From Biological Brain to Mathematical Mind: The Long-term Evolution of Mathematical Thinking

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1. Introduction

In this paper we consider how research into the operation of the brain can give practical advice to teachers and learners to assist them in their long-term development of mathematical thinking. At one level, there is extensive research in neurophysiology that gives some insights into the structure and operation of the brain, for example, magnetic resonance imagery (MRI) gives a three-dimensional picture of brain structure and fMRI (functional MRI) reveals changes in neural activity by measuring blood flow to reveal which parts of the brain are more active over a period of time. But this blood flow can only be measured to a resolution of one or two seconds and does not reveal the full subtlety of the underlying electrochemical activity involved in human thinking which operates over much shorter periods.

Here we use available information about the brain to consider aspects of mathematical thinking that can be observed by teachers and learners. For example, by understanding how the brain interprets written text and hears spoken words, it becomes possible not only to reveal *why* individuals have difficulty in making sense of expressions in arithmetic and algebra but also *how* sense making can be improved at every level from the full range of young children to the varied needs of adults. One possibility involves noticing aspects that are intuitively grasped by more successful thinkers that give them advantage and introducing these insights explicitly to improve mathematical sense making for the broader population.

Another aspect relates to the difference between the way that the eye reads text and follows moving objects. This offers fundamental insight into human perceptions of constants and variables that are foundational in the calculus. At a higher level of abstraction there is the manner in which a written proof may be scanned by someone attempting to make sense of it. These diverse insights are used to build a coherent picture of how the biological brain can develop into a mathematical mind capable of contemplating and sharing increasingly subtle mathematical theories at all levels from new-born child to adult.

Building such a theory must take into account that different communities of practice may interpret situations in ways that may be in conflict with one another so that the conclusions in one community may not be appropriate for another. This applies to many different communities engaged in mathematical activity, including mathematicians in different specialisms, teachers in various educational contexts, philosophers, psychologists, neurophysiologists, curriculum designers, politicians, and so on.

In addition to considering competing theories, the proposed framework seeks a higher level multi-contextual overview that takes of account of the natural ways that mathematical thinking develops over the longer term, both in the individual and also corporately in different communities. The evidence presented here suggests that accepted approaches to teaching and learning mathematics by established communities of practice may be counter-productive in supporting the long-term development of mathematical thinking.

2. Differing conceptions of mathematics and a multi-contextual overview

The term 'community of practice' was initially introduced by Lave & Wenger (1991) as 'a group of people who share a craft or a profession.' In any community of practice, individuals

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may have differing personal viewpoints but, overall, they agree (or believe that they agree) to certain shared principles. Communities of practice incorporate a wide range of participants, including 'experts' who are well-versed in the practices and 'novices' who are being introduced to the practices of the community in various contexts.

In the long-term learning of mathematics, the contexts encountered will change substantially. For the purposes of this paper the term 'mathematical context' will refer to a specific mathematical topic being experienced by a particular individual or group of individuals in a specific community of practice. The topic may relate to a single example or to a longer-term sequence of activities.

As mathematics becomes more sophisticated, some ideas that worked well in a previous context may continue to be *supportive* in a new context, while others may become *problematic*. (Tall, 2013). For instance, simple number facts such as '2+2 makes 4' encountered with whole numbers continue to be supportive when dealing with more general numbers, such as fractions or signed numbers. Other experiences, such as the fact that the product of two whole numbers gives a bigger result but the product of two fractions can be smaller, may be problematic for many learners.

Our main strategy is to seek fundamental principles that remain supportive through many contexts over the long-term, so that they can be used as a stable basis for learning, while identifying successive problematic aspects that arise as the context changes to help learners become aware of them and address changes appropriate to support long-term learning.

Problematic changes in context often occur as mathematical thinking evolves, both in history and in the individual. This can be seen in the language relating to new kinds of number — positive and *negative*, rational and *irrational*, real and *imaginary* — which involve significant boundaries in the evolution of ideas that need to be addressed.

Crossing a boundary may be termed a 'transgression' from the Latin for 'going across', which carries with it a sense of moving to previously unacceptable territory (Kozielecki, 1987; Pieronkiewicz, 2014). It is used not only in a religious context, but also in a geographic context such as when water flows across a flood plain. It is also appropriate in a historical or personal transition across a boundary in mathematics.

The changes in context may be interpreted in different ways by different communities. If a given community A has a particular belief that is problematic for community B, and an individual or subgroup S in community A changes to adopt the beliefs of community B, then this change will be seen by community A as a *transgression* while community B will see it as an *enlightenment*. (Tall, 2019).

In historical development, such transitions from transgressions to enlightenments occurred with the introduction of negative or complex numbers, or the use of infinitesimals in seventeenth century calculus, which was criticised and later rejected by the introduction of epsilon-delta analysis at the beginning of the twentieth century, then re-introduced, subject to great dispute, in non-standard analysis in the 1960s.

Similar conflicts occur in individual learning as mathematics shifts to new contexts, say from whole number arithmetic to fractions, to signed numbers, to finite and infinite decimal expansions, to real and complex numbers and from various contexts in arithmetic to algebra.

It is not simply a matter of shifting from one level of insight to a higher level. Often it is important to be aware that apparently conflicting possibilities can coexist in different contexts at the same time. For instance, in whole number arithmetic there is a theory of unique factorization into prime numbers which can be extended to fractions and signed numbers by allowing the powers to be positive or negative, and to factorizing polynomials in algebra. But, for highly technical reasons, it fails for certain more general algebraic numbers that mix whole numbers and square roots. (Stewart & Tall, 2015).

Why should the average reader care? The answer is that average readers are unlikely to encounter this particular problem in algebraic number theory, but they *will* encounter many examples where new experiences conflict with previous experience that makes them feel *uncomfortable*, *unwilling*, even believing that they are *incapable* of thinking about mathematics. Matters are made worse when learners are subject to the beliefs of communities of experts that are at variance with their own current level of development.

The response to these conflicts is to identify their possible sources not only in the thinking of the student or the teacher, but in the mathematics itself as it develops in sophistication. This offers new ways of addressing the problem of making long-term sense of mathematical thinking.

We begin by considering

- how the biological brain operates as it encounters increasingly sophisticated mathematical constructs in successive contexts over the long term.

Then we consider

- how the brain makes sense of space and number,
- how the eyes and brain interpret written text,
- how the brain interprets spoken and aural expressions.

This information will be used to formulate a framework for the long-term meaningful interpretation of expressions in arithmetic and algebra.

More generally, we will briefly consider

- how the eye follows a moving object, giving meaning to constants and variables,
- how the eye reads through a written proof to make it meaningful.

This will be shown to be part of an overall framework for the long-term evolution of mathematical thinking in the individual (and in corporate society) that takes account of cognitive and affective growth through increasingly sophisticated mathematical contexts.

3. The Biological Brain

The biological brain is far too complicated to describe in detail in a paper such as this. It has evolved over many years where more successful variants in individuals are passed on to later generations without any overall grand design. The individual grows from a single fertilized cell and develops by successive cell subdivision guided by the genetic structure from the parents to construct an essentially symmetric brain in two halves with complex links between them.

Evolution works in unexpected ways. For example, the left side of the human brain receives signals and sends output to the right side of the body and the right side of the brain deals with the left side of the body. The two halves cooperate together: almost all right-handed individuals and most left handers deal with sequential operations such as language, speech and calculation in the left brain while the right brain focuses on global aspects such as interpreting visual information and estimating size.

Neuroscience studies the brain in a variety of ways. These include the use of electrodes on the scalp to detect electrical activity in the cortex (the ‘grey cells’ on the surface where sophisticated thinking takes place). Magnetic resonance image scanners (MRI) take cross-sectional scans of the brain to give a three-dimensional picture of brain structure including the internal connections. Functional MRI scanners (fMRI) trace blood flow over a period of two seconds or so, as the blood carries more oxygen to areas where the brain is more active. Both give valuable insight into brain structure and a broad view of its operation, while being too coarse to trace the detail of human thinking occurring in milliseconds.

Initially fMRI studies of mathematical activity focused mainly on simple arithmetic tasks. More recent studies (e.g. Almaric & Dehaene, 2016) suggest surprising possibilities in the relationship between language and mathematical thinking. They say that Chomsky (2006) declared that 'the origin of the mathematical capacity [lies in] an abstraction from linguistic operations', while Einstein insisted: 'Words and language, whether written or spoken, do not seem to play any part in my thought processes' (quoted in Hadamard, (1945), pp. 142–3). Of course, different individuals may think in different ways and Einstein certainly used imaginative thought experiments in developing his theories of relativity.

