be inappropriate for others. Data collected in major studies may be usefully interpreted from different viewpoints to offer insight into how implicit beliefs in different communities may cause difficulties in teaching and learning.

An example that has proved helpful for me is the major MAA report on College Calculus (Bressoud, Mesa, Rasmussen, 2015). This gathers comprehensive data on the teaching of calculus in the USA at all types of institution from two-year colleges to PhD-granting universities. It begins with the following statement:

> Calculus occupies a unique position as gatekeeper to the disciplines in science, technology, engineering, and mathematics (STEM). At least one term of calculus is required for almost all STEM majors. For too many students, this requirement is either an insurmountable obstacle or—more subtly—a great discourager from the pursuit of fields that build upon the insights of mathematics.

It goes on to discuss the Calculus Reform movement that sought to remedy the problem and comes to the conclusion, "Many decades later, we seem to have made little progress." It also reports that AP calculus has grown substantially, so that around three-quarters of all calculus students take their first calculus course in high school. Given that the College Board AP calculus syllabus (2019) has assessment only using multiple-choice tests and does not mention any use of meaningful embodiment or local straightness, it is no wonder that there is little progress in improving calculus teaching and that difficulties continue in university mathematical analysis.

Intuitive ideas in analysis often prove false, so it is natural for mathematicians to culturally resist ideas based on intuitive embodiment. However, embodiment plays a major role in making sense of new ideas which is fundamentally more humanly meaningful than theoretical definitions and deductions. Consider, for example, the meaning of the operation of taking a fraction of a quantity. Operations such as $^{2}/_{4}$, $^{3}/_{6}$ are considered to be 'equivalent' fractions. But, when embodied as points marked on a line, they are *the same point*. Likewise, $2(x + 3)$ and $2x + 6$ are different processes but, embodied as a graph, they are *the same graph*.

Embodied representations allow an enormous compression to think about multiple processes as a single object, offering a far simpler way to contemplate more sophisticated ideas.

The notion of 'local straightness' is an aspect of embodiment that acts as a meaningful foundation of the more sophisticated development of mathematical analysis, such as the theory of multi-dimensional manifolds that are 'locally Euclidean'. In the long-term, local straightness generalises mathematically to more formal notions in real and complex analysis in differentiation, differential equations, partial derivatives and multi-dimensional vector analysis.

The current curriculum is constructed from the formal limit definition of the derivative re-formulated as an informal process getting as close as is desired to the derivative, without being related to a practical embodied meaning. Culturally, the AP calculus curriculum is a massive document, created by a committee, listing all kinds of detail required from different perspectives. Where is the simple insight to make sense from the viewpoint of the learner? As Hilbert said in his plenary presentation to the International Congress of Mathematicians in 1900 when he presented his famous list of problems for the twentieth century:

> An old French mathematician said: "A mathematical theory is not to be considered complete until you have made it so clear that you can explain it to the first man whom you meet on the street." This clearness and ease of comprehension, here insisted on for a mathematical theory, I should still more demand for a mathematical problem if it is to be perfect; for what is clear and easily comprehended attracts, the complicated repels us. (Hilbert, 1900, p. 407)

Surely it is simpler to encourage students to begin their study of the calculus by looking closely at a graph that is locally straight to *see* its changing slope function, rather than offer them and their teachers the daunting compendium of ideas listed in the AP calculus curriculum.

7. The Meanings of Concepts in the Calculus

The calculus is encountered by learners who have studied mathematics for a decade or so. The example of differentiation through local straightness is only part of the story.

Traditionally, the concept of continuity, in terms of drawing a curve dynamically on a piece of paper without lifting the pencil may be used as an intuitive introduction to formal idea. However, in a traditional approach the two meanings are not coherently linked. This is a particular problem for communities that may wish to make some sense of integration and the fundamental theorem without the technicalities of formal analysis.

I offered a solution back in the 1980s by using a different magnification factor on the horizontal and vertical axes (Figure 3).

