Making sense of elementary arithmetic and algebra for long-term success

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This chapter is written to encourage elementary teachers to make sense of mathematical ideas for themselves and for their learners. Mathematics grows more sophisticated as it develops in history and in the learning of the individual. In the individual, it begins with practical activities —recognising shapes, describing their properties, learning to count, performing simple arithmetic, recognising general relationships— and evolves into more theoretical ideas based on definition and deduction. Broadly speaking, this may be seen as a steady growth from the *coherence* of *practical mathematics*, where ideas fit together in a meaningful way, to the *consequence* of *theoretical mathematics*, where mathematical ideas are formulated as definitions and properties are proven from those definitions.

As new contexts are encountered, sometimes the individual has the necessary experience to make sense of the new situations and sometimes ideas that worked before become an impediment to future learning. For example, young children may perform calculations with small numbers using their fingers, but this will not work with larger numbers and will need rethinking to deal with fractions or signed numbers. Powers of numbers such as 2^2, 2^3, … may be introduced as repeated multiplication 2×2×2, … but this will no longer work for fractional or negative powers.

Measuring various quantities, such as time, distance, speed, area, volume, require new ways of thinking. If we start with a length of 4 metres and take away 1 metre, we are left with 3 metres, but we cannot start with 4 metres and take away 5 metres, because we can't have a length less than zero. Or can we? If we measure 4 metres in one direction then move back 5 metres, the net result is simply 1 metre in the reverse direction. Here it is possible to rethink the mathematics and see new possibilities working with signed numbers to extend the system of counting numbers.

In other contexts, as we generalise operations, the meanings may require us to think beyond our practical experience. For instance, if we measure a length of 2 metres, its square, 2^2 is an area and its cube 2^3 is a volume, but what kind of figure is represented by 2^4? Does it involve a fourth dimension when we only live in three?

Different individuals meet these challenges in different ways and this applies to teachers as well as learners. Each of us has a personal history and we have learnt to cope with mathematics in our own way. It is essential for us to understand how we make sense of mathematics so that we can seek to become aware of our own strengths and weaknesses and how these may help us mentor new generations to make sense of mathematical ideas.

Changes in meaning in the long-term development of mathematical thinking

In this chapter, we consider the long-term development of arithmetic and algebra from the point of view of the learner, to gain insight into the changes of meaning that are encountered. Some of these lead to new ways of thinking that offer power and insight while others cause difficulties that impede progress.

The way in which we *speak* a mathematical expression can affect its meaning. For instance, if we say '2+2×2' as 'two plus (pause) two times two' then this means something quite different from 'two plus two (pause) times two'. (Say it to yourself and think about it.) By taking account of how we speak mathematical expressions, we can begin to make sense of new meanings introduced in new contexts.

While some meanings change, other properties remain stable throughout school mathematics, particularly those of the foundational operations of addition and multiplication. For example, if we add a collection of numbers such as 2 + 5 + 3 +1, then it does not matter in what order we work out the result. On the other hand, a mixture of operations such as 9–5+3 is affected by the way in which we group the terms, so that (9–5)+3 is 7 while 9–(5+3) is 1. However, adding together mixed types of numbers such as 9+(–5)+3 is independent of the order of operation, whether the terms are added in the given order or as 9+3+(–5), or in some other sequence. This continues to work with sums involving mixtures of whole numbers, decimals, fractions, and negative numbers, such as 2+5+(-7)+1.4+¾. The same is true for the product of numbers: the order of multiplication does not affect the result.

These general principles also apply in algebra, though here we use various conventions for writing expressions in more compact ways. A product of numbers and variables, say '*x* times 2 times *b* times 3', can be written in a standard way by working out the product of the numbers first and then following it with the letters, usually in alphabetical order, as '6*ax*'. Using the index notation x^2 to represent $x \times x$ allows us to write expressions such as the quadratic ax^2+bx+c and mentally manipulate them in a more flexible way. We may 'factorise $3x^2–5x–2$' to get $(3x+1)(x–2)$ or 'multiply out brackets' in $(3x+1)(x–2)$ to get $3x^2–5x–2$ or $(x–2)(3x+1)$. The various expressions $3x^2–5x–2$, $(3x+1)(x–2)$ and $(x–2)(3x+1)$ look different and involve different sequences of calculation, but fundamentally, they are just different ways of writing the same underlying expression.

Mathematicians and mathematics educators interpret what is going on in various different ways. Mathematicians develop sophisticated meanings that can be written in subtle, flexible ways, but educators study what is happening in the classroom and report the observed development of learners. Each of us, with our own particular history of personal development, will be sympathetic to some aspects while being less sympathetic to others. It is important to keep an open mind and attempt to grasp the relevance of various viewpoints.

In earlier times the mathematics curriculum in arithmetic and algebra was often seen as an accumulation of techniques for making calculations in increasingly complicated situations, including conventions relating to the use of brackets and the order of precedence of operations. This can lead to arithmetic and algebra being seen as increasingly difficult so that some children, and also some teachers, believe that mathematics is too complicated for them to be able to make sense of what is going on. This is a belief that needs to be challenged.

Progress can be made to improve both the learning experience of children and the conceptions of teachers by reflecting on how learners make sense of mathematics at a given time and how these ideas evolve in sophistication over the long term. As a by-product, this not only helps us to teach more sensitively, it may also help us understand the sources of our own beliefs and attitudes to help us to make sense of mathematics for ourselves.

Individuals think very differently. Some children have special educational needs related to physical, social and psychological difficulties. Others may be categorised as gifted or talented in a wide array of areas. This will lead to a broad range of success in making initial sense of ideas and drastically affect how individuals cope with successive levels of sophistication in the longer term. In this chapter, we seek underlying principles to give us greater insight into those ideas that are supportive in long term learning and the changes in context and meaning that may impede progress.

