Long-term effects of sense making and anxiety in algebra

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For some students algebra involves pleasurable activities in seeing patterns and building the foundations for more advanced mathematics. For others it is a source of anxiety with little use in everyday life. This chapter offers a 'joined up' framework for mathematics in general, and algebra in particular, to help understand the reality of the full range of experience of the whole population, in particular, why some find algebra a pleasurable activity while others find it a source of anxiety.

Personally I am committed to see algebra introduced in a meaningful and positive manner. However, this has yet to be achieved in the wider population despite many years of seeking ways of improving sense making in algebra all over the world.

Research has shown that as individuals attempt to make sense of increasingly sophisticated ideas over the longer-term, new ways of thinking are required that may be enlightening for some and problematic for others. These have a long-term cumulative effect that widens the gap between those who are successful and enjoy the mathematics, those who are determined to cope with new ideas that they may not fully understand, and those who become increasingly disaffected. While curriculum designers may focus on the positive sequence of achievement that is desired, the implementation of the curriculum depends not only on the teacher's understanding of mathematics but also on how individual learners attempt to make sense of the new ideas and how teachers can mentor them to develop more powerful understanding.

Long-term sense making

While mathematics may be seen to be coherent from an expert viewpoint, this is not true over the longer term for most learners, nor is it true at the boundaries of mathematical research where mathematicians are grappling with new ideas. It is a salutary fact that all mathematicians enter the world as new born children with brains as yet not sufficiently formed to make subtle connections. So everyone goes through a long-term process of making sense of increasingly subtle ideas.

At each stage, new mathematical ideas often require more sophisticated ways of thinking. In considering how we may make algebra more meaningful, it is not sufficient just to look at the status quo. It is essential to see how different individuals make sense of mathematics over the longer term. What is learnt at earlier stages and how it is interpreted by learners at each successive stage continually build up a broader spectrum of different ways of working that affect the attitudes and understandings of new ideas as they are encountered.

For example, in early arithmetic the process of addition becomes more compressed, starting from three counting processes (count one set, count the other, put them together and 'count all') to a single counting process ('count on' the second number after the first), then to a known fact that can then be used as part of a flexible knowledge structure to derive new facts from known facts. This produces a long-term bifurcation in performance as some students continue to use less sophisticated counting procedures that become more complicated while others benefit from the flexible use of more sophisticated ways of thinking that are more productive (Gray & Tall, 1994).

This bifurcation continues as whole number arithmetic develops through more sophisticated topics: making sense of place value, multi-digit addition, subtraction, multiplication, long division of whole numbers, introducing fractions and decimals, negative numbers, and so on. As number systems become more sophisticated, the generalized properties of the operations are seen first as generalized arithmetic, and then as manipulation of variables in algebra, combining operations with symbols and visualization using graphs.

It is inevitable that long-term learning involves successive encounters with new ideas that behave differently from previous experience. This involves not just what the teacher intends to teach, but also what the learner senses in the ideas encountered. These ideas may not be explicitly taught, but they may have a profound effect in the learner's sense of security in handling new experiences. For example, taking away a whole number always gives a smaller result, but this is not always true for signed numbers. Multiplication of whole numbers gives a bigger result, but this doesn't always happen with fractions. In each case the operations become more complicated and may cause uneasy feelings as the nature of number is generalized. How can taking something away give more? How can 'two minuses make a plus'? The square of a non-zero number is always positive, so how can there be a number i such that $i^2 = -1$?

When mathematics is extended to new situations, research tells us that students often have 'misconceptions'. The literature is so vast that it would be invidious to give a single reference. However, the same data may be analyzed in a new way to find that the 'misconceptions' often involve using methods that worked perfectly well at one stage, yet, without reconstruction, fail in a more sophisticated situation.

For instance, the difference between two single digit numbers effectively means 'take the smaller one from the larger', but when a child meets two digit subtraction written in columns, taking the smaller from the larger in each column may lead to an erroneous answer such as concluding that 43 – 27 is 24 because the difference between 4 and 2 is 2 and the difference between 3 and 7 is 4. In such a situation, it seems evident that the learner needs to be taught to use the correct procedure, but is this sufficient to deal with successively more sophisticated procedures? Simply being taught how to perform an appropriate procedure without understanding may lead to greater difficulties being encountered at a later stage, causing even greater confusion over the longer term.