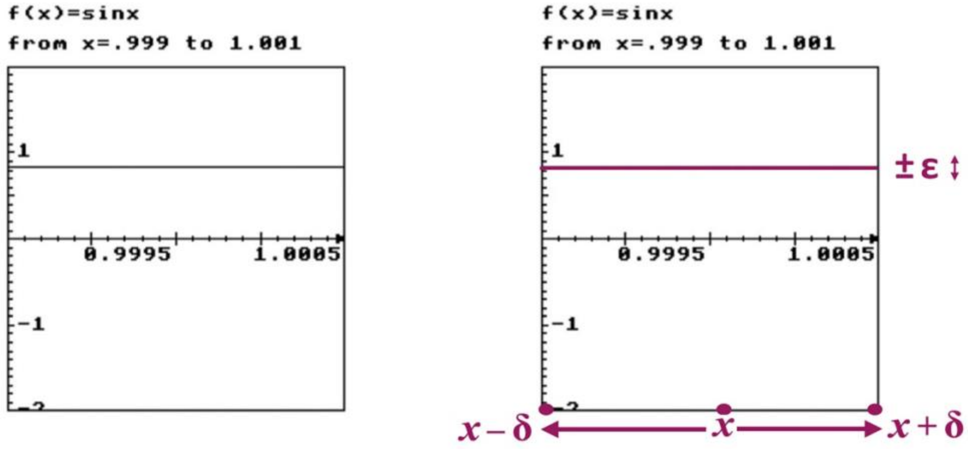

Figure 3: stretching a graph horizontally while maintaining the vertical scale (Tall, 1986).

On the left is a screen dump from my Graphic Calculus software and on the right is a dynamic embodied interpretation of its meaning. The horizontal line is a line of pixels over a point x on the x-axis in the centre of the box window where the pixel line is centred vertically on $f(x)$ and lies in the vertical range $f(x) \pm \varepsilon$. To 'pull the graph flat', requires us to satisfy the following:

> Given $\varepsilon > 0$, we need to find a $\delta > 0$, so that when the graph is pulled horizontally between $x - \delta$ and $x + \delta$, then it lies within the pixel height $f(x) \pm \varepsilon$.

This links directly to the formal definition of continuity:

> Given any $\varepsilon > 0$, there is a $\delta > 0$, such that
> if t satisfies $x - \delta < t < x + \delta$ then $f(x) - \varepsilon < f(t) < f(x) + \varepsilon$.

The embodied idea of 'pulling flat' now leads naturally to the formal definition of continuity. Unlike the traditional approach to the calculus, for the first time, the embodied sense of drawing a continuous curve provides a foundation for the formal theory.

Of course, there are subtleties that need to be addressed at some stage. The formal definition starts by fixing x and interpreting the definition as *continuity at a point*, then varies x over a domain D to speak of *(pointwise) continuity over a domain*. Then there are other possibilities. Is the domain *connected?* Is it *compact* (to lead to the notion of *uniform continuity*)? Is the number system *complete* (in one of several different formulations)? Is the number system a *continuum* (meaning that a moving point on the number line changes imperceptibly in some sense)? There are so many variants in subtlety of meaning that a pure mathematician can build a whole career out of proving various possible theories with subtle changes in the formal definition. A mind full of so many possibilities needs to reflect very carefully on how to present fundamental ideas of the calculus to a learner.

Today's interactive retinal displays offer new ways of looking at the notion of continuity and its relationship with integration by using a different magnification factor on the horizontal and vertical axes. In the design of the TI-92, at my suggestion, different scale factors were introduced with the intention of supporting the approach visualised in figure 3, but this did not lead to a significant change in the syllabus.

Now that the display on a smart phone has been improved to 'retinal' level, I imagine a new app as represented in figure 4, which I unapologetically call a 'Tall-scope'. A vertical strip can be selected in the left box and simultaneously stretched horizontally in the right box. As the value of x is moved to the left or right, the horizontal line will move up and down.