However, when Almeric and Dehaene studied mathematicians working in very different research areas (abstract algebra, analysis, geometry, topology), they found that all of them activated areas of the brain related to spatial sense and number which are present in young children before they develop language and are also found in many other non-human species.

Apart from linguistic memory for arithmetic facts, these areas rarely link to areas processing language (Dehaene et al, 1999; Shum et al. 2013; Monti et al. 2013). In addition, brain-imaging studies of nested arithmetic expressions reveal little or no links with language areas (Maruyama et al., 2012; Nakai & Sakai, 2014).

While language may be used as scaffolding to link different aspects of mathematical thinking, deeper levels of mathematical thought link with spatial imagery and mathematical operations. In this paper we seek to link the natural use of language to fundamental human ways of thinking flexibly about spatial imagery and number.

Brain activity, as a whole, deals not only with cognitive issues. The limbic system[1] in the centre of the brain handles a complex array of tasks including laying down and retrieving long-term memories; it also reacts immediately to threats in a primitive 'fight or flight' mechanism. This suffuses the brain with bio-chemicals (neurotransmitters) that enhance or suppress connections that can affect mathematical thinking in emotional ways. These may be positive in terms of determination and resilience or negative in terms of anxiety or avoidance.

To make sense of how the human brain builds mathematical connections, it is therefore important to complement what is known about cognitive development with affective reactions to mathematical ideas.

3.1 How the brain makes sense of spatial information and number

In the early years a young child develops the capabilities to recognise a given object seen from different viewpoints and in different orientations as being consistently the same. In mathematics, over a period of years, the child builds what Piaget (1952) called 'conservation of number'. This means that a given collection of objects has a consistent number attached to it and that, if the objects in the collection are rearranged spatially or if they are counted in a different way, then the number of objects remains the same.

Mathematicians formulated the properties of number and arithmetic using rules such as the 'commutative', 'associative', and 'distributive' laws for addition and multiplication. This was taken as a foundation of the 'New Math' of the 1960s, but failed to take account of the reality of the development of mathematical thinking in the learner. On the other hand, mathematics educators studied the difficulties encountered by learners and formulated more child-centred approaches including the elaboration of different methods of counting and whole number arithmetic, such as count-all, count-on, known facts, derived facts. International

[1] Wikipedia: Limbic System: https://en.wikipedia.org/wiki/Limbic_system.

comparisons such as TIMMS (2015) and PISA (2015) brought politicians into the act as they sought to improve international competitiveness. Multiple communities of practice sought to influence the curriculum in very different ways that could be in conflict.

In this paper we will not enter into a comparison between the practices of different communities. Instead we focus on the increasing sophistication of mathematical structures and operations and how they develop from fundamental human ideas of time, spatial sense and number.

The concept of number does not start with rules of arithmetic. Instead it builds from a sense that when a collection is reorganised in space or counted in different ways, then some things remain the same. The most important of these, which is not immediately obvious to the child, is that the number of objects in a collection remains the same, no matter how it is rearranged or how it is counted. Figure 1 (taken from Tall, 2019) shows how a collection of six objects has the same number of objects no matter how it is arranged or counted. The number 6 is chosen because it is the smallest number that allows not only different methods of counting and addition, but also two different methods of multiplication.

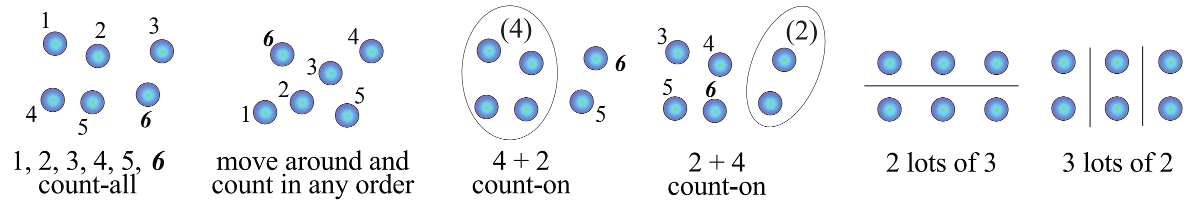

Figure 1: conservation of the number 6

Young children will have many life experiences that contribute to the development of mathematical thinking, including shared singing and dancing with rhythmic representations of the number sequence: 'One, two, three, four five, once I caught a fish alive; six, seven, eight, nine, ten, then I let him go again.'

As children mature, they will have many experiences, playing games, practising arithmetic techniques, exploring patterns. Opportunities arise to focus on an increasing awareness of the conservation of number. For example, in counting a collection of objects in different ways, the total number remains the same; in adding together two or more collections, the total number of items is the same, no matter how it is calculated.

$3 + 4 + 6 + 15$ gives the same result as $4 + 15 + 6 + 3$.

This is a fundamentally important principle over the long term. It applies not only to whole numbers, but also to fractions, signed numbers, decimal notation, infinite decimal expansions, real numbers and even complex numbers. For instance:

$7 + \frac{3}{4} + 1.414 + (-5)$ gives the same result as $1.414 + 7 + (-5) + \frac{3}{4}$.

This leads to a major underlying principle that is supportive throughout the number systems encountered in school mathematics.

> **The General Principle of Addition for Numbers**: A finite sequence of additions of numbers is independent of the order of calculation.

For individuals who attain more sophisticated levels of mathematical thinking, this can lead to a further generalisation in algebra and calculus:

> **The General Principle of Addition**: The sum of a finite collection of constant or variable quantities is independent of the order of calculation.

There are corresponding principles for multiplication, such as:

The General Principle of Multiplication: The product of a finite collection of constant or variable quantities is independent of the order of calculation.

The multiplication principle works for most situations in school mathematics, though it fails in more sophisticated contexts such as matrix multiplication.

Both principles can be extended to other operations, such as subtraction, division, powers and this will be addressed later.

To support meaningful learning of mathematics over the long term, the aim is for teachers and learners to become aware of properties that remain supportive through several changes in context to give a stable foundation for new learning. The plan is to use the sense of security in such general principles to encourage learners to address situations where the context changes and previously supportive ideas become problematic, to seek meaningful reasons why changes in meaning need to be incorporated into long-term thinking. This can be assisted by reflecting on how we humans make sense of our perceptions and actions.

3.2 How the eyes read text and symbolic expressions

When we read text on a page, we do not scan the lines smoothly. Instead the retina in the eye has a small area called the macula which registers much higher detail and takes in successive parts of the text in a succession of jumps (called ‘saccades’) that the brain puts together to build up the meaning of the text. Figure 2 (taken from Tall, 2019) shows printed text on the left and a representation of how human vision focuses on a small part of the text on the right. Read the clear text on the left, several times to sense how your eye jumps along the lines to make build the meaning of the text. *Do this now…*

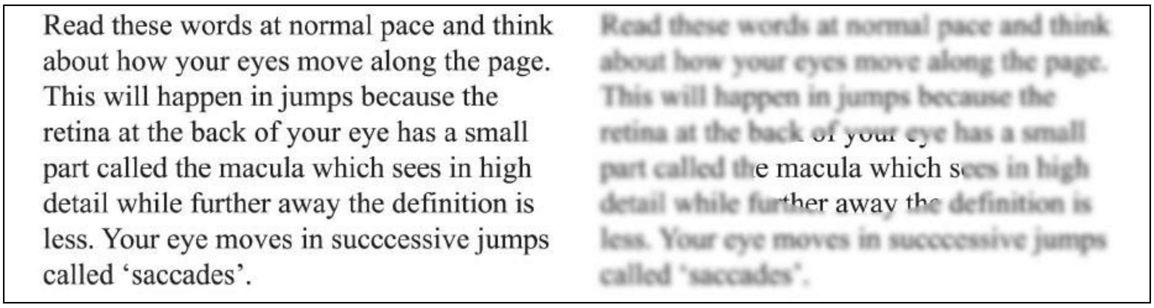

Figure 2: reading text

It transpires that when we speak words, we do so as a sequence in time, and when we read text, we do so in an ordered sequence in a direction dependent on the language concerned — usually left to right in Western languages (Tall, 2019). This sets an implicit mode of thinking that can become problematic when interpreting expressions in arithmetic and algebra.