Subtle changes in meaning occur throughout the mathematics curriculum. For example, in shifting from arithmetic to algebra, a sum such as 2+3 always has an answer (in this case 5), but in algebra, an expression involving letters as variables such as $2+3x$ does not. Here it is possible that a particular learner who has not yet made sense of the meaning of algebraic notation may look at $2+3x$ and recognize the first part 2+3 as an arithmetic operation that *can* be performed to give 5, but then the remaining x cannot be incorporated, so the learner leaves the answer as $5x$.

In attempting to help learners make sense of new ideas, it is important for the teacher, as mentor, to be aware of the possible effects of previous learning. For instance, the famous 'students and professors problem' (Clement, Lochhead & Monk, 1981) – in which the number of students is S, the number of professors is P and there are 6 students for each professor – is often written as 6S = 1P. This provoked a whole array of research papers to analyze what is happening and how to deal with it, when the main reason is there for all to see. Letters are often used as units, for example

1m = 100cm

to represent 1 meter is 100 centimeters. Interpreting S as 'students' and P as 'professors', it is evident that 6 students correspond to 1 professor, so this leads to

$6S = 1P$.

Often letters are used as objects to introduce students to manipulation of expressions, so 6a+3b is six apples and 3 bananas, which allows an expression such as 6a+3b+2a to be simplified to 8 apples and 3 bananas or 8a + 3b. The student now *seems* to be able to manipulate algebraic expressions in simple cases and may use this interpretation to have initial success in manipulating algebra. However, in the longer term, this is likely to sow problematic seeds that can grow into the student-professor problem.

This is a recurring phenomenon throughout the curriculum as each individual interprets new ideas in terms of previous experiences that are sometimes supportive, giving increasing mathematical power, and sometimes problematic, causing increasing difficulty.

Supportive and problematic met-befores

As we build on our previous experience, we use ideas that are familiar to interpret new experiences. Having developed language to describe certain familiar circumstances, the same language is available to describe similar ideas in new contexts. This has led to a range of theories in which human thinking is expressed in terms of metaphor (e.g. Lakoff & Johnson, 1980, Sfard, 2008).

The theory of metaphor can be very helpful in gaining insight into puzzling situations in mathematics learning. However, it has an Achilles' heel: it is often used to consider the problem from the sophisticated viewpoint of the expert, and this may be very different from the wide range of thinking of different learners.

As I sought to understand what was happening from the viewpoint of the learner who has yet to develop the sophistication of an expert, I played with the sound of the word 'metaphor' and invented the new word 'met-afore' using the old English word 'afore' to refer to ideas that the learner had met before. Then I replaced 'met-afore' by the new term 'met-before' which distinguished 'metaphor' and 'met-before' in sound as 'metAphor' and 'metBefore'. It can be formulated as follows:

> A met-before is a mental construct that an individual uses at a given time based on experiences they have met before. (Lima & Tall, 2008).

The first publication using the term met-before (Tall, 2004) focused mainly on met-befores that cause difficulty, giving the limited impression that they simply refer to misconceptions. However, it is important to balance the positive and negative aspects to give a balanced view of how we use previous experience in new situations. I defined a *supportive* met-before to be a previous experience that supports learning in a new situation and a *problematic* met-before as a previous experience that causes difficulties. It is essential to see the met-before operating in a specific new context, as a particular previous experience may be supportive in one context and problematic in another. For instance, 'take away gives less' is supportive in whole numbers and fractions (without sign), but problematic in signed numbers and much later in handling infinite cardinal numbers. The way in which a learner copes with a met-before can cause very different emotional reactions and consequent differences in future progress.

This phenomenon is not just restricted only to students, it occurs in experts too, as can be seen throughout history when firmly held beliefs are challenged by new possibilities that prove difficult to grasp. Our language is littered with terminology that reveals these transitions, from *natural* numbers to introduce *negative* numbers, from *rational* numbers to introduce *irrational* numbers, from *real* numbers to *complex* numbers that have real and *imaginary* parts.

As we consider the long-term sense making of our students, we need to be aware that the same mechanisms operate at different levels in the minds of all of us, including teachers, curriculum designers, and expert mathematicians. Moreover, the fact that we have each had our own personal developments in different communities means that we may see the learning of mathematics from very different perspectives and the way in which one community interprets mathematics may be appropriate or entirely unsuitable for another.

Mathematicians who have reorganized their thinking to a powerful expert level may have ways of operating that are good for them yet prove to be problematic for learners, while educators and teachers who present mathematical ideas to learners at a given stage may be unaware of the later consequences of their teaching. It is therefore important to develop an overall picture of mathematical development so that those in different communities of practice can be sensitive to both the long-term goals of learning and using mathematics and also to the personal development of the individual.