We begin by studying the broad sequence of development from the child's earliest experiences with number, considering the practical activities that lay the foundation of the later development of more theoretical aspects of arithmetic and algebra.

In today's complex world, it is common for an individual teacher to concentrate on the detail of what they teach at a particular stage of development to become an expert in early learning, or in a particular stage in the primary, secondary or advanced school curriculum, or in various topics at undergraduate, graduate or research mathematics. Here we emphasise the need for teachers at any given stage to be aware of the supportive and problematic aspects that learners may bring from their previous experience and also be aware of how learning at a particular stage plays its part in longer-term development.

Early experiences with numbers

A child's early experiences with number starts with practical activities, such as singing songs that incorporate numbers in sequence, or counting the stairs going up or down a staircase. This is in itself part of the process of associating words with people, objects and actions, 'mummy', 'daddy', 'dog', 'cat', 'come here'. Pointing at a person or an object and saying a word, is intended to associate the word with the person or object. Counting is somewhat different. Pointing at the objects in a collection, saying the words 'one', 'two', 'three', … is intended to count the collection, rather than associating the last word said with the last object. To count the number of objects in a collection requires more than being able to perform the complex act of counting, it requires grasping the principle that whatever order the objects are counted, when it is done correctly, the resulting number is always the same.

Practical activities soon include the operations of adding and take away. Not only does this include developing specific knowledge of number facts such as 2+2 is 4 or 5+3 is 8, it also builds an implicit sense that adding gives a larger result, that 'take away' gives a smaller result, and that it is not possible to take a larger number from a smaller one. Such subtle subconscious ideas become part of the child's mental imagery for whole number arithmetic. They may then act as barriers to learning operations with new kinds of numbers. How can multiplication make *less* when multiplying fractions? How can 'take away' give *more*? What does it mean to multiply negative numbers? How can the product of two negatives be positive?

The curriculum is organised to build mathematical ideas from simple addition of two single digit numbers, then numbers up to 100, introducing place value, learning multiplication of whole numbers as repeated addition, using multiplication tables to remember products, then applying these ideas in various contexts such as measuring length, time, speed, weight, volume, and other mathematical and scientific concepts. Now the product of two lengths gives an area, of three lengths gives a volume, but how can we visualise the product of four lengths?

The fact is that, with *numbers*, powers are not a problem. We can calculate powers of a number simply by repeated multiplication, so that $2^2 = 4$, $2^3 = 8$, $2^4 = 16$, and so on. This suggests that a good way forward is to begin with meaningful experiences of simple counting and then to focus on the properties of operations with numbers.

Flexible properties of addition and multiplication

Making sense of mathematics in the long term is likely to be enhanced by starting with experiences that are designed to make sense to the learner at the time, with the teacher, as mentor, being aware of those aspects that remain stable in the long term and those that involve subtle changes in meaning.

Two fundamental properties of addition and multiplication remain stable throughout school mathematics. One is that when we add a finite collection of numbers together, such as 5+3+26+44, then it does not matter in what order the operations are carried out. We can add them together in sequence, 5+3 is 8, then 8+26 is 34, then 34+44 gives the total 78. Or we could add 26+44 to get 70, add 5+3 to get 8, then add 70 and 8 to again get 78. With experience, we can begin to build up a belief in what we might term the General Principle of Addition:

> (GPA) The sum of a finite collection of numbers is independent of the order of calculation.

There is a corresponding General Principle of Multiplication:

> (GPM) The product of a finite collection of numbers is independent of the order of calculation.

Both of these principles arise in practical mathematics of experience and coherence. They may be seen to always work. However, in theoretical mathematics, it is more usual to formulate the more fundamental commutative rules (C) and associative rules (A):

(C) $a+b = b+a$, $ab = ba$,

(A) $(a+b)+c = a + (b+c)$, $(ab)c = a(bc)$.

It is, of course, possible to prove (GPA) and (GPM) from the rules (C) and (A), but this requires a huge conceptual leap which, if done fully, requires the use of proof by induction. The general principles (GPA) and (GPM) are more useful in practical mathematics than the effort required in theoretical mathematics. In particular they are very helpful in the longer term because precisely the same properties hold as we move from whole numbers to fractions, signed numbers, decimals, real numbers and even complex numbers. A sum such as 8+(–5)+12 has the same result whatever order the calculation is performed: it can be calculated as 8+(–5) is 3, then 3+12 is 15, or as (–5)+12 is 7, then 8+7 is 15, or in any other order. These underlying principles also work in algebra and provide foundational principles for long-term learning in arithmetic and algebra.

Long-term success in mathematics is enhanced by realising the flexible relationships that occur naturally in simple arithmetic and also realising subtle differences in meaning that occur when expressions involve different operations or when arithmetic and algebra are used in different contexts.

We begin with a simple example that can be used to illustrate flexibility of whole number arithmetic. It can be used at a stage when a child has experience in counting to ten (perhaps by singing nursery rhymes involving counting) and has the distinction that it shows the flexibility of addition and multiplication while laying the foundations for later evolution of meaning in more sophisticated number systems.

The flexible number 6

The number 6 is chosen because it is the smallest number that is the product of two different whole numbers (not including 1).

A set of six objects can be moved around and counted in many different ways, each time getting the same total ‘6’. It is possible to count all the objects in a set of six objects, which can be broken down into subsets, say of 4 and 2, to see that counting four objects, adding a second set of two objects allows us to ‘count all’ to see that 4+2 is 6. It can be ‘counted on’, starting with a set of four objects and counting on two to get ‘five, *six*’ giving 4+2 is 6. The same two subsets can be left in the same place, but counted in a different order to get 2+4, or they can be moved around and recounted in any order, again getting the same number. By placing them in 2 rows of 3 or 3 rows of 2, we get 2 lots of 3 or 3 lots of 2 is 6.