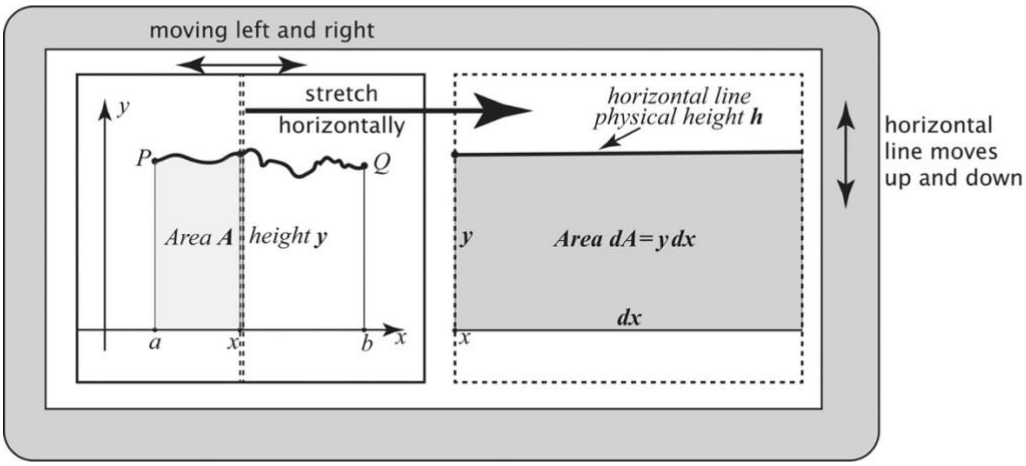

Figure 4: Stretching a continuous graph horizontally to see in 'pull flat' (Tall, 2019a, p.19)

Martin Flashman has produced a useful prototype using Geogebra (see Tall, 2019a, p.20). What would be useful is an app that offers equal magnification on both axes (to deal with local straightness and differentiation) or horizontal stretching (to deal with continuity, integration and the fundamental theorem). To my current knowledge, such an app does not exist. It would provide a supportive embodied environment for local straightness. However, horizontal stretching changes the look of the area $y\,dx$, giving subtle problems of interpretation of the embodiment. The solution lies in noting that, while the visual area changes, the symbolic numerical calculation remains the same. Even so, this interpretation may cause subtle difficulties for some learners.

Others may benefit from the approach offered in my approach in Graphic Calculus in the eighties which not only gave an embodied visual interpretation of differentiability and continuity, it also presented a visual example of a function that is continuous everywhere but differentiable nowhere. This is supported by an embodied explanation because the function is built up by sequentially by adding successive half-size saw-teeth: repeated magnification reveals smaller saw-teeth so it is nowhere locally straight (Tall, 1982). By the fundamental theorem, the integral of this continuous function is differentiable once but not twice and, by successive integration, it is theoretically possible to construct a function that is differentiable n times where its nth derivative is continuous everywhere and differentiable nowhere.

This proved interesting for teachers in England, especially at a time when there was excitement generated by drawing beautiful fractal pictures generated using simple iteration. It fell out of use because the syllabus was designed to give students practice using calculus techniques where such strange functions were irrelevant. Instead, even in university analysis, 'non-differentiable functions' were usually limited to examples such as $x\sin(1/x)$ and $x^2\sin(1/x)$ which are given by a formula with just an isolated problematic point.

The mental imagery generated by these limited experiences suggest a culturally shared notion of discontinuity that only occurs at isolated points. Cultural stability maintains such examples from historical development while impeding more sophisticated meanings.

Future evolution of ideas depends on considering different possible meanings. My doctoral supervisor, Michael Atiyah, jointly received the Fields Medal and the Abel Prize by proving the Atiyah-Singer Index Theorem relating analytic and topological characterisations of solutions of certain differential equations, saying:

> Any good theorem should have several proofs, the more the better. For two reasons: usually, different proofs have different strengths and weaknesses, and they generalize in different directions— they are not just repetitions of each other. And that is certainly the case with the proofs that we came up with. There are different reasons for the proofs, they have different histories and backgrounds. Some of them are good for this application, some are good for that application. They all shed light on the area. If you cannot look at a problem from different directions, it is probably not very interesting; the more perspectives, the better!
>
> (Atiyah, quoted from Raussen & Skau, 2004, p. 24)

8. Towards a more comprehensive framework for meaningful mathematical thinking

The idea of seeking many perspectives is implicit in developing a theoretical framework for mathematical thinking.