Long-term development of mathematical thinking

As mathematics grows in sophistication, both corporately in history and individually in each one of us, changes of meaning occur to deal with new situations. To understand how individuals cope with such changes, it is necessary to consider

- the increasing sophistication of mathematics,
- the personal interpretations of the individual,
- the long-term effects of sense making in increasingly sophisticated contexts.

Long-term development of mathematical thinking begins with practical experiences that develop into more theoretical ways of reasoning. As learners meet new ideas, their personal interpretations are affected by a succession of previous supportive and problematic met-befores that subtly affect their thinking.

Problematic ideas in algebra may have their origins in early arithmetic and accumulate through successive experiences over the years. If the problematic aspects remain unresolved, the spectrum of difficulties may become so complicated that it may no longer be easy to resolve problems arising in a particular topic because their origins are so deeply embedded in the subconscious mind of the learner.

For example, McGowen and Tall (2013) reveal how the minus sign changes its meaning as the curriculum advances through the years, first as an operation of subtraction, then as a sign to denote a negative number such as –3, then as a sign to denote the additive inverse $-x$ which could have a positive value if x is negative. Combine this with another operation, such as squaring, then '–3 squared' may be 'the square of –3' or 'the additive inverse of 3 squared' which is resolved by an appropriate use of brackets, but may, in practice, be confused by students struggling with college algebra. (See McGowen, chapter xxx.)

This general phenomenon occurs throughout learning. At any stage a given group of learners will contain individuals at different stages of development, so what some may grasp easily will be difficult or even impossible for others. Over time, as new challenges are encountered in new contexts, the spectrum of possibilities may become more diverse and the teacher's task in helping students make sense of new ideas becomes more complicated.

Visualizing and symbolizing

Manipulating objects to 'see' relationships is a powerful way of getting a sense of more general properties in arithmetic such as commutativity of addition and multiplication. However, some visual ways of symbolizing algebraic relationships may be supportive at one stage but become problematic in more sophisticated situations.

Tall (2013), chapter 7, considers the case of the algebraic identity

$$a^2 - b^2 = (a - b)(a + b).$$

This can be represented visually as a large square side a, taking away a smaller square side b to see the relationship perceptually (Figure 1).

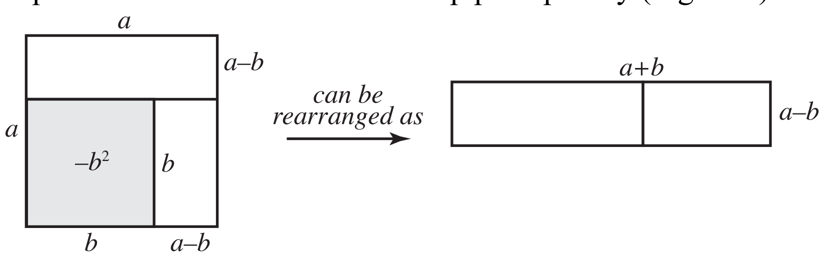

Figure 1: Rearranging the difference between two squares $a^2 - b^2$

This picture of the difference between two squares can be used to visualize the meaning of the equation. However, implicitly, the values of a and b are positive. What happens if one or both of a and b are negative, or if a is *less* than b?

In this case, it is possible to 'see' a change in sign as 'turning over' the square to see the other side, interpreting one side as positive and the other as negative. We can now see that the operation of turning it over changes the sign, then turning it again returns to the original sign, so 'two minuses make a plus' (Figure 2).

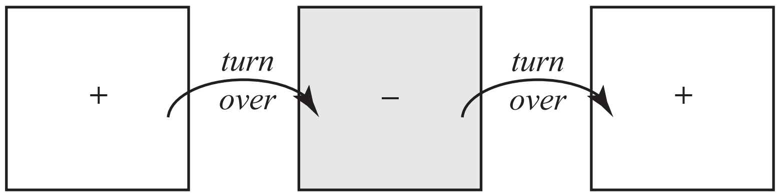

Figure 2: Changing signs by turning over to see the reverse side

When we shift from the difference of two squares in two dimensions to the difference between two cubes $a^3 - b^3$ in three dimensions, we can still see what happens when a smaller cube b^3 is removed from a larger cube a^3 (for a and b both positive) (Figure 2).