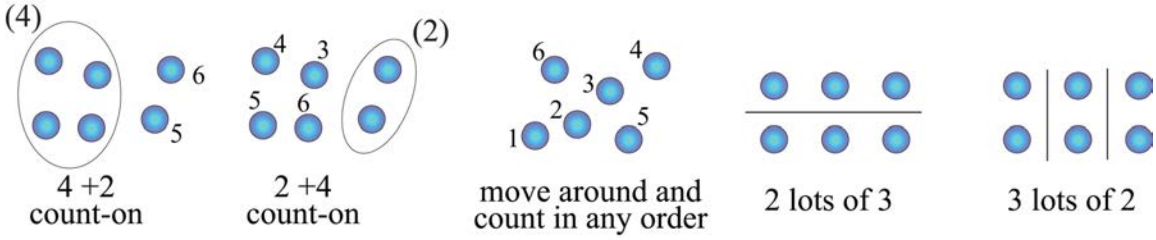

Figure 1: The flexible number 6

The essential foundational idea is that, for a given collection of objects, it does not matter in what order they are counted, or how the set is subdivided, the number of objects is always the same. This underlying flexibility of addition and multiplication of whole numbers proves to lay a sound long-term basis for learning arithmetic. However, as soon as an expression involves more than one operation, the situation becomes far more complicated.

Combining two operations

When an expression combines addition and multiplication, as in 2+2×2, then the result depends on how the expression is interpreted. Is it ‘2+2 (which is 4) times 2’, which gives 8? Or is it ‘2 plus 2×2’, which is ‘2 plus 4’, giving 6? It turns out that the meaning can be made clearer by the way the expression is spoken. If we say ‘2 plus 2’ (pause) ‘times 2’, then this evidently means 2+2 is 4, then ‘times 2’ gives 8. But if we say ‘2 plus (pause) 2 times 2’ then this means 2+4 which is 6.

This difference is part of an amusing video in which a student stands in front of a wall moving towards her, with a swimming pool behind. On the wall is the problem ‘2+2×2’ and two doors, one marked ‘6’, the other ‘8’. If the student makes the ‘correct’ choice ‘6’, then the door opens and she is safe. Otherwise, the ‘incorrect’ choice ‘8’, does not open the door and she is pushed into the pool.

Figure 2: What is 2+2×2?

To be able to answer correctly, the student only needs to remember the rule that ‘multiplication takes precedence over addition’. But many students fail: reading the expression from left to right gives ‘2 plus 2’ (which is 4), then 2+2×2 is 4×2, which is 8. The experience of reading symbols sequentially from left to right makes it natural to first perform 2+2, then multiply the result 4 by 2. The rule that multiplication is performed before addition involves calculating the expression in a less natural sequence, first performing the calculation at the end (2×2) and then adding the first part (2+) to get the final ‘correct’ result.

Mathematics educators over the years have analysed the ways in which children count to add numbers, for instance, calculating 8+2 by counting a set of 8 objects, then a set of 2, then putting the objects together to count them all. A more efficient method is to start with the 8 in the first set, and simply ‘count on’ two more numbers to get ‘nine, *ten*’. This leads to a difference in meaning between 8+2 and 2+8. The first of these, 8+2, only requires starting at 8 and counting on two numbers while 2+8 starts at 2 and counts on eight numbers as ‘three, four, five, six, seven, eight, nine, *ten*’.

In the UK in the seventies and eighties, the common practice was to regard 8+2 in terms of ‘count-on’ starting with an initial number ‘8’ followed by the operation ‘+2’ to give the final answer ‘10’. Essentially this focuses on addition as ‘count on’. It developed into a theory called ‘state-operator-state’, beginning with the first state ‘8’, applying the operator ‘+2’ to count on two to get the final state ‘10’.

The same theory was then used for multiplication in which ‘3×2’ starts with the first ‘state’, ‘3’, performing the operator ‘×2’ giving the final state ‘6’. This led to the interpretation of 3×2 as ‘two threes’ or 3+3. On the other hand, tables were learnt by saying 3×2 as ‘three twos’ so that the ‘two times table’ was said as

One times two is two, two twos are four, three twos are six, …

and written as

$1\times2=2,\ 2\times2 = 4,\ 3\times2 = 6, \ldots$

Here 3×2 interpreted as ‘three twos’, which is 2+2+2.

For a time, while teaching simple arithmetic, a distinction has been made between 3×2 and 2×3. This happened not only in England, but in Japan and other East Asian Communities, such as Hong Kong. My Hong Kong colleague Chun Chor (Litwin) Cheng explained this to me by suggesting that it was possible to have three ducks each with two legs but not two ducks with three legs.

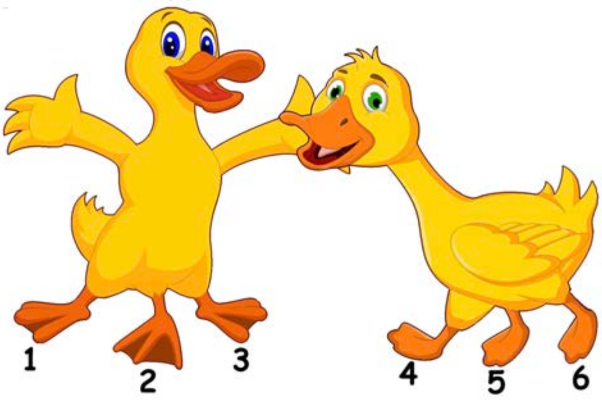

Figure 3: Two ducks each with three legs

Although in the real world, one cannot have two ducks with three legs, in the world of imagination, it is possible to count the number of legs in one’s mind’s eye to see that two lots of three is again six. In arithmetic, what matters is the relationship between the *numbers*, where 3×2 and 2×3 are simply different ways of calculating the *same* result.