Mathematics evolves over time to develop more sophisticated ideas that simultaneously make the theory simpler, as explained by Atiyah:

> If we have to start from the axioms of mathematics, then every proof will be very long. The common framework at any given time is constantly advancing; we are already at a high platform. If we are allowed to start within that framework, then at every stage there are short proofs.
>
> One example from my own life is this famous problem about vector fields on spheres

> solved by Frank Adams, for which the proof took many hundreds of pages. One day I discovered how to write a proof on a postcard. I sent it over to Frank Adams and we wrote a little paper which then would fit on a bigger postcard. But of course that used some K-theory; not that complicated in itself. You are always building on a higher platform; you have always got more tools at your disposal that are part of the lingua franca which you can use. In the old days you had a smaller base: if you make a simple proof nowadays, then you are allowed to assume that people know what group theory is, you are allowed to talk about Hilbert space. Hilbert space took a long time to develop, so we have got a much bigger vocabulary, and with that we can write more poetry. (Atiyah, quoted from Raussen & Skau, 2004, p. 29)

The same evolution is necessary for the development of a theoretical framework for long-term mathematical thinking using information from different areas of expertise. It will take time and deep reflection to develop a 'lingua franca' that expresses the theory in a way that the essence of these disparate ideas can be made available to others:

> The passing of mathematics on to subsequent generations is essential for the future, and this is only possible if every generation of mathematicians understands what they are doing and distills it out in such a form that it is easily understood by the next generation. Many complicated things get simple when you have the right point of view. The first proof of something may be very complicated, but when you understand it well, you readdress it, and eventually you can present it in a way that makes it look much more understandable—and that's the way you pass it on to the next generation! Without that, we could never make progress.
> (Atiyah, quoted from Raussen & Skau, 2004, p.28)

To make progress in the teaching and learning of university mathematics using a range of different forms of expertise, we need to distill the essential ideas in a form that is meaningful for the evolving university mathematics community. In the next section I will summarise the aspects that have been discussed so far and outline others with the potential to offer insight into the evolution of ideas.

9. Essential elements of the framework in progress

So far, I have outlined a number of aspects of the long-term meaningful development of mathematical thinking, based on the increasing sophistication of mathematical ideas, including

- Practical, theoretical and formal evolution of sophistication,
- Conceptual embodiment, operation symbolism, axiomatic formalism,
- Structure theorems to build from formal theory to more sophisticated embodiment and symbolism,
- Cultural evolution, diffusion, lag, resistance and stability.

In recent times I have moved on to consider a range of other aspects that have much to offer in the improvement of long-term meaningful mathematical thinking. Some involve subtle aspects of brain structure and operation that have a substantial effect on the way different individuals think mathematically. Others involve new ways in which we can make sense of mathematical expressions that we can manipulate flexibly in our minds. Some of these ideas are simple to observe and explain for teachers and learners and, even more importantly, can be introduced into current practices in teaching and learning to improve long-term flexible thinking at all levels, in school, and in undergraduate, graduate and research mathematics. These will be outlined briefly in the following sub-sections prior to summing up the implications for the evolution of future teaching and learning of mathematics at university level.

9.1 Embodied foundations

An essential part of the earlier discussion in this chapter is the importance of embodiment in the long-term. While symbolic mathematics offers procedures for calculation and manipulation and formal mathematics codifies mathematics into a structured framework, embodied mathematics gives human *meaning* to fundamental ideas. At university level, intuitive ideas often prove to have subtle exceptions. There is a clear need for a refined interpretation of intuition that offers forms of justification and proof appropriate for each particular community. However, there are two essential reasons why embodiment should be seen as an essential foundation. First, the embodied pictures of successive number systems enable them to be seen as successive points on a number line and even in the complex plane, including real and complex infinitesimals as processes or as objects. Second, the notion of structure theorem takes formal mathematical theory on to more sophisticated forms of symbolism and embodiment.