3.3 how the brain interprets spoken and aural symbolic expressions

When a mathematical expression such as $1 + 2 \times 3$ is spoken or heard, it occurs *in time*, as ‘one plus two times three’. The traditional sequence of spoken and written language suggest that the operations should follow the sequence in time: first carry out the operation ‘$1 + 2$’ which is ‘3’, then ‘3×3’ which gives **9**.

Children are given a different convention in mathematics that contradicts this natural sequence with the rule ‘multiplication takes precedence over addition’. This requires first performing the second operation ‘3×3’ to get ‘6’ and then calculating ‘$1 + 6$’ to get the ‘*correct*’ answer, **7**.

If children learn to follow the rule without reason, as expressions become more sophisticated and the rules more complicated, then, over the longer term, arithmetic, and subsequently algebra, becomes increasingly difficult. Simply learning by rote fails to take into account the subtle ways in which language is expressed. It is not just a matter of *what* we say, but *how* we say it.

In written text we use punctuation to distinguish subtle differences in meaning. In spoken text, we can use tone of voice and articulation to express meaning. If we say mathematical expressions, by leaving gaps in different places, we can emphasise which operations are linked together. For example, saying

'one plus [gap] two times three'

emphasizes that the words in the phrase 'two times three' are to be taken together, suggesting that the result is 'one' plus 'two times three', which is 'one' plus 'six', which is 'seven'.

In Tall (2019) I played with the idea of writing two dots (..) to denote a gap, so that

1 + 2 .. × 3 means 3 × 3, which is 9,

while

1 + .. 2 × 3 means 1 + 6, which is 7.

At this point it is helpful to speak the two expressions '1 + 2 .. × 3' and '1 + .. 2 × 3' out loud to yourself and, if possible, to another person, to see how these two ways of speaking give two clearly different meanings, not only to the person speaking, but also in communication with others.

Do this now before proceeding. It is essential that you experience this for yourself.

This reveals that the meaning of a sequence of operations in arithmetic, and also later in algebra, depends on the way it is spoken. It can be formulated as:

> **The Articulation Principle**: The meaning of a sequence of operations can be expressed by the manner in which the sequence is *articulated*. (Tall, 2019)

It is essential to realise that this principle does not act like a definition in mathematics that can be used to make a formal deduction or to prove a theorem. It makes us aware that we need to think very carefully how we interpret and communicate mathematical expressions.

The principle operates with other expressions. For instance, it clarifies the possible meanings of an expression such as 2 × 3 + 4 which could be interpreted as

2 × 3 .. + 4 which gives 6 + 4, which is 10

or

2 .. × 3 + 4 which gives 2 × 7, which is 14.

This makes it imperative to introduce suitable conventions to clarify the precise meaning, such as introducing brackets around operations that should be performed first. Thus

2 × 3 .. + 4 can be written as (2 × 3) + 4, which is 10

and

2 .. × 3 + 4 can be written as 2 × (3 + 4), which is 14.

It is then possible to introduce further conventions to reduce the length of expressions. For instance, the convention 'multiplication takes precedence over addition' allows us to remove brackets around a product, in the knowledge that the convention requires multiplication to be calculated before addition to rewrite

(2 × 3) + 4 as 2 × 3 + 4

while retaining the notation for 2 × (3 + 4).

The principle of articulation is widely applicable throughout mathematics. For instance, my 11-year old grandson surprised me one day when he asked me

‘What is the square root of 9 times 9?’

Knowing that he was familiar with negative numbers, I replied that the answer could be +9 or –9.’ ‘No,’ he replied, ‘it’s 27.’ Then he explained that he meant

‘the square root of 9 .. times 9’

which gives 27 (Tall, Tall & Tall, 2017).

Subsequently, we found that the principle works not only for simple arithmetic expressions but throughout the whole range of mathematical expressions used to specify mathematical operations as mental objects.

3.4 Flexible use of symbolism dually representing process or concept

The idea of an operation becoming a mental object of thought has permeated research on mathematical thinking for many years. Piaget referred to this transition at ‘reflective abstraction’ and many other authors have formulated similar ideas using different terminology. (See Tall et al. 2000 for a general discussion.) Broadly speaking, there are two essentially different mental constructs — a *process* (or *operation*) which occurs in time, either as a procedure with a specific sequence of actions, or as a more general input-output process — and a *concept* (or mental *object*) that can be conceived as a mental entity that can be manipulated in the mind. In what follows, when referring to mathematical expressions, the terms ‘process’ and ‘operation’ will be used interchangeably as will the terms ‘concept’ and ‘object’. Often the situation is seen as having two different states one as process (or operation), the other as concept (or object) with distinct acts of passing from one to the other.

Gray & Tall (1994) realised that an expression such as 2 + 8 can be conceived either as a process to be carried out, such as ‘add 2 and 8’, or as a concept, the ‘sum of 2 and 8’, which is 10. They responded to this dual and ambiguous use of the symbol by naming it a ‘procept’. This offers a major theoretical advance because it refers to the possible use of the symbol flexibly, either as a process that could be carried out in a variety of ways, or as a single mental entity that can be manipulated as a mathematical object, whichever is more useful in a given context.

Sometimes it is important to distinguish between the two meanings. As operations, two different operations can give the same object, so we often speak of them as ‘equivalent operations’. For instance, when we speak of fractions, we say that $3/6$ and $2/4$ are ‘equivalent fractions’ because they are certainly different as operations, but they are the same rational number, represented on the number line by a single point.

This flexible duality of expressions as process or concept occurs throughout arithmetic, algebra, calculus, and more sophisticated use of symbols. Often the curriculum is designed to start with examples of specific procedures to convert one expression to another. For instance, algebraic expressions such as $(x + 1)(x – 1)$ can be multiplied out to get $x^2 – 1$, and this can be factorised to return to $(x + 1)(x – 1)$. Initially these two expressions may be seen as ‘equivalent’ but they are also different ways of representing the same underlying mathematical object which has a single graph.

This is not the only way in which sophisticated ideas evolve. It is also possible to begin with an intuitive sense of a concept and then seek ways of constructing and calculating it. Applied mathematicians do it all the time. They start with a situation that they seek to model and use mathematics to construct and test the model to see if it gives a good prediction.

3.5 Making sense of mathematical expressions dually representing operation or object

Given the way in which rules of precedence violate the directional way of reading text, Tall (2019) proposed a simple notation to use the distinction between process and concept to give a natural meaning to the rules of precedence. Starting with a single operation such as $2 + 8$, simply put boxes round the objects.

$\boxed{2} + \boxed{8}$ is the operation of adding the objects $\boxed{2}$ and $\boxed{8}$.

If the whole expression is conceived as an object, put the box round the whole expression:

$\boxed{2 + 8}$ is the object which is the result of adding 2 and 8.

This relates directly to the different ways we articulate an expression to indicate which operations should be performed first and then the result should be considered as an entity to be operated upon. For instance

2×3 .. $+ 4$ can be interpreted as $\boxed{2 \times 3} + 4$ which is $6 + 4$, giving 10,

while

2 .. $\times 3 + 4$ can be interpreted as $2 \times \boxed{3 + 4}$ which is 2×7, giving 14.

The general principle of addition tells us that if there are several additions in a box, then the order does not matter, so

$2 \times \boxed{3 + 4 + 5}$ is the same as $2 \times \boxed{4 + 5 + 3}$.

There is a corresponding principle for a box containing several multiplications.

There are a few conventions that require individual treatment. For example, if letters are involved in an algebraic expression, such as $2 \times a \times b$, then the convention is to omit the multiplication signs, writing it as $2ab$. There is no problem here:

the operation can be written as $\boxed{2}\,\boxed{a}\,\boxed{b}$ and the object as $\boxed{2\,a\,b}$.

In dealing with the contraction 2½ for 2 + ½, boxing the expression as an operation requires an explicit addition sign $\boxed{2}+\boxed{\tfrac{1}{2}}$ as the symbol $\boxed{2}\boxed{\tfrac{1}{2}}$ could be confused with the product.

The exponent notation for x^2 can be written as $\boxed{x}^{\boxed{2}}$ as an operation and as $\boxed{x^2}$ an object.