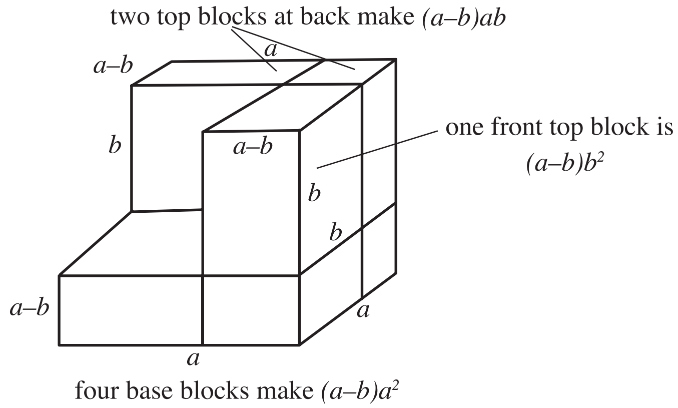

Figure 3: $a^3 - b^3 = (a - b)ab + (a - b)\, b^2 + (a - b)\, a^2 = (a - b)(ab + b^2 + a^2)$.

This allows us to visualize the formula $a^3 - b^3 = (a - b)(ab + b^2 + a^2)$ which can also be found easily using symbolic algebra. But if we attempt to 'see' the formula when a or b is negative or when $b > a$, then the idea of seeing the minus sign as 'turning over' involves reflecting a cube in a mirror which can no longer be performed by a physical movement in space. When we generalize further to the difference between two fourth powers $a^4 - b^4$ in four dimensions, this is even more problematic for creatures living in three-dimensional space.

As we generalize the picture becomes more difficult to see. Yet if we focus on factorizing algebraically, it is possible to factorize

$$a^3 - b^3 = (a - b)(a^2 + b^2),$$

and to factorize $a^4 - b^4$ is even easier because it can be rewritten as

$$(a^2)^2 - (b^2)^2$$

and we can re-use the formula for the difference of two squares to get

$$\begin{aligned} a^4 - b^4 &= (a^2)^2 - (b^2)^2 \\ &= (a^2 - b^2)(a^2 + b^2) \\ &= (a - b)(a + b)\,(a^2 + b^2). \end{aligned}$$

So now symbolic operations seem to be more appropriate than visual representations.

However, the advantage of symbolization over visualization is short-lived. When we consider $a^5 - b^5$, the factorization turns out to be

$$a^5 - b^5 = (a - b)(a^2 - 2\cos(72°)ab + b^2)\,(a^2 - 2\cos(144°)ab + b^2).$$

This factorization is unlikely to be found by manipulating algebraic symbols.

At a much later stage, when we have complex numbers at our disposal, we can 'see' the complex roots of $z^n = 1$ which turn out to be the complex roots of unity of the form $e^{2\pi i/n}$. In the case of $n = 5$, this is where the values cos(72°) and cos(144°) arise as the values of $2\pi/5$ and $4\pi/5$ expressed in degrees. (Tall, 2013, pp. 168-171)

The moral of this story is that, in the long-term development of sophistication in mathematics, visual ideas can give insight, while symbolic ideas give increasing power that take us beyond our original perceptions. In later contexts, new kinds of visualization may give insights that shift us on to higher levels of operation.

Embodied, Symbolic and Formal development

So far our analysis has been expressed in terms of visualization and symbolization, growing from practical activities in arithmetic where we can 'see' more general relationships then on to generalized arithmetic that can be symbolized using algebra and visualized using pictorial representations. However, these mathematical activities involve other forms of human perception and action than just visualization. They are based not only on our physical senses and actions, but also our mental imagination. I use the term *conceptual embodiment* to include the full range of physical and mental conception and action (Tall, 2004). The inclusion of mental imagination in our embodied thought is essential to take us from our perception and actions in the actual world to more sophisticated mathematical concepts. For example, it includes the way in which we sense general properties of arithmetic operating on objects that form a basis for properties in algebra.

For instance, by acting on a set of, say, six objects physically or mentally, we may see that $4 + 2$, $2 + 4$, 2×3, 3×2 are precisely one and the same number. This allows us to sense that the results of addition and multiplication are unaffected by changing the order.

Embodied operations on objects such as counting, adding, subtracting, and so on, may then be symbolized to develop a distinct form of mathematical thinking. Instead of imagining objects being moved around, the focus of attentions switches to operating with the symbols themselves. This gives a new way of thinking that may be termed *operational symbolism*.