9.2 Reading, speaking and hearing expressions: the principle of articulation

When we read text or mathematical expressions (left to right in most Western languages), the eye does not do so smoothly. Only a small part of the retina, called the fovea, has sufficiently accurate vision to take in detail and this is only around two hundred photoreceptors in diameter The eye focuses momentarily on a piece of detail to take in information, then jumps rapidly to the next position to focus on the next piece. You can sense this for yourself by reading this paragraph and become aware of the jumps (called saccades).

The meaning of an expression such as 2 + 2 × 2 can be changed by the manner in which it is spoken or heard, depending on the articulation. Leaving a small gap after 2 + 2, denoted by an ellipsis '…', the expression

2 + 2… × 2 can be interpreted as 4 × 2, which is 8,

while

2 + … 2 × 2 can be interpreted as 2 + 4, which is 6.

This distinction can be understood by almost anyone, yet, the standard convention, that 'multiplication takes precedence over addition', given as a 'rule' that makes 2 + 2 × 2 equal to 6, is highly confusing because it contravenes how we naturally interpret the operations in sequence, performing the operation '2 + 2' before the operation '× 2'.

This may be formulated as:

> **The Articulation Principle**: The meaning of a sequence of operations can be expressed by the manner in which the sequence is *articulated*. (Tall, 2019a, p.14)

This is not a mathematical definition, nor does it explicitly say how the sequence of operations should be performed in a mathematical expression. However, it unlocks a principle that enables the learner to articulate expressions in different ways to *give meaning* to the use of brackets to reveal the sequence in which operations should be performed. More significantly, it can be shown to generalise the meaning of operational symbolism throughout the whole of mathematics as outlined in several recent papers on my website (see, for example, Tall, 2019a, 2019b, 2020a, 2020b).

Early practical experience can be used to recognize the principle that the sum of a collection of numbers is independent of how it is calculated, and a similar principle applies to the result of multiplying a collection of numbers. This can be extended to more complicated symbolic expressions taking into account the order of precedence of operations. The more general situation can be given meaning by a flexible interpretation of sub-expressions as processes (operations) or concepts (objects).

9.3 Symbols as operations and objects: parsing mathematical expressions

My original approach to the calculus in terms of local straightness had the advantage of giving embodied meaning to the changing slope of the graph of a function. However, it did not extend to giving meaning to the interpretation and manipulation of the symbolism. Recent developments offer new ways of parsing mathematical expressions by seeing sub-expressions as mental operations or mental objects. This can be seen by introducing a simple notation using the duality of symbolism as process (or operation) and concept (mental object) based on the procept theory of Gray & Tall (1994). An expression such as 3 + 2 can be considered either as an operation (addition) to be performed in various ways in time, or as a mental object (the sum). This can be notated by placing an object in a box so that

$\boxed{3} + \boxed{2}$ is the process of addition of the numbers 3 and 2

$\boxed{3 + 2}$ is the object, the sum 3 and 2.

This idea extends to all operational expressions such as 2^3 which can be written as $\boxed{2}^{\boxed{3}}$ or $\boxed{2^3}$ and more general expressions written spatially using TeX or MathType. Over the long term, it is important to build the meaning in simple stages, starting with simple addition, subtraction and multiplication of whole numbers. The articulation principle already alerts the learner to different meanings of expressions such as $6 - 3 - 2$ and $2 + 2 \times 2$. It later applies to powers such as -3^2, which can be interpreted as $(-3)^2$ or as $-(3^2)$.

Over the longer term, the principle of long-term meaningful learning can use the supportive ideas that the sum and the product of collections of numbers are both independent of the order of calculation from whole numbers, through fractions, signed numbers, real and complex numbers and also generalised to expressions in algebra. These supportive ideas have the potential to offer a sense of stability and security, allowing problematic changes in meaning to be considered explicitly to address new meanings required in new contexts.

These ideas, including handling operations with different orders of precedence, are considered in more detail in previously mentioned papers (Tall, 2019a, 2019b, 2020a, 2020b). They are still under development and require further distillation to handle situations encountered in different cultural settings. In particular, I do not advocate teaching the use of boxes in a procedural way because the technique soon becomes over-complicated. The focus should always be on simplifying the *meaning* of the mathematics to be able to imagine the parsing of sub-expressions as process or object.