What is important for the human brain is to reduce the complication by not using unnecessary notation. What matters is the principle of seeing operational symbols flexibly as process or concept and to interpret the operations in an expression according to their precedence. For instance, in the quadratic expression

$2x^2 + 7x + 6$

it is not necessary to put boxes round the numbers. Visualising the terms x^2, $2x^2$ and $7x$ as single objects, the expression can be seen as

$2\boxed{x^2} + 7x + 6$

or, in the usual notation, as

$2x^2 + 7x + 6$

where now the reader can flexibly see x^2 as an object and $2x^2$ and $7x$ as objects which are also the product of objects. The expression is now the sum of three terms and the general principle of addition allows them to be written in any order. Individual terms can be manipulated to see $5x$ as $3x + 2x$ and 6 as 3×2 and the expression $2x^2 + 7x + 6$ can be factorised as $(2x + 3)(x + 2)$.

To be able to manipulate expressions in this way requires considerable flexibility on the part of the individual. In a traditional algebraic curriculum, the reading of more complicated expressions is often guided by mnemonics such as PEMDAS in the USA or BIDMAS in the UK to specify successive levels of precedence.

PEMDAS, remembered as ‘Please Excuse My Dear Aunt Sally’, sets the order of precedence as ‘Parentheses, Exponents, Multiplication, Division, Addition, Subtraction’; BIDMAS gives ‘Brackets, Indexes, Division, Multiplication, Addition, Subtraction’. The situation is more complicated because the order is actually P > E > M = D > A = S or B > I > D = M > A = S where > denotes a higher level of precedence and = denotes an equal level. The rule states that higher precedence operations are performed first and equal precedence are performed left to right.

The use of this mnemonic proves to be highly problematic. Brain research reported earlier shows that merely learning the mnemonics by rote may link to language areas in the brain but not to the areas involved in fundamental human sense of space, time and number. Our new view of understanding meaning through articulation and flexible interpretation of symbol as process or concept now offers a new way of linking visual symbolism to fundamental human ideas of spatial layout and number.

There is also a further limitation of the mnemonics PEMDAS and BIDMAS because they only apply to binary operations $a + b$, $a - b$, $a \times b$, $a \div b$, a ^ b (written as a^b) and not to unary operations such as the additive inverse, $-a$, square root $\sqrt{a}$, nor to more sophisticated operations such as matrix multiplication, limits, differentiation, integration and other more advanced symbolism that require new rules of operation.

The principle of articulation generalises naturally to give meaning to more advanced concepts. A typical instance is the square of a negative quantity $-x^2$ which can be articulated as

‘minus *x* [gap] squared’, or as ‘minus [gap] *x* squared’.

These give the two different meanings:

$(-x)^2$ and $-(x^2)$.

The same idea also clarifies the meaning of x^2 when a negative number is substituted for *x*. College students may find difficulty in substituting ‘*x* equals minus 2’ in ‘*x* squared’. Is it ‘minus *two squared*’ as –4 or ‘*minus two* squared’ as +4? (McGowen & Tall, 2010). The articulation principle clarifies this distinction.

What becomes apparent in this long journey through sense-making in arithmetic and algebra is that it is possible to make sense of the conventions adopted in traditional algebraic notation by building from the principle of articulation, the general principles of addition and multiplication and the duality of expressions as process and concept. This approach links naturally to what has been discovered about the workings of the brain where mathematical thinking at all levels benefits from making mental links between concepts in space, time and number.

What is even more remarkable is that this analysis generalises to more sophisticated expressions written spatially using templates as laid out in modern digital software.

We have already seen a spatial layout when a power is written raised as a superscript. Possibilities proliferate with symbolism for limits, summation, integrals, matrix layouts and so on. These can be written by hand or built up using software templates such as MathType or specified symbolically using languages such as TeX. Figure 3 shows the layout for the general solution of a quadratic equation.

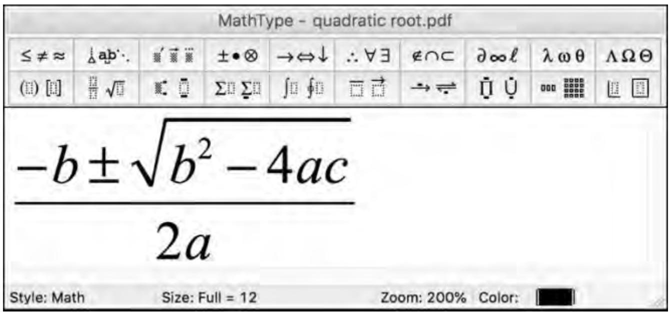

Figure 3: spatial layout of an expression

Reading an expression involves scanning the spatial layout to make sense of it. By starting with the whole expression as an object, it is possible to see it as flexibly as a process with sub-expressions as objects in boxes, and then to dig hierarchically down into the objects re-imagined as processes to give flexible meaning for the whole expression (Figure 4).

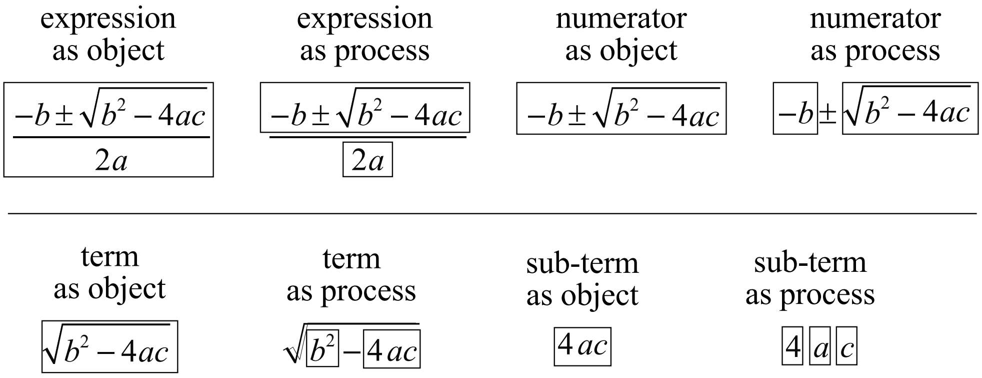

Figure 4: sub-expressions as operation or object

This successive focus on the whole as an object then as a process, to see the constituent parts of the process as objects that can then be further broken down, is essentially how successful thinkers can intuitively see the hierarchical structure of the expression. Now it can be explained explicitly to encourage a broader range of the population to make sense of expressions.

3.6 Equations

Once teachers or learners have the insight afforded by the meaning of the principle of articulation and the flexibility of expressions as process or object, this can give meaningful new ways of interpreting equations. An equation consists of two expressions with an equal sign between them. The new insight allows an equation to be seen in different ways, depending on whether either or both of the two expressions are process or concept.

For a young child, an equation in arithmetic such as

$$2 + 3 \ = 5$$

is usually read from left to right as an *operation* in which 2 + 3 is seen to give the result 5. This is in the form "process = number". An algebraic equation in the same form, such as

$$2x + 3 = 9$$

can be seen as a process to produce the output object $\boxed{9}$:

$$2x + 3 = \boxed{9}.$$

Seen as a succession of steps, the process can be written as:

$$\boxed{x} \xrightarrow{\times 2} \boxed{2x} \xrightarrow{+4} \boxed{10}$$

The process can then be undone by reversing the steps:

$$\boxed{3} \xleftarrow{\div 2} \boxed{6} \xleftarrow{-4} \boxed{10}$$

which immediately tells us that the original input x must be 3.
However, an equation with an expression on both sides, for example:

$$3x - 2 = 2x + 1$$

cannot be 'undone' in the same way. This might be solved by guessing a value for x which works, or by seeing both sides as the same object, written as:

$$\boxed{3x - 2} = \boxed{2x + 1}.$$

We can then operate on the equation by 'doing the same operation to both sides' which retains the equality of the new sides. Once the original equation can be imagined as having an object on either side, we can do this in standard notation, by adding 2 to both sides to get:

$$3x - 2 + 2 = 2x + 1 + 2.$$

Using the general principle of addition, this simplifies to:

$$3x = 2x + 3$$

and, taking $2x$ from both sides gives

$$x = 3.$$

A student who sees an expression only as a process and not as an object is more likely to be able to solve an equation of the form 'expression = number' by 'undoing' than solve an equation with expressions on both sides. This is studied extensively in the literature and was named 'the didactic cut' (Filloy & Rojano, 1989).

A teacher who has given meaning to expressions using the principle of articulation and has grasped the flexibility of expression as process or concept has a new way of giving meaning to equations. 'Doing the same thing to both sides of an equation' in the form 'object = object' either ends up with both sides always being the same (an 'identity') or with the equation only being satisfied by certain values of the unknown (an 'equation'). The first case occurs with an equation such as

$$2\,(x + 3) = 2x + 6$$

or

$$(x + y)(x - y) = x^2 - y^2.$$

In our new way of thinking, this is a single object (a procept) with different processes to calculate it.