Relationships between numbers in whole number arithmetic may be visualised using counters. For example, by looking at the array of counters in Figure 4, it is possible to see that the calculation of 2 times 3+4 gives the same result as 2×3 plus 2×4.

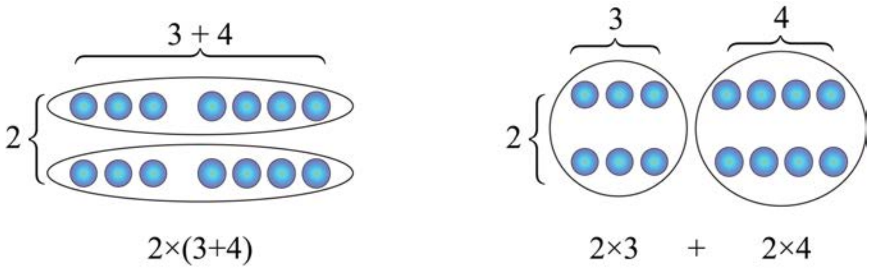

Figure 4: Two times 3+4 gives the same result as 2×3 plus 2×4

Using algebraic notation, this later generalises to *the distributive law* that multiplication is distributive over addition

$a\times(b+c) = a\times b + a\times c$

where a, b, c stand for any numbers. We may even compress the symbolism a little by using the convention that the multiplication sign × may be omitted in products such as $a\times b$ where there is no ambiguity to represent the distributive law as

$a(b+c) = ab + ac.$

Such conventions need to be handled with care. We can omit the multiplication sign in products such as $a\times b$, $2\times x$ or $3\times a\times b$ to write ab, $2x$ or $3ab$, but not in 2×1 or 3×½ where 21 and 3½ represent different conventions. Where we have a mixed product of numbers and variables such as $2\times a\times 3\times b$ we also organise the terms in a standard way by multiplying the numbers and placing the product in front of the letters as $6ab$.

The way in which we interpret these conventions is extremely important. If a learner imagines the symbol 3+2 as an instruction to perform some counting operation (e.g. count-all or count-on), then the symbol $a+b$ can have no meaning, because, if you don't know the actual values of a and b, then you can't perform the actual counting process. It is therefore essential to be aware of how learners build up meaning for mathematical expressions over the long term.

Speaking mathematically

Focusing on how we speak mathematical expressions allows us to clarify the different ways in which expressions may be interpreted. This happens not only with expressions involving a mixture of addition and multiplication, but whenever an expression involves other operations, such as '7–3+2' which can be 'seven take away three (pause) plus two' which is 4+2 = 6 or 'seven take away (pause) three plus two' which is 7–5 = 2. Likewise, 'the square root of nine times nine' can be 'the square root (pause) of 9×9' which is 9, or 'the square root of nine (pause) times nine' which is 27.

Even expressions involving a single operation other than addition or multiplication depend on the way that they are spoken. For instance, 'Five take away three take away one' would normally be interpreted in sequence as it is spoken to start with 5, take away 3 to get 2 and then take away 1, to get 1. But 'Five take away (pause) three take away one' could also mean '5 take away 1' to get 4.

In general, interpreting expressions in the sequence in which they are spoken is more natural, so when writing from left to right, as in English, it is natural to interpret 5–3–1 as 5 take away 3, then take away 1, to get 1. This is why 2+2×2 causes problems, because the accepted meaning (with multiplication taking precedence over addition) requires working out the second operation 2×2 first, which contradicts the natural order reading left to right.

There are, of course, differences in the direction of writing various languages. Most European languages read from left to right down the page while Arabic and Hebrew read from right to left. Meanwhile Chinese script is written down vertical columns starting on the right of the page with columns moving successively to the left. Traditional Japanese script (*tategaki* 縦書き) follows the Chinese convention, while more modern technical Japanese in scientific and mathematical texts is written following the European convention in rows from left to right (*Yokogaki* 横書き).

However, when an expression is *spoken*, the terms follow one another in time, so they offer a natural sequence of operation, which can be further clarified by the way in which the words are grouped together when they are spoken or written. In all languages, it is useful to think carefully about how expressions are spoken and how this is translated into written symbolism.

Giving meaning to spoken and written expressions

Once the learner is aware that the way in which an expression is spoken can affect its meaning, it becomes more appropriate to introduce brackets to distinguish between meanings. The convention is that operations inside brackets are performed first. So 2+(2×2) means 2+4 which is 6 while (2+2)×2 is 4×2, which is 8. Using the same idea, (7–3)+2 is 4+2 = 6, while 7–(3+2) is 7–5 = 2. Earlier, we spoke of the meaning of 'the square root of 9×9' which now has two distinct meanings as $(\sqrt{9})\times 9 = 3\times 9 = 27$ and $\sqrt{(9\times 9)} = \sqrt{27} = 9$.

It is important to begin with simple examples so that when a learner reads an expression this is performed with an understanding of its meaning. For instance, if a child does not make sense of the conventions, an expression like $2+3x$ might cause bewilderment. What does it mean? Reading it from left to right, it first says '2+3', which has a meaning in itself as '5', but the 'x' is, literally, 'unknown', so the child may simplify $2+3x$ to carry out the part of the expression that can be calculated and leave the rest as it is to write down the 'answer' as $5x$.

There is a vast literature documenting children's 'misconceptions' in arithmetic and algebra. This is useful data. However, the problem lies in how this data is used to attempt to help children make sense of the mathematics. It is not sufficient simply to say that these interpretations are wrong and to attempt to teach the children the right conventions. Long-term difficulties in arithmetic and algebra remain throughout the world. To seek a better understanding of the situation, we focus on the underlying development of mathematical thinking as the child encounters new ideas.