9.4 How the eye follows a moving object and concepts of constants and variables

When the eye follows a moving object, it jumps in a single saccade to focus on the object, then moves smoothly with the object as it moves. You can test this for yourself by placing a finger a short distance away from your eye and moving it from side to side. It doesn't make any difference if you hold your head still and turn your eye or if you turn your head to follow the movement, you can still keep your moving finger in focus while the background is blurred.

Now imagine a moving point marked as a blob on a line on a retinal display. Your eye *sees* a point moving smoothly and you can imagine the distinction between a constant point and a variable point. The notions of 'constant' and 'variable', including the idea of a variable that can get arbitrarily close to a constant, are built into your working brain. By mentally 'zooming in', you can imagine a variable getting as close to a given constant as desired.

The structure of the human brain has had little time in evolutionary terms to change substantially in the time that mathematical thinking has developed over the last few thousand years. This offers an alternative way of analysing the historical development of mathematical thinking in terms of the effects of the structure and operation of the human brain.

9.5 Language and mental links with space and number

Language is essential for the development of more sophisticated theoretical ideas. It enables us to *name* ideas, to formulate its *properties* and to talk about *relationships* with other ideas. Without language there would be no sophisticated mathematics. Yet Dehaene and his colleagues assert that mathematical thinking does not build substantial links with the areas of the brain that deal with language:

> By scanning professional mathematicians, we show that high-level mathematical reasoning rests on a set of brain areas that do not overlap with the classical left-hemisphere regions involved in language processing or verbal semantics. Instead, all domains of mathematics we tested (algebra, analysis, geometry, and topology) recruit a bilateral network, of prefrontal, parietal, and inferior temporal regions, which is also activated when mathematicians or nonmathematicians recognize and manipulate numbers mentally. Our results suggest that high-level mathematical thinking makes minimal use of language areas and instead recruits circuits initially involved in space and number. (Amalric & Dehaene, 2016)

The methodology for this research uses an fMRI scanner to measure the flow of magnetised blood that only registers changes of at least two seconds or so. While this conclusion questions the links between a mnemonic such as PEMDAS ('Please Excuse My Dear Aunt Sally') and the order of operational precedence

> Parentheses, Exponents, Multiplication/Division, Addition/Subtraction,

it does not have the resolution to distinguish essential changes in mathematical thinking where links are made in around 40 milliseconds.

It is not necessary to have expensive equipment to be aware of subtle changes in meaning that give rise to emotional changes. These can be observed by teachers and learners in their everyday mathematical activity. They arise as a result of the structure and operation of the human brain through linking cognitive activity to emotional response.

9.6 Mathematics and emotion

In the centre of the brain is a collection of structures called 'the limbic system' (from the term 'limbus', meaning 'border'), bridging diverse connections between brain activity and bodily function. The limbic system reacts subconsciously to incoming data before the conscious forebrain has time to receive information that passes to the hindbrain to be interpreted and forward to the forebrain to take decisions. This gives rise to the 'thinking fast, thinking slow' phenomenon (Kahneman, 2011) where an initial intuitive reflex reaction occurs before a more reflective decision process. Under stress, this causes to a 'fight or flight' reaction affecting the whole human system, flooding the brain with neurotransmitters and the body with hormones that either heighten or suppress activity. Neurotransmitters may set the mind on alert to enhance thinking processes, or suppress connections, causing mathematical anxiety that interferes with mathematical thinking. Meanwhile, hormones affect autonomic functions such as maintaining body temperature, blood pressure, heart rate and so on, and the resulting physical sensations may be felt by those affected and seen by sensitive observers.

This is particularly important when there is a change in context, both in the change in topic for a learner and in the cultural difference between two mathematical communities.

9.7 A meaningful overview of change in context

The whole ethos of this chapter is that long-term mathematical learning depends on compressing ideas to sophisticated concepts that can be easily manipulated in the human mind.