Here the focus of attention changes from physical perception and operation on objects to operating with the symbols themselves. The symbols then may be conceived as mental objects that may be operated upon and these new operations become mental entities that can be operated upon, and so on. Counting becomes number, addition of numbers becomes sum, repeated addition becomes multiplication, generalized operations become algebraic expressions.

Initially, an equation of the form $3x + 1 = 7$ may be seen as an operation "3 times a number plus one is 7". This may be "undone", first by taking off the 1 to get "3 times the number is 6" from which we can see that the number x must be 2. It only involves operations on *numbers*, taking 1 from 7 to get 6, then dividing by 3 to get the answer 2. It can be solved because there is a single algebraic operation '$3x + 1$' which has a numerical result and the solution can be found purely in terms of the operations of arithmetic.

However, an equation such as $3x +1 = 2x +3$ involves different operations on the two sides and the arithmetical operation of 'undoing' cannot be applied in such a simple way. On the other hand, we may embody this equation as a physical or mental 'balance' where x is a quantity and 3 lots of x plus 1 balances 2 lots of x plus 3 (Figure 3).

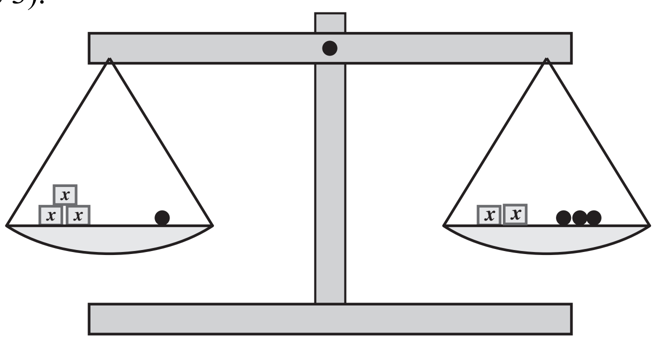

Figure 3: Embodying the equation $3x +1 = 2x +3$ as a balance

It is then possible to 'do the same thing to both sides', first taking off 2 lots of x from both sides to get $x + 1$ balancing 3, then taking off 1 from both sides to get $x = 2$.

This seems to support the idea that an embodied approach using a balance is more general than a symbolic approach because it can enable the learner to cope with more general equations. However this is not so because an equation such as $2x - 1 = 5$ cannot easily be represented as a physical balance as it has $2x - 1$ on the left and we cannot take away the 1 from $2x$ as a physical move when we don't yet know what x is as a number.

As a consequence, we see that symbolic undoing and embodied balance are each supportive for some types of equation but problematic for others. Furthermore, there are some equations such as

$$3x - 1 = 2x + 1$$

which are not suitable for either symbolic undoing (it has expressions on both sides) or a balance model (it has a minus sign on the left).

A more sophisticated technique is the principle of 'doing the same thing to both sides'. First 'add 1 to both sides' to get

$$3x = 2x + 2$$

then 'take $2x$ from both sides' to get

$$x = 2.$$

A student who grasps this principle is likely to have an all-inclusive strategy to solve more general equations. However, it turns out that this more general approach is problematic for many students. When three classes of teenage students had been taught to find the solution by 'doing the same thing to both sides', Lima (2007) found that none of them mentioned the general principle when interviewed at a later stage. Instead many referred to the use of specific rules such as 'change sides, change signs'. The equation '$3x - 1 = 5 + x$' was solved by shifting the '-1' to the right hand side to get '$3x = 5 + x + 1$', rearranging it to get '$3x = x + 6$', then, after shifting the x to the right and simplifying to get '$2x = 6$', using a second rule: 'shift the number 2 over the other side and put it underneath' to get '$x = 6/2 = 3$'. Common errors included misremembering the rules and mixing them up, included solving $2x = 6$ by 'moving the number underneath and changing its sign' leading to the error $x = 6/{-3} = -2$.

This symbol shifting involving rules that may or may not be understood is termed 'procedural embodiment'. It is a procedure that is embodied by using the mental action of shifting symbols. If used correctly, procedural embodiment will lead to correct solutions. However, it can also break down and lead to error.

Procedural embodiment causes even more complicated difficulties at the next stage. For example, when the students in this study moved on to solve quadratic equations, the teachers, aware of the students' difficulties, focused mainly on giving them an apparently guaranteed solution by solving an equation in standard form $ax^2 + bx + c = 0$ using the formula

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This failed because most of the students could not cope with the algebraic manipulation required to transform a quadratic equation into standard form (Tall, Lima & Healy, 2014). The failure to make sense of symbol manipulation in linear equations has even more serious consequences in quadratic equations.