Long-term compression of symbols from operations to mental objects

As mathematics becomes more sophisticated, expressions which stand for operations to be carried out can be interpreted in different ways. For instance, we have already seen that the expressions 8+2 and 2+8 are calculated in very different ways using 'count-on'. So, as operations carried out in time they are quite different, yet they give the same numerical result, so that 8+2 is the same number as 2+8. As calculations they are different, but as numbers they are the same.

In multiplication 2×3 involves a different calculation from 3×2: one is 3+3, the other is 2+2+2, but the result is the same. Again 2×3 and 3×2 are different as calculations but represent the same number.

In fractions, $^{2}/_{4}$ and $^{3}/_{6}$ are different as operations, one involves dividing a unit into 4 equal pieces and selecting 2, the other divides the unit into 6 equal pieces and selects 3. They lead to a different number of pieces of different sizes, but the quantity in both cases is a half. Our use of language even emphasises this: we speak of *equivalent* fractions, $^{2}/_{4}$ and $^{3}/_{6}$. By saying they are 'equivalent', we are implicitly suggesting that they are not 'the same'. Yet, when we mark them on the number line, they are marked at the same point and are seen as two different ways of representing the single 'rational number', ½.

In dealing with negative numbers, 'taking away +2' and 'adding –2' are different operations, but they have the same effect. The result of the two operations $5-(+2)$ and $5+(-2)$ is the same as $5-2$, giving 3 in each case.

Throughout the long-term development of the curriculum we learn new operations where different sequences of calculation lead to the same underlying concept. This allows a greater flexibility in the use of spoken and written symbolism. A spoken symbol such as 'three plus two' written as '3+2' can refer to a *process*, 'add three and two', or to a *concept*, 'the sum of three and two'. Not only can the same symbol refer to different processes, but different symbols, such as '3+2' or '2+3' can be interpreted flexibly as different processes or as the same concept. This flexible use of symbolism as process and/or concept is named a *procept* (Gray & Tall, 1994, Tall, 2013). As arithmetic and algebra develop in sophistication, it is the flexible use of symbolism that allows more complicated ideas to be imagined in ways that are both simpler to speak and write, yet also more powerful in use.

This becomes more apparent as we move from arithmetic to algebra. If x represents a number, then we write twice the number as $2\times x$, and often shorten this by missing out the multiplication sign between a number and a letter to write it more compactly as $2x$. The expression $2x + 1$, then means 'twice x plus one'. The question to ask is 'is $2x+1$ a *process* of calculation, or is it a *concept* that can be imagined as something that can be manipulated mentally?" The answer is that it can be *either* or *both*. We not only need to think of $2x+1$ as a process of calculation for a given number x, or as a more general formula to calculate the value for *any* particular value of x, but also as an entity in itself that can be manipulated. For example, we may wish to multiply it by x+3, writing the expressions 2x+1 and x+3 in brackets to show how the operations are performed as

$$(2x+1)(x+3).$$

It is helpful to think of 2x+1 as a single entity and use the distributive law to get

$$(2x+1)\times(x+3) = (2x+1)\times x + (2x+1)\times 3$$

then use the distributive law again for each of the brackets on the right-hand side to get

$$(2x+1)\times x + (2x+1)\times 3 = 2x\times x + 1\times x + 2x\times 3 + 1\times 3$$

then rearranging terms, multiplying numbers together, writing $x\times x$ as x^2 and combining terms where possible gives

$$(2x+1)(x+3) = 2x^2 + 7x + 3.$$

Once again, we have the same phenomenon. The expressions on each side of the equation can be seen as different processes of calculation yet represent the same algebraic concept. Such an equation is called an *identity*. It works for *any* value of x.

Other equations such as

$$3x+2 = 11$$

only work for certain values of x and we are invited to 'solve' the equation to find specific values of x that make it true. In this particular case, we can think of the left hand-side as a process, starting with x, multiplying by 3 to get 3x then adding 2 to get 11. We can then reverse these steps to solve the equation. If 3x+2 is 11, then 3x must be 11–2, which is 9 and so x must be 3.

This technique only works if we have a number on the right-hand side. If we are faced with an equation such as

$$3x+2 = x+8$$

then we cannot treat the two sides as operations 'to be undone' as we can't simultaneously take 2 from the left-hand side and 8 from the right-hand side. Now we need to think of each side as a *number* that is written in two different ways. We can then take 2 from both sides to get

$$3x+2-2 = x+8-2$$

which can be simplified to get

$$3x = 6,$$

so, in this case, x is 2.

Anyone who has qualified to teach mathematics will have met these ideas and yet the mathematics education literature is full of the recorded difficulties that so many learners experience. The step from solving an equation of the form 3x+2 = 11 with a number on the right-hand side to solving an equation such as 3x+2–2 = x+8–2 has been shown to be of significant difficulty for over thirty years (see, for example, Filloy & Rojano, 1989). This relates to the fact that the first equation looks like a *calculation* that gives a *number* as an answer, which can be undone by simple arithmetic. The second equation involves operations on the unknown value x which cannot be performed purely using arithmetic when x is not a known number.