A teacher who belongs to a community of practice that makes sense of expressions in this way may offer enlightenment where others may only see many complications arising from the didactic cut. But whether this transition to a new way of thinking is possible for teachers depends on their current beliefs and whether the transition, as seen from their current practice, is a transgression or an enlightenment.

The ability to see the equals sign used in a flexible way has further benefits as the mathematics evolves in sophistication. An equation in the form 'variable = expression' may take the form of a *definition* of the variable (as a mental object) given by a process. For instance, the equation $y = x^2$ defines the (dependent) variable y in terms of a process operating on the (independent) variable x.

This applies in more sophisticated theory, such as infinite sums in the calculus where

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

defines the mathematical expression sin x computed as a potentially infinite process. The left-hand side is now a mental object which can be calculated as accurately as required by adding enough terms.

Over the long term, making sense of expressions, initially through the principle of articulation and then through the flexibility of expressions as process or concept, offers a consistent approach that encourages the learner to build on supportive ideas with confidence and deal with problematic aspects as they arise. Whether this approach is successful or not will depend on how current communities of practice see it as a transgression from their accepted practices or an enlightenment to move forward into the future.

4. Building a new framework for long-term of mathematics

While various approaches to the curriculum have led to 'Math Wars' arguing between approaches to mathematics learning, we can now shift to a higher multi-contextual level where learning 'the basics of arithmetic' can be related flexibly to the meaning of expressions.

As children experience mathematical ideas in practical contexts, they will naturally pick up aspects related to each context. Making sense of different contexts to draw out common ideas is more complicated than having available simple principles that work in multiple contexts.

This is part of a much broader framework for making long-term sense in mathematics as a whole. In *How Humans Learn to Think Mathematically* (Tall, 2013) I formulated a framework for long-term mathematical thinking beginning from the child's perceptions and operations with the physical world and with others in society. One strand of development senses the properties of objects, initially physical, then constructed mentally, which I termed *conceptual embodiment*. Another strand focuses on the properties of operations that I termed *operational symbolism*. Both of these develop in sophistication from *practical mathematics* based on the *coherence* of properties that occur in practice to *theoretical mathematics* where properties are defined and relationships are deduced one from another in what may be termed *consequence*.

At the turn of the twentieth century, a further strand developed based on *properties* defined using set theory or logic which I termed *axiomatic formalism*. For many mathematicians, formal mathematical proof starts with Euclidean geometry. But there is a huge difference between mathematics based on properties of pictures or on known calculations and mathematics based on formal definition and proof. Prior to the end of the nineteenth century, the study of mathematics and science based on naturally occurring phenomena was described as 'natural philosophy'. I therefore distinguish 'theoretical mathematics' based on 'natural phenomena' from 'axiomatic formal mathematics' based on set-theory and logic (Figure 5).

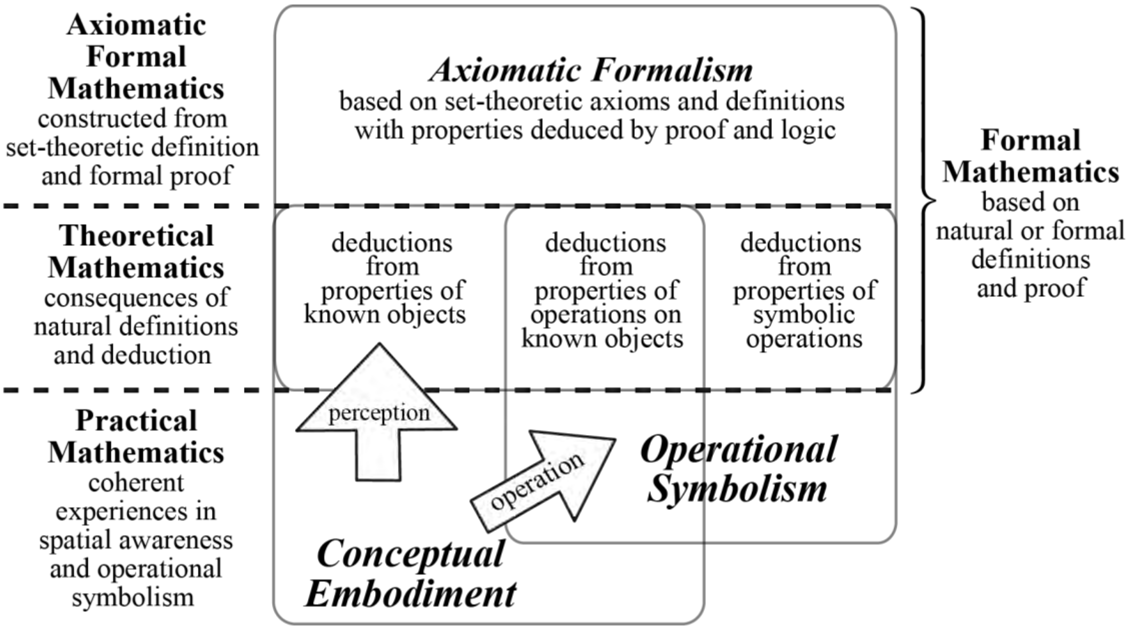

Figure 5: the long-term development of mathematical thinking

Figure 5 is a much-simplified view of the theoretical framework developed in Tall (2013), based on the new information available from neuroscience. I termed the three main strands as 'worlds of mathematics' because each world represents a fundamentally different way of thinking that evolves both in history and in the individual. Conceptual embodiment exists in many species and in human ancestors several hundred thousand years ago. Operational symbolism evolved in Homo Sapiens in the last fifty thousand years, proliferating in various communities in Egypt, Babylon, India, China around five thousand years ago, becoming increasingly theoretical in Greek mathematics with the first flowering of mathematical proof two and a half thousand years ago. Axiomatic formal mathematics has been around for little more than a century. Now new possibilities are emerging in our digital age enabling Homo Sapiens to use new digital tools to enhance the possibilities of enactive interface, dynamic visualisation, symbolic computation and the emergence of new forms of artificial intelligence.

In this ongoing evolution, the biological brain evolves slowly. There is no reason to suppose that the biological brain of the ancient Greeks is substantially different from our own. In contrast, the technical evolution of digital tools available to support the mathematical mind that have occurred within a generation is immense. Although we now know that the biological brain is more complex than a simple duality between left and right brain, it still continues to support conceptual embodiment and operational symbolism with the fore-brain taking an increasing role in integrating mathematical thinking in new forms of axiomatic formalism.

It is interesting to note that the diagram in figure 5 nowhere explicitly mentions the role of language. Instinctively, when I originally thought about the framework, I saw mathematical thinking to be related to the complementary roles of visual imagination, sequential symbolic operation and later logical deduction, with verbal language being used to describe connections between different parts of the framework.

The resulting two-dimensional picture gives only a partial idea of the broader complexity of the workings of the human brain. For example, it focuses on cognitive aspects that occur in the surface areas of the cortex and says little about the activity of the limbic system in the centre of the brain that not only performs many cognitive tasks relating to short and long-term memory but also responds emotionally to supportive and problematic aspects of mathematical thinking.

Individuals do not operate in all areas of the framework. For everyday mathematics, as used by the vast majority of the population, all that is required is practical mathematics focusing on the coherence of spatial perception of the properties of objects and symbolic operation.

Those involved in mathematical applications including STEM subjects (Science, Technology, Engineering, Mathematics) usually only require practical and theoretical mathematics.

Only a small percentage of the population studying pure mathematics and logic use axiomatic formal mathematics.

4.1 Extending the framework

Although the picture places axiomatic formal mathematics at the top of the figure, this is by no means the end of the story. Among the properties proved in axiomatic systems, certain theorems called 'structure theorems' prove properties that reveal new forms of conceptual embodiment and operational symbolism. Sometimes the structure is unique in the sense that any two structures satisfying the definition have the same properties (said to be 'isomorphic'). We can now see that the two structures may not only be 'essentially the same', they may also be conceived as a single entity that can be represented in different ways.