It is an advantage to be able to carry out increasingly sophisticated procedures as the mathematics becomes more complex, but this is not sufficient. The principle of long-term meaningful learning emphasises the importance of identifying ideas that are supportive through several changes of context, to give the learner confidence to address problematic aspects that need to be modified to be meaningful in a new context.

The transition of an individual to a new context and the cultural transition between communities follow a common pattern (Tall, 2019b). When a new context is encountered with problematic aspects that do not make sense, this gives rise to an *impediment* that can be interpreted as a boundary that cannot be crossed. In the case of differing communities, such as different religions, the possibility of crossing the boundary from community A to community B may be seen as a *transgression* by those in A, but those in B may consider it as an *enlightenment*. This may occur in the transition between school mathematics and formal mathematics at university where university mathematicians seek to enlighten students to the formal viewpoint that many students see as a boundary that they are unable to make meaningfully, and so resort to rote-learning. To seek an appropriate resolution requires a reflective *overview* to appreciate both viewpoints (Figure 5).

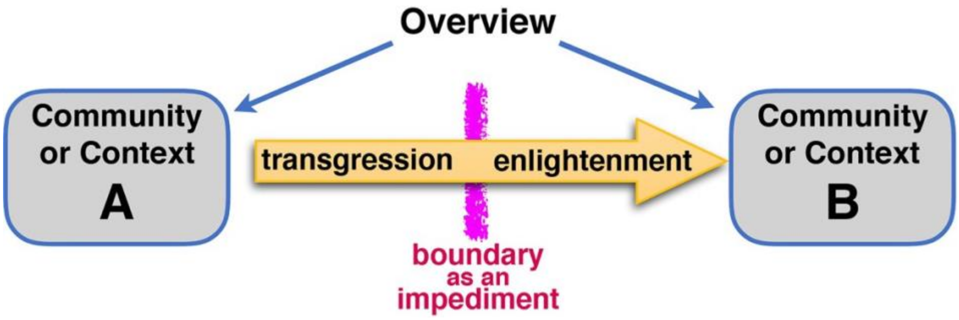

Figure 5: Transition between contexts for individuals and communities (Tall 2019b)

In the case of two communities, the possibilities may be characterised as:

- Impediment: *inability* to leave the current community to cross over a boundary
- Transgression: crossing *out of* the current community over a boundary
- Enlightenment: crossing *into* a new community over a boundary
- Overview: encouraging communication *between* communities.

Such differences may exist between pure and applied mathematics, between mathematicians and educators, between politicians who prescribe the curriculum, curriculum designers, teachers and assessors, or between different levels of teaching in early learning, primary, secondary, university and different theoretical areas in mathematics.

For an individual seeking to make a change in context, the possibilities are:

- Impediment: *inability* to change context
- Transgression: *unwillingness* to change context
- Enlightenment: *ability* to change context
- Overview: ability to switch *between* contexts.

Examples include generalising number systems from counting numbers to fractions, to signed numbers, to rational numbers, reals, complex numbers, from arithmetic to algebra, from practical drawing to Euclidean proof, from school mathematics to university, and so on.

Of particular importance is the ability to switch *between* contexts. For example, to switch between whole number contexts with prime numbers and uniqueness of factorization to algebraic numbers where factorization has a different definition and different properties.

9.8 Making sense of mathematical proof in undergraduate mathematics.

Different contexts have different properties and different communities in a complex society have differing needs. The question arises as to what kind of proof is appropriate for each community.

One method that has widespread implications in encouraging students to make sense of proof is the notion of 'self-explanation'. Using eye-tracking techniques, Hodds, Alcock & Inglis (2014) confirmed that undergraduates devoted more of their attention to parts of proofs involving algebraic manipulation and less to logical statements than expert mathematicians. They developed materials to encourage 'self-explanation' by reading a proof line by line, to identify the main ideas, get into the habit of explaining to themselves why the definitions are phrased as they are and how each line of a proof follows from previous lines. Students were counselled not to simply paraphrase the lines of the proof by saying the same thing in different words, but to focus on making connections to grasp the main argument and explain how the given assumptions and definitions in previous lines led to the current line and contribute to the following lines. This led to a significant improvement in subsequent reading of proofs.