Although it is possible to teach children to manipulate equations by learning rules by rote, this causes problems, beautifully expressed by Pierre Van Hiele, who wrote;

> When I wanted to learn algebra myself for the first time – I was ten years old and I had found a textbook on that subject – I was overwhelmed by the vast number of rules you

had to apply. You were told that $3a+4a = 7a$, but $3a\times4a = 12a^2$. To make it more difficult $a^3\times a^4 = a^7$. In secondary school, I got another textbook of algebra but it was not much better. However, I had no more difficulties with algebra because I had overcome them when I was ten. Van Hiele (2000) p.27

This comment expresses two sides of the phenomenon: the bewilderment of the young child presented with too many rules that do not make sense and the eventual success of a child who later became a leading thinker in mathematics education. However, many children do not make sense of this way of thinking. Some find the algebraic manipulation too complicated and others may be able to carry out the operations while feeling uncomfortable about what they are doing, as typified in the following student comment:

> I was always good at math. But, I didn't really like it. [….] Why? I don't know. I guess I always felt like I was getting away with something, you know, like I was cheating. I could do the problems and I did well on the tests, but I didn't really know what was going on. Wilensky (1998) p.184

Different individuals develop in different ways. Children with learning difficulties may not be able to remember simple arithmetic facts, some may find procedures involving several steps to be too complicated to carry out in full and, even if a child can carry out a particular procedure to get a specific answer, this activity may be seen in terms of procedures to be carried out in time without seeing an expression as a single entity that can be manipulated as a mental object.

Finding mathematics difficult is not something that occurs only in children. It occurs in virtually all of us at some stage or other. (It happened to me personally in my second year as an undergraduate at Oxford. In the first year, I had some difficulties with some of the courses, but when I revised them for examinations, I believed I understood all of the material. However, even though I was awarded a prize given to the top three students out of 150 in the final year, I found that I could only cope with enough questions to gain a high mark overall and could not answer others. If this happened to a prize-winning student, what happens to the rest?)

In recent years, research in neurophysiology has begun to uncover reasons for these emotional reactions.

Mathematical Thinking and the Brain

The human brain is an immensely complex organ. Yet the whole individual begins as a single cell uniting the ovum of the mother and the sperm of the father, which goes through a successive sequence of cell subdivision guided by the individual's DNA to develop into the new-born child. These successive subdivisions produce a human brain with two symmetric halves which evolve to perform specific tasks in ways that enable the whole brain to operate as a single unit (Figure 5).

One half— usually the left for almost all right-handers and for the majority of left-handers—handles most of the linear input and output of speech and sequential operations such as counting. The corresponding parts of the brain on the other side usually operate in a more general way, noticing aspects of global imagery, such as estimating number by visual appearance. In some individuals, the division of activity between the two halves may be shared in various ways, for example if the left side of a young child's brain is injured, speech may be reconstructed on the right.

Evolution has constructed connections in the brain that are idiosyncratic, even bizarre, for example, the left side deals with sensory input and action of the right-hand side of the body and vice-versa. Meanwhile, input from the senses is passed to the back of the brain for perceptual interpretation and then to the front of the brain for physical action and mental reason.

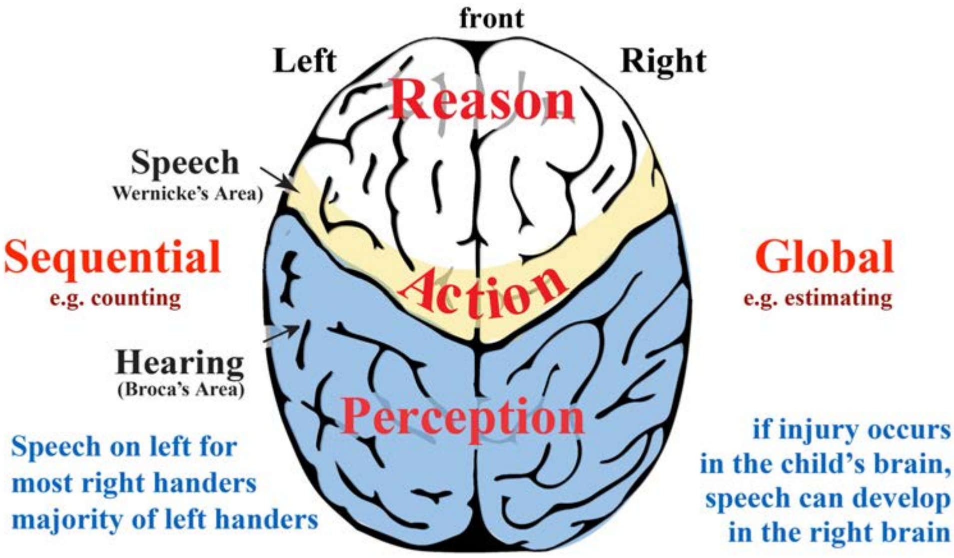

Figure 5: The human brain from above

In the centre of the brain, straddling both sides, is the limbic system. (Figure 6.) This includes a range of functions including links to long-term memory and also a primitive 'fight or flight' system that heightens or suppresses neuronal links. Perceptual data that has been passed to the back of the brain for processing, passes through the limbic system before being processed logically by the prefrontal cortex at the front of the brain.

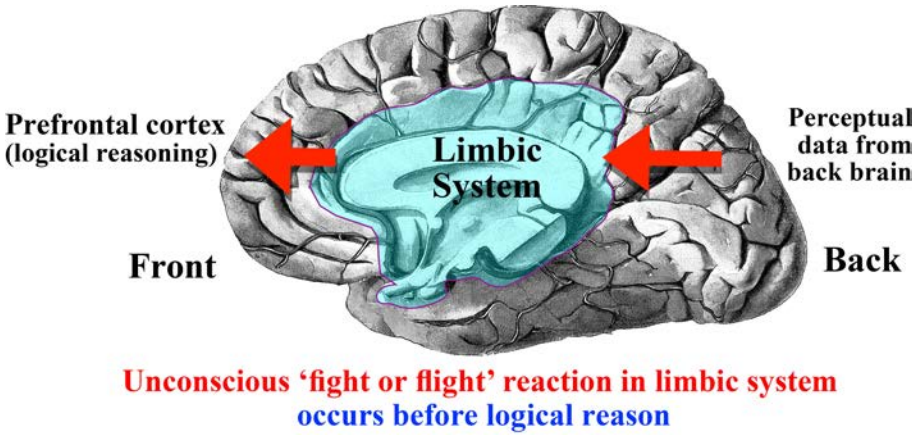

Figure 6: Cross-section of the brain seen from the left

This passage from perception to logical reasoning through the limbic system may cause an immediate emotional reaction which puts the individual in 'fight or flight' mode to deal with what may be interpreted, in a primitive way, as a threat. A biochemical reaction occurs which floods the brain with chemicals that may excite or inhibit connections between neurons. A confident individual may go on high alert to respond positively to a challenging situation. Another may feel threatened and the inhibited brain may be unable to think about the problem at all.