Two examples of such unique structures are 'the natural numbers' and 'the real numbers' which can be represented visually as points on a number line and symbolically using decimal notation. Other axiomatic systems may have many different examples, such as the concepts of 'group' or 'vector space'. These have structure theorems that allow them to be classified and represented as mental objects or as operational structures. For example, a 'finite dimensional vector space' can be proved to have a coordinate system, visualised as two or three-dimensional space or imagined mentally in higher dimensions, where the coordinates allow the vectors to be manipulated symbolically (Tall, 2013).

5. How the brain makes sense of more sophisticated mathematics

Knowledge of the workings of the brain can now explain in simple terms how the biological brain can make sense of sophisticated mathematical ideas that are considered to be problematic for some and enlightening to others. Again, this explains evolution of ideas corporately in history as well as in the growing individual.

5.1 How the eye follows a moving object, giving meaning to constants and variables

When the eye follows a moving object, it starts using the same initial action as reading text, with a single jump to focus on the object, but then it follows the object smoothly as it moves. You can sense this by holding a finger in front of your eye at a comfortable distance away and move it sideways, keeping your gaze on the finger as it moves. The finger stays in focus while the background is blurred. In this way the eye is set up to follow moving objects smoothly. It is therefore natural to imagine a point on a line which moves. It is also natural to distinguish between a fixed point on a line (a constant) or a moving point on a line (a variable). (Figure 6.)

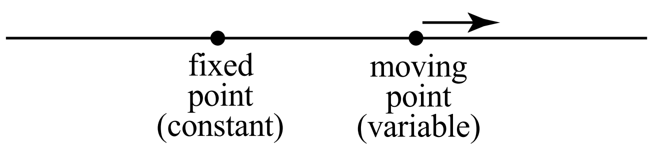

Figure 6: constant and variable points on a line

This has profound implications for the historical and individual imagination for constant quantities and variable quantities, including variables that can become arbitrarily small. In history this gave rise to ideas of indivisible quantities that are small but no longer further divisible and infinitesimal quantities, either as potential never-ending processes or as actual mental objects. This interpretation of infinitesimals as variable quantities offers a new way of considering the Greek arguments about potential and actual infinity. It sheds new light on the development of infinitesimal ideas in the calculus, in particular, how Leibniz may have imagined different orders of infinitesimality (Tall, 2013, chapter 13) or how Cauchy imagined infinitesimals as sequences that tend to zero (Katz & Tall, 2012; Tall & Katz, 2014).

5.2 How dynamic movement can represent infinitesimals as process and concept

An infinitesimal may be visualised as a variable point on a line. For example, consider a rational function $f(x) = p(x)/q(x)$ where p and q are polynomials with q non-zero. Draw the graph of $y = f(x)$ and the vertical line $x = k$. Figure 7 shows the vertical line intersecting the graphs of $y = c$, $y = x$, $y = x^2$ at heights c, k, k^2. For constant $c > 0$, as k decreases to zero, the points height k and k^2 fall below the point height c and the variable points k and k^2 are eventually less than any positive real number c. In this sense they are *infinitesimal*. Moreover, k^2 is a smaller than k. If we think of k as being of order 1, then k^2 is of order 2 and, in general, as n increases, k^n is an even smaller infinitesimal of order n. Using such a visual representation, we can imagine infinitesimals of any order.

Of course, this argument may be rejected as a transgression, as it was by many contemporary critics of the early calculus. But for others, it offers enlightenment.

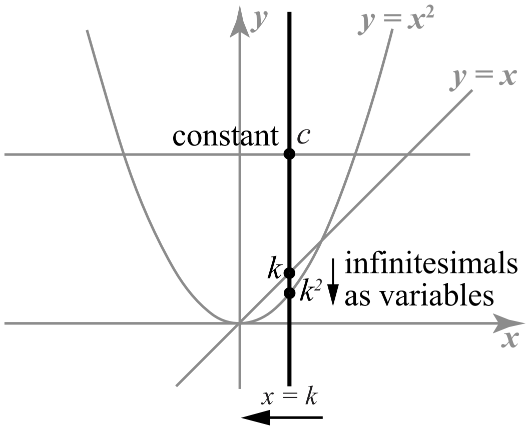

Figure 7: infinitesimals as variables tending to zero

Using axiomatic formal arguments, we can go even further. Consider any ordered field K that contains the real numbers as an ordered subfield. (Remember that the field of real numbers is unique as the one and only *complete* ordered field, in the sense that any non-empty subset of real numbers has a least upper bound.) Any element x in K can be compared with any real number c, so we know that either $x > c$ or $x = c$ or $x < c$. We can then separate the elements of K into three distinct categories:

(1) those x in K such that $x > c$ for all real c

(2) those x in K such that $x < c$ for all real c

(3) those x in K which lie between two real numbers, $a < b < c$.

We call those in (1) *positive infinite* elements, those in (2) *negative infinite* and those in (3) *finite*. If x is finite, then the set of real numbers less than x is non-empty (it contains a) and bounded above (by b), therefore, by the completeness axiom for the real numbers, it has a least upper bound c. It is then a straightforward deduction to prove a structure theorem that the element $e = x - c$ is either a (positive or negative) infinitesimal or it is zero. This tells us that every finite element x is (uniquely) of the form $x = c + e$ where c is a unique real number and e is an infinitesimal or zero. We call this number c the *standard part* of x.

This transition to the new axiomatic formal context transforms a transgression into an enlightenment. It opens the flood gates. The structure theorem enables us to visualise infinitesimals (and their multiplicative inverses which are infinite) using simple algebraic maps (which I term *optical microscopes*) mapping x onto the standard part of $(x - c)/e$. This maps c to 0 and $c + e$ to 1, spreading out infinitesimal detail near c so that we can see it as a real picture! It also reveals that, seen through an optical microscope, the image of an infinitesimal part of a differentiable function is a real straight line. This links the intuitive idea that a differentiable function is 'locally straight' under high magnification to a perfect visualisation in the axiomatic formal world (see Tall, 2013, chapter 11 or Stewart & Tall, 2014, chapter 15, for details).

These ideas easily extend to see infinitesimal detail for complex functions (Stewart & Tall, 2018, chapter 15). They are part of a much bigger framework of multi-dimensional analysis including visualisations of analysis in higher dimensions and sensible meanings for partial derivatives (Tall, 2013, chapters 11, 13).

5.3 How the eye reads through a written proof to make it meaningful

In addition to the way the brain can interpret pictures, it can also scan a written proof, not just line by line, but also by looking back at significant steps and getting an overall grasp of the proof structure. Using eye-tracking techniques, Inglis and Alcock (2012) confirmed that undergraduates devoted more of their attention to parts of proofs involving algebraic manipulation and less to logical statements than expert mathematicians. Hodds et al. (2014) developed a technique of 'self-explanation' in which students were encouraged to read a proof line by line, to identify the main ideas, get into the habit of explaining to themselves why the definitions are phrased as they are and how each line of a proof follows from previous lines. They were counselled not to simply paraphrase the lines of the proof by saying the same thing in different words, but to focus on making connections to grasp the main argument and explain how the given assumptions and definitions in previous lines led to the current line and contribute to the following lines. Students who had worked through these materials before reading a proof scored 30% higher than a control group on a subsequent occasion.

Notice that, in this case, the explanations were expressed linguistically, but the focus once more is on the relationships between ideas. A focus on making personal links is more likely to give a more coherent personal knowledge structure in the longer term.

6. The role of the limbic system in enhancing and inhibiting mathematical thinking

Up to this point, the presentation has focused on

- *cognitive* aspects of how individuals think about mathematical structures.

To gain a broader understanding of the long-term development of mathematical thinking, it is also essential to consider:

- *affective* aspects that enhance and suppress the making of mental connections.

Mathematics evokes a wide range of emotions in different individuals. Some experience great pleasure in solving a difficult problem, even relishing the challenge. Others suffer a sense of tension and anxiety that interferes with their ability to answer a mathematical question or manipulate numbers. The anxiety can range from a mild sense of insecurity to a full-blown fear and loathing of mathematics.

These emotions arise in the limbic system in the centre of the brain. This is a collection of structures that support a variety of functions, including cognitive links between short-term and long-term memory, but also gives rise to primitive emotional responses of pleasure or pain. In particular, it responds to challenges or to danger with an immediate 'fight or flight' reaction that suffuses the whole brain with neurotransmitters that excite or inhibit mental connections.

Confident students who rise to the challenge are placed on alert, ready to tackle the situation. Those who find the mathematics difficult or even impossible are likely to have their mental connections suppressed, causing them to freeze mentally and even be unable to respond. It is not just that students suffering from mathematics anxiety are unwilling to think mathematically. When their mental connections are depressed, they may not be able to think about mathematics at all.