'Self-explanation' is relevant to different cultural approaches, not only in different specialisms such as pure mathematics and engineering, but also between differing preferences within a specialism. The framework of three worlds of embodiment, symbolism and (axiomatic) formalism suggested a distinction between *natural* mathematics (based on theoretic origins in embodiment or symbolism) and *formal* mathematics; an individual may have preferences that use any combination of these. This leads to a variety of different approaches, depending on the individual and the specialism. For instance, the theory of whole numbers and of real numbers both study a system which is unique up to isomorphism, whereas the theory of groups has many different systems satisfying the group theoretic axioms, some of which may be embodied as groups of transformations or symmetries of a set or classified in terms of generators and relations.

The long-term framework of meaningful mathematical development proposed here values the use of embodiment to give an initial human meaning to mathematical theories which later develop into formal axioms, definitions and proofs. Our fundamental embodied experience arises from our lives in three-dimensional space where time moves inexorably forward and not backwards. However, we now have tools such as videos which allow us to reverse time by playing the video in reverse, allowing us to see a vehicle moving backward reversed in time to see it moving forward, so the product of two negative quantities is positive.

We also record our ideas on two dimensional paper which is static, or on a two dimensional visual display which can be programmed to change dynamically. A two-dimensional static display such as figure 1 which represents the interaction of embodiment, symbolism and formalism with the increase in sophistication through practical, theoretical and formal mathematics is limited because it does not include any explicit reference to emotional or cultural aspects. This is particularly relevant when these aspects suggest serious flaws in a particular area of study, such as the teaching and learning of calculus.

The proof of structure theorems moves formal mathematics on to cycles of more sophisticated embodiment and symbolism leading to even more sophisticated formal theory. This requires an awareness of forms of embodiment that can be used as a basis for more formal development and of those aspects of limited embodiment that need to be modified to support formal development.

10. Implications for Future Teaching and Learning of Mathematics at University Level

At this point I return to the original request from the ICME-14 Survey Team concerning the significant advances, changes and or gaps in the field of university mathematics education in recent years. My concern is that the discussion will focus on specific aspects of teaching and learning rather than seeking a bigger picture. It will provide an opportunity for individuals to report their specific experiences and even to compare a range of different aspects.

However, there are broader issues involved which require an overview to be able to recognize that different parts of a complex society require approaches that are appropriate for their own needs but which may seem to conflict with the needs of others. This chapter offers a broader analysis, revealing evidence that current approaches are affected by human thinking processes and cultural aspects that are not being taken into account.

I have proposed an overall *principle of long-term meaningful growth* of mathematics that builds on the natural structure and operation of the human brain and the changes in meaning of concepts that occur as the mathematics grows more sophisticated and shifts to new contexts. I also offer new ways of making meaningful sense of long-term growth of mathematical thinking that can be implemented into today's experiences of teaching and learning. These include the *principle of articulation* and the *meaningful interpretation of mathematical expressions* by seeing sub-expressions flexibly as processes or objects.

At university level, I use *structure theorems* to reveal formal systems have more sophisticated forms of embodiment and symbolism to develop even more sophisticated formal structures.

Mathematical research benefits by having different forms of mathematical thinking that opens up new ways of formulating and solving mathematical problems. Advances are made by encountering conflicts and looking at situations in different ways.

New tools offer new ways of thinking, such as smart phones with the internet to communicate ideas around the world and new ways of performing and representing dynamic mathematical ideas. After billions of years of development of the universe, millions of years of evolution of life on earth and a few thousand years of human civilisation, in a single lifetime we have experienced epoch-making changes that occur so fast that society has insufficient time to grasp their implications.

At such a time, it is essential to focus on those aspects that remain relatively stable to provide a foundation for building new ideas. In particular:

> Mathematics is an evolution from the human brain, which is responding to outside influences, creating the machinery with which it then attacks the outside world. It is our way of trying to reduce complexity into simplicity, beauty, and elegance.
>
> (Atiyah, quoted from Raussen & Skau, 2004, p.26)

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