The brain acts as a whole as perceptual inputs and information retrieved from various parts of the brain are linked together. The electrochemical signals in the brain are far slower than in a computer, taking several milliseconds to react, so that initial reactions occur before the longer time it takes to make logical decisions (Kahneman, 2011).

Given the emotional intervention of 'fight or flight' reactions, it may happen that those who are inhibited by a negative reaction may be unable to make sense of the situation. As a consequence, it is not that they are simply 'lazy' or unwilling to work. If they are unable to make sense of the ideas, they may attempt to learn procedures by rote, which may give short-term success but is less likely to support long-term sense making.

These initial reactions depend on the subconscious links that have been formed over the years as we mature. To enhance mathematical thinking in the long-term it is important to enhance the positive connections and to address the difficulties that impede progress.

These links may focus on the *properties of objects*, as happens in geometry and other contexts involving physical objects and mental imagery, or on the *properties of the operations*, that are symbolised in arithmetic and algebra. I use the term 'conceptual embodiment' to refer to the first of these and 'operational symbolism' for the latter, shortening them to 'embodiment' and 'symbolism' as appropriate.

Coherence and Consequence

In the opening paragraph of this chapter, I proposed that the long-term development of mathematical thinking evolves from the *coherence of practical mathematics* to the *consequence of theoretical mathematics*. The young child may begin interacting with physical objects illustrated in figure 7 by a collection of shapes.

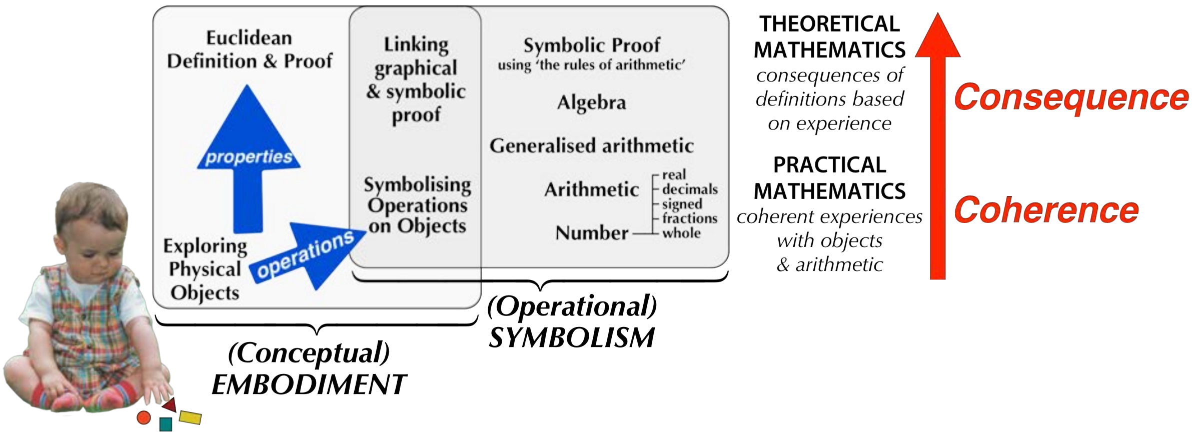

Figure 7: The child's journey through coherence and consequence

A distinction is made between a focus on the *properties* of the objects such as their shape, colour, size, and so on, and on the *operations* that are performed on the objects. Both of these involve both perception and action, but the former concentrates more on global aspects relating to physical perception and mental thought experiment while the latter concentrates more on the sequential properties of processes such as counting, leading to the operational symbolism of whole numbers, fractions, signed numbers, decimals, real numbers, developing a sense of the generalised properties of arithmetic that lead to algebra.

Coherence involves noticing and describing relationships between properties and operations, including constructions in geometry and calculations in arithmetic and recognising general properties of arithmetic that later generalise to algebra. Consequence involves the formulation of specific definitions and the deduction of other properties from the definitions.

Theoretical mathematics can operate in various ways. One involves working with practical ideas, such as drawing geometric figures, or noting the generalised properties of arithmetic and using them to guide operations in algebra. A more sophisticated version involves making step-by-by step proofs based only on the definitions as given.

In arithmetic and algebra, the main definitions are specified as:

(C) the commutative laws: $a+b = b+a$, $ab = ba$ for specific numbers a, b
(A) the associative laws: $(a+b)+c = a+(b+c)$, $(ab)c = a(bc)$
and (D) the distributive law: $a(b+c) = ab + ac$.

The usual approach in school mathematics is to take these rules in a practical sense and use their evident coherence to extend them to the more general properties of arithmetic (GPA) and (GPM) that the operations of addition and multiplication are each independent of the order of calculation, then introduce brackets to specify the order of operations, subject to general rules of precedence, such as PEMDAS (in the USA) and BIDMAS (in the UK). PEMDAS is remembered as 'Please Excuse My Dear Aunt Sally' and specifies the order of preference as 'Parentheses, Exponents, Multiplication, Division, Addition and Subtraction', while BIDMAS puts the order as 'Brackets, Index, Division, Multiplication, Addition, Subtraction'. These rules also involve additional subtleties. For instance, multiplication and division are regarded as of equal precedence and are read as they arise reading from left to right. The same is true for addition and subtraction. This suggests that the rules might be more clearly written as P-E-MD-AS and B-I-DM-AS, which explains why the first has the order MD while the second has DM. With the arrival of the computer, the rules of precedence that are used in different software programs and computer languages increases the complexity.