Research identifies many diverse factors related to mathematics anxiety, including negative experiences of mathematics, fear of being asked questions in front of others, social deprivation, poor self-image, poor memory, and so on. Here we are only concerned with one aspect: the long-term relationship between the individual and mathematics. A biological brain which has rich flexible connections and an awareness of the need to deal with problematic aspects of new contexts is much more likely to succeed than one which has limited rote-learned knowledge. A brain suffused with neurotransmitters that enhance mental connections is better placed to construct new meanings than a brain with mental connections that are suppressed.

7. Strategies for enhancing long-term mathematical thinking

The evidence presented here shows the value of making sense of symbolism by realising how verbal articulation can clarify the meaning of ambiguous expressions in arithmetic and algebra. Then symbolism used to denote a mathematical operation can also be envisaged as a mathematical object in its own right to be manipulated symbolically, leading to a meaningful way of making sense of the hierarchal structure of expressions.

This approach has positive advantages in transitioning from arithmetic to algebra. By focusing on the meaning of expressions in arithmetic in terms of how they are processed, it also focuses on the underlying meanings that give rise to algebra. Instead of having many different experiences in arithmetic and basing algebra on seeking for the underlying patterns, it offers a a focused sense of the general principle that applies to all these experiences.

There are opposing tendencies in operation here: one involves realizing that different experiences can give rise to the same underlying idea, the other that a general underlying idea can apply to many different experiences. Is it better to have a wide range of experiences from which one attempts to abstract more sophisticated principles or is it better to seek to focus on sophisticated principles to apply in a range of increasingly sophisticated contexts?

Clearly, the young child will encounter a range of different experiences each of which will have its own characteristics, some of which may be supportive in another specific context, but some will involve mental links that become problematic. Is it even possible to help the young child to be sensitive to more sophisticated principles at an early stage?

I suggest that it is not a question of one *or* the other, but that *both* should be addressed in parallel. As the child encounters specific examples in arithmetic, I hypothesise that it is of value to relate them to the general principles that have been highlighted in this paper. In particular, the articulation principle can be used at every level of development to give meaning to fundamental principles not only in arithmetic, but also in preparation for algebra. It can be used for young children in early arithmetic, for adult learners struggling with mathematics, or even for experts who have yet to see the power of this simple idea.

I have found by informal discussion with many individuals, that this principle is widely understood implicitly by children and teachers of all ages and by adults in different professions, but it is not yet the subject of any research that I know. The reader can participate in testing this by asking anyone, 'What is $2 + 2 \times 2$?' to see whether the answer given is '8' (following the spoken sequence) or '6' (using the rule 'multiplication must be performed before addition').

Continuing the conversation by speaking the question in two different ways (as ‘two plus two [gap] times two’ or ‘two plus [gap] two times two’ will give practical insight into the principle.

However, the implementation of this, or of any other idea, in a curriculum will depend very much on the communities of practice involved. For some, their established practices may render a new direction to be problematic and the change to be a transgression. For others, the new direction may offer highly supportive insight.

7.1 Moving to the future in different communities of practice

Recent international comparisons in TIMSS (2015) and PISA (2015) reveal widespread differences in long-term mathematical competence. PISA shows East Asian countries scoring highly in the first seven places out of sixty-five participants, with the Netherlands (10th) among those above average, the UK (26th) being average, the USA (36th) slightly below average, with a long tail including Brazil (58th). I selected these countries because they include some of the areas where I have had direct research experience.

As a consultant in a project involving twenty economic communities around the Pacific Rim, it was my privilege to participate in a multi-cultural overview of different communities developing Japanese Lesson Study and editing (but not writing) the English version of the first three volumes of the Japanese Junior High School mathematics (Isoda & Tall, 2018). These books are written by mathematicians and teachers to encourage students to think for themselves, informed by research in mathematics education. The lesson sequence is organised to give the students experiences that will be useful for solving problems encountered later in the sequence. The sequence is then modified over successive implementations to build a stable version intended for general use.

The development of Lesson Study is broadly consistent with the framework formulated here with some differences. For example, mathematics education research distinguishes between ‘three twos’ and ‘two threes’ and the curriculum initially retains this difference as processes rather than seeking their unity as an object. Perhaps the next iteration of the curriculum will address this aspect.

The building of the long-term curriculum reveals a problematic transition from practical to theoretical mathematics. In the Netherlands, ‘realistic mathematics’ introduces children to make sense of practical situations as active participants solving meaningful problems in imaginative ways (Van den Heuvel-Panhuizen & Drijvers, 2014). This approach has spread internationally with widely acclaimed success. Yet it proved to lead to a situation where students in the Netherlands going to university were less well prepared.

Advocates of realistic mathematics investigated this phenomenon in three PhD studies involving ‘subtraction under 100’, ‘fractions’ and ‘algebra’, to show that:

> Dutch students' proficiency fell short of what might be expected of reform in mathematics education aiming at conceptual understanding. In all three cases, the disappointing results appeared to be caused by […] the textbooks' focus on individual tasks […] with a lack of attention for more advanced conceptual mathematical goals, constitut[ing] a general barrier for mathematics education reform.
>
> (Gravemeijer et al., 2016, p. 25.)

The authors came to the conclusion that it is not a weakness in the theory of realistic mathematics, but in the implementation of theory: that ‘realising’ mathematical ideas needs extending to grasp the underlying theoretical ideas in more advanced mathematics.

An attempt to use Lesson Study in the Netherlands to address the problem for teenagers studying calculus proved initially to be problematic as the teachers followed their experience of Dutch culture including ‘following the textbook closely, the strict school guidelines and the pressure for high exam results' (Verhoef & Tall, 2011). Only in the second year of the study did teachers begin to grasp the students' personal ways of thinking to make sense of the relationship between dynamic visualisation and symbolism using *Geogebra* (Verhoef et al, 2014).

In the USA there is a vast quantity of research literature studying the complication of ideas in arithmetic, fractions and algebra. In general, this literature focuses more on the complications of mathematics and its implementation in the classroom. But where is there extended research to consider how to make sense of the simple idea of the principle of articulation and its resulting flexibility of symbolism as process or concept?

In the UK with a maximum political cycle of five years, politicians need results that vindicate their policies within such a period. Given the perceived lack of competitiveness in international comparisons, they sought to find how the more successful countries operate, seeking insights from Singapore, Shanghai, Finland and elsewhere, finding that different social and cultural attitudes made it problematic to transfer the expertise.

In Brazil, which scores low in PISA studies, research revealed teachers teaching students rote-learnt rules to pass tests which work in simple cases but fail in general. For example, solving a quadratic equation using the formula, when many students could not manipulate a quadratic into the form $ax^2 + bx + c = 0$ to use the formula (Tall, de Lima & Healy, 2014).

In both high-scoring and low-scoring communities on the PISA scale outside East Asia, the desire to 'teach to the test' may offer some short-term success, but over the long term, rote-learning of a range of disconnected methods may act as a barrier to the development of more sophisticated long-term mathematical thinking.

8. Reflections

This paper has offered evidence relating to how the human brain makes sense of increasingly sophisticated mathematical ideas by referring to neurophysiological research and simple ideas that can be observed by teachers and learners in the classroom.

It acknowledges changes in meaning over the longer term as the learner encounters more sophisticated contexts. To offer positive support to address these changes, it focuses on fundamental aspects that remain supportive over several changes of context as a secure foundation to help learners make sense of problematic changes in meaning.

Focusing on how we articulate mathematical expressions can offer profound insight into the long-term learning of arithmetic and algebra. Other observations into the workings of the human brain offer insights into how we think about mathematical ideas at all levels from new-born children to the wide variety of adult thinking. This is part of a broader theory of long-term mathematical development including both historical and individual growth that takes account of cognitive, affective and social aspects.

However, participants in different aspects of the enterprise will have their own views on how they should proceed. Different communities of practice may have radically different approaches that conflict with each other and a change in meaning that one community may see as an enlightenment, another community may see as a transgression. This has led to widespread differences involving 'math wars' between different approaches and it is highly unlikely that a single approach will provide a universal solution.

The contribution of this paper is to reflect on simple yet profound ideas that may enlighten different communities in ways that offer each community appropriate insight.

Difficulties encountered by young children in arithmetic may grow into to mathematics phobia in adults. Mathematics educators often focus on creativity, encouraging young learners to see a specific pattern in many imaginative different ways. The framework recommended here uses the principle of articulation to clarify the meanings of expressions and, by interpreting expressions flexibly as process or object, it goes on to show how equivalent, but different, processes can be conceived as a single object. This makes explicit a long-term implicit development in the curriculum, where equivalent fractions are later seen as the same rational number marked as a single point on the number line, and equivalent algebraic expressions are later seen as a single entity with the same graph. A parallel focus on specific examples and underlying structure offers the possibility of a closer relationship between arithmetic and algebra.

At a more sophisticated level, by realising how the human eye sees variable quantities, the framework offers a new understanding of the use of infinitesimals in the calculus, linking together the different approaches in pure and applied mathematics.

At an even more advanced level of thought, the notion of 'structure theorem' links set-theoretic mathematics and logic back to visual intuition and meaningful symbol manipulation.

The possibilities are immense, especially at a time in history where new technology enables the fundamental operation of Homo Sapiens to function in new ways that not long ago would have been inconceivable. Digital technology offers enactive control of dynamic imagery to support visual intuition, symbolic manipulation to support operational symbolism, together with the ongoing evolution of artificial intelligence that currently falls short of the full capacity of the human brain. It is an exciting time to see how the biological brain uses new facilities to operate ever more powerfully as a mathematical mind.

References

Amalric, M. & Dehaene, S. (2016), Origins of the brain networks for advanced mathematics in expert mathematicians, *Proceedings of the National Academy of Science of the US*A. 016 113 (18) 4909-4917. https://doi.org/10.1073/pnas.1603205113

Chomsky, N. (2006). *Language and Mind*. Cambridge: Cambridge University Press.

Dehaene, S., Spelke E., Pinel, P., Stanescu, R., Tsivkin, S. (1999). Sources of mathematical thinking: Behavioral and brain-imaging evidence. *Science* 284 (5416): 970-974.

Filloy, E., & Rojano, T. (1989). Solving equations, the transition from arithmetic to algebra, *For the Learning of Mathematics*, 9 (2), 19-25.

Gravemeijer, K., Bruin-Muurling G., Kraemer J. M. van Stiphout, I. (2016). Shortcomings of Mathematics Education Reform in The Netherlands: A Paradigm Case? *Mathematical Thinking and Learning*, 18:1, 25-44, DOI: 10.1080/10986065.2016.1107821.

Gray, E. M. & Tall, D. O. (1994). Duality, Ambiguity and Flexibility: A Proceptual View of Simple Arithmetic, *Journal for Research in Mathematics Education*, 26 2, 115-141.

Hadamard, J. (1945). *The Psychology of Invention in the Mathematical Field*, Princeton University Press, (Dover edition, New York 1954).

Hodds, M., Alcock, L., & Inglis, M. (2014). Self-explanation training improves proof comprehension. *Journal for Research in Mathematics Education, 45*, *62*-101.

Inglis, M., & Alcock, L. (2012). Expert and novice approaches to reading mathematical proofs. *Journal for Research in Mathematics Education, 43*, 358-390.

Isoda, M. & Tall, D. O. (editors), (2018). *Mathematics: Junior High School 1, 2, 3*, English Edition. Tokyo: Gakkotosyo.

Katz, M. & Tall, D. O. (2012). The tension between intuitive infinitesimals and formal analysis. In Bharath Sriraman, (Ed.), *Crossroads in the History of Mathematics and Mathematics Education, (The Montana Mathematics Enthusiast Monographs in Mathematics Education 12),* pp. 71--90.

Kozielecki, J. (1987). *Koncepcja transgresyjna człowieka* [Transgressive concept of a man]. Warszawa: PWN.

Lave, J. & Wenger, E. (1991). *Situated Learning: Legitimate Peripheral Participation*. Cambridge: Cambridge

University Press. ISBN 0-521-42374-0.

Maruyama, M., Pallier, C., Jobert, A., Sigman, M., Dehaene, S. (2012). The cortical representation of simple mathematical expressions. *Neuroimage* 61(4):1444-1460.

McGowen, M. & Tall, D. O. (2010). Metaphor or Met-before? The effects of previous experience on the practice and theory of learning mathematics. *Journal of Mathematical Behavior*, 29, 169-179.

Monti, M. M., Parsons, L. M., Osherson D. N. (2012) Thought beyond language: Neural dissociation of algebra and natural language. *Psychological Science*, 23(8), 914-922.

Nakai, T., Sakai, K. L. (2014). Neural mechanisms underlying the computation of hierarchical tree structures in mathematics. *PLoS One* 9(11): e111439.

Nieder, A., Dehaene, S. (2009). Representation of number in the brain. *Annual Review Neuroscience*, 32, 185-208.

Piaget, J. (1952). *The Child's Conception of Number*. New York: Norton.

Pieronkiewicz, B. (2014). Report of Mathematical Transgressions Conference, *Annals of the Polish Mathematical Society, 5th Series: Didactica Mathematicae* 36, 163-168.

PISA (2015). *Programme for International Student Assessment.* Retrieved from https://en.wikipedia.org/wiki/Programme_for_International_Student_Assessment, November 30, 2018.

Shum, J., Hermes, D., Foster, B. L., Dastjerdi, M., Rangarajan,V., Winawer, J., Miller, K., Parvizi, J. (2013). A brain area for visual numerals. *Journal of Neuroscience*, 1, 6709-6715.

Stewart, I. N. & Tall, D. O. (2014). *Foundations of Mathematics*, Second Edition. Oxford: OUP. ISBN: 9780198531654.

Stewart, I. N. & Tall, D. O. (2015). *Algebraic Number Theory and Fermat's Last Theorem*, Fourth Edition, CRC Press. ISBN: 9781498738392.

Stewart, I. N. & Tall, D. O. (2018). *Complex Analysis*, Second edition. Cambridge: Cambridge University Press. ISBN: 978-1108436793.

Tall, D. O, Thomas, M. O. J., Davis, G. E., Gray, E. M., Simpson, A. P. (2000). What is the object of the encapsulation of a process? *Journal of Mathematical Behavior*, 18 (2), 1-19.

Tall, D. O. (2013). *How Humans Learn to Think Mathematically*. New York: Cambridge University Press. ISBN: 9781139565202.

Tall, D. O. (2019). Complementing supportive and problematic aspects of mathematics to resolve transgressions in long-term sense making. *The Fourth Interdisciplinary Scientific Conference on Mathematical Transgressions*, Krakow (to appear).

Tall, D. O. & Katz, M. (2014). A Cognitive Analysis of Cauchy's Conceptions of Function, Continuity, Limit, and Infinitesimal, with implications for teaching the calculus. *Educational Studies in Mathematics*, 86 (1), 97–124.

Tall, D. O., Tall, N. D. & Tall, S. J. (2017). Problem posing in the long-term conceptual development of a gifted child. In Martin Stein (Ed.) *A Life's Time for Mathematics Education and Problem Solving*. *On the Occasion of Andràs Ambrus' 75th Birthday*, pp. 445-457. WTM-Verlag: Münster.

Tall, D., O., de Lima, R. & Healy, L. (2014). Evolving a Three-world Framework for Solving Algebraic Equations in the Light of What a Student Has Met Before. *Journal of Mathematical Behavior*, 34, 1-13.

Van den Heuvel-Panhuizen, M. & Drijvers, P. (2014). Realistic Mathematics Education, in S. Lerman (ed.), *Encyclopedia of Mathematics Education*, DOI 10.1007/978-94-007-4978-8, Springer: Dordrecht. https://www.icrme.net/uploads/1/0/9/8/109819470/rme_encyclopaediamathed.pdf.

TIMSS (2015). *TIMSS 2015 results*. Retrieved from http://timss2015.org, November 30, 2018.

Verhoef, N. C., & Tall, D. O. (2011). Teacher's professional development through lesson study: effects on mathematical knowledge for teaching. *Proceedings of the 35e Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4 (pp. 297-304). Ankara, Turkey.

Verhoef, N. C., Tall, D. O., Coenders , F. & Smaalen, D. (2014). The complexities of a lesson study in a Dutch situation: Mathematics Teacher Learning. *International Journal of Science and Mathematics Education* 12: 859-881. doi: 10.1080/19415257.2014.88628.

Wikipedia: Limbic System. Retrieved from https://en.wikipedia.org/wiki/Limbic_system. November 30, 2018.