In the long-term development of mathematical thinking, educational research has shown the difficulties that are encountered by different students. Fundamentally the manipulation of symbols depends on how they are processed by the human brain with or without the assistance of computer technology. Logically, one may reason that this is performed using the rules of arithmetic, but mentally, much of the activity occurs subconsciously.

While pure mathematicians affirm the need for formal proof from minimal definitions such as (C), (A), (D), the step-by-step proof of general principles of arithmetic from these axioms requires proof by induction, which greatly raises the level of difficulty from practical to theoretical. The majority of the population are content to leave the formal proof to the pure mathematicians and focus much more on using the general principles to model problems and solve them using agreed techniques.

The manipulation of symbols depends on the way in which they are processed mentally by the human brain. Logically, one may reason that this is performed using the rules of arithmetic, but mentally much of the activity occurs subconsciously. What we actually 'see' in our mind may be different from what is written. For example, a symbol such as

$$\frac{2}{4}$$

may be seen by many as 'two over four' or 'two fourths', yet others see it automatically as '$\frac{1}{2}$' or 'a half'. When I see '$2(a+b)$', I also see it in my mind's eye as '$2a+2b$'. In the same way, an expression such as $2x^2 + 7x + 3$ may be imagined mentally as a sum of three parts, $2x^2$, $7x$ and 3, that can be moved around, decomposed and recomposed in the mind at will. The symbol 'x' and the power '2' in the term x^2 may be mentally bound together so strongly that they can be manipulated mentally as a single unit so that $2x^2$ is two of these units, while $7x$ is 'seven lots of x' (whatever x is) and the three terms $2x^2$, $7x$, and 3 are added together in any order. At the same time, the term x^2 may be seen as the product of x times x, as required). Even though it is an effort to factorise it to get $(2x+1)(x+3)$, the factorisation leads to a different way of writing the same underlying concept as the product of two mental objects (each in brackets).

Long-term sense making may be assisted in a number of ways. Procedures of calculation can be made more efficient by evolving new ways of working using fewer steps, but the ability to see an expression not just as a procedure to carry out, but as a mental object that can be manipulated makes a major step forward. By taking into account the articulation of spoken symbols it is natural to introduce brackets to give meaning to the sequence of operations and, by using convention to write expressions in more compact notation, it becomes possible to articulate subtle meanings in simpler ways.

Looking to the Future

The approach advocated here involves making sense of ideas in a way that is appropriate for the learner at the time, but also realises what aspects may be stable in the longer term and what aspects may later cause impediments that will need to be addressed at a later stage to make sense of a new situation.

As society evolves, developments in information technology are changing the nature of mathematics as it offers new ways to make calculations and to provide dynamic visual ways of representing ideas. There are also many other factors that affect the long-term development of mathematical thinking. These include the different genetic inheritance of individuals, from those in special needs to those who may be gifted and talented. Individual development is also affected by the social environment in the home and at school. The curriculum is affected by all kinds of input from politicians, parents, teachers, mathematicians, educators, and so on.

The broad approach here focuses on *making sense* of mathematical ideas. Individuals vary in the way that they attempt to make sense. A child with special educational needs finding it difficult to recall simple number facts or to process multi-step tasks will be faced with very different challenges from those with a more coherent grasp of relationships.

Nevertheless, although the degree of success will vary, in arithmetic, most children are likely to benefit from exploring the relationship arising from playing with physical objects and investigating the properties of arithmetic, starting with whole number relationships. In the longer term, successful development of ideas benefits from the immediate recall of simple addition facts and the flexible relationships between them.

In addition to 'making sense' in a given context, such as learning simple arithmetic with whole numbers, it will also be necessary at a later stage to realise that what makes sense in one context may cause subconscious emotional reactions that become an impediment at a later stage. For instance, 'multiplication makes bigger' and 'take away gives less' both work for whole numbers but impede learning of fractions and signed numbers. At such a time, it is important for the teacher as mentor to be aware of changes in meaning and to guide the learner to make sense of the new meaning. Conventions such as PEMDAS or BIDMAS may be of help to codify the conventions of precedence for the flexible learner, but they may be totally meaningless to those who only learn by rote.

There are many pressures in mathematics education today, from politicians who wish to educate their population to compete in the global economy, parents who want their children to have every possible advantage, teachers who wish their students to be successful in examinations, mathematicians who see proof as the essential foundation of their theory, businessmen who require future employees to have certain prescribed skills, and so on.

International comparisons such as TIMMS and PISA suggest that, in the longer term, East Asian countries outperform many western nations. One factor that may contribute is the concentration in countries such as China on fundamental fluency with simple arithmetic. This has longer-term consequences, for example, fluency with whole number arithmetic is a helpful foundation for equivalence of fractions and for more general arithmetical proficiency leading on to flexible relationships in arithmetic and algebra.

In Japan, Lesson Study has been developed to encourage children to work together in carefully designed sequences of lessons to make sense of mathematical ideas. Outside school, many attend juku lessons to develop fluency in mathematical calculations.

This dual focus on flexible understanding of arithmetic and fluency in calculation has a powerful effect on long-term learning. It can be enhanced further by focusing on the spoken and written meaning of expressions, leading to the meaningful use of brackets and conventions that simplify mental manipulation of symbols. It is essential to distinguish between those aspects of addition and multiplication that remain stable in the long term and also to recognise the emotional reactions to changes in meaning that enhance or impede long-term learning.

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