Level Reduction

We have seen how students encounter new situations that are either too complicated for them to make sense of the new ideas or where they learn a technique that works at one level but becomes problematic at the next. Concerned teachers, who wish to help their students reach attainment levels, may turn to routine practice of test items to encourage them to pass the next test. This may succeed in the short-term but in the long-term successive focus on procedural learning may prove increasingly problematic.

In his book *Structure and Insight*, Pierre Van Hiele (1986, p. 39) formulated the notion of successive levels of thinking:

> You can say someone has entered a higher level of thinking when a new order of thinking enables him, with regard to certain operations, to apply these operations to new objects. (translated from the Dutch original in Van Hiele, 1955, p. 289).

His work is usually associated with geometry where he introduced five levels, but in *Structure and Insight*, he was well aware that his broad framework of levels applies widely throughout mathematics and in other domains of knowledge. He declared that the number of levels was not important. What matters are the changes that occur in the shift in level from a familiar situation to a new level of thought. He settled on three basic levels: *visual*, *descriptive* and *theoretical*. First situations are seen as a whole, then their properties are described, then definitions are formulated in ways that can be used to build a theory deduced from those definitions.

This can be applied more generally by broadening the notion of visualization to incorporate the full range of embodiment using perception and action. In the development of arithmetic and algebra, the first basic level starts with practical arithmetic operating on physical objects in which we build the operations of arithmetic. In terms of the properties of arithmetic, we first *recognize* fundamental underlying properties, at the next level we *describe* those properties, then we move to a *theoretical* level of definition of properties that are used to deduce a theory.

Practical examples may have limitations that are not found at the theoretical level. For example, temperatures above and below zero or as heights above and below sea level may be represented as signed numbers. In these practical examples we can perform various operations. For example we can shift up or down by a certain amount, but we do not actually add temperatures or elevations. We certainly do not multiply them. As a consequence, the theoretical level may involve new ideas that may not be represented in specific examples. On the other hand, the practical level may have differences that do not occur at the theoretical level. For example, in practice three lots of two may be different from two lots of three (think, for example, of three ducks with two legs or two ducks with three legs). The shift from the practical to the theoretical involves various subtleties that may be problematic. But at the theoretical level, as pure numbers, 2×3 is precisely the same as 3×2. Hence if thinking operates at this higher level, the ideas are simpler than at the practical level. Students adhering to the specifics of practical examples therefore may find it problematic to shift to the theoretical level while those who achieve the shift have a more flexible way of operating mathematically.

Van Hiele (1986, p.53) used the term 'level reduction' to describe how 'it is possible to transform structures of the theoretical level with the help of a system of signs, by which they become visible' (Van Hiele, 1986, p. 53). By this statement he refers to the possibility of manipulating symbols in a way that can be seen but not necessarily understood. For example, a student may learn to reproduce a geometric theorem by rote learning without grasping the underlying structure of the proof. In arithmetic or algebra, level reduction occurs when learning a procedure to carry out an operation or solve an equation without understanding what it means or why it works. A typical example is the use of procedural embodiment in solving equations.

The literature has long debated the distinction between different forms of knowledge and understanding, such as instrumental and relational understanding (Skemp, 1976) or between procedural and conceptual learning (Hiebert & Lefevre, 1986). Both types of learning and understanding play important roles in mathematics. Procedural learning is important to develop the necessary skills while conceptual understanding provides a grasp of the bigger picture in formulating and solving problems.

If level reduction occurs at successive levels, the difficulties may be compounded. While learning procedures may give short-term success in passing the next test, the cumulative effect of several stages of level reduction may have more serious effects in subsequent learning. The short-term success focusing on procedural learning may lead to teachers and students changing their goals to the immediate priority to pass the test. In the longer term it may become the default goal, with successive level reductions leading inevitably to a form of mathematics that is less suitable for more sophisticated thinking.

This is a phenomenon that can be seen around the world as comparison of success on various levels of testing causes the need to pay attention to achieving the highest possible grades at each level. This pressure can have an enormous detrimental effect on those who are not making sense of the mathematics at successive levels and so build a sequence of short-term rote learning to pass the test, with possible short-term success, but with longer term consequences.

Where do we go from here?

Currently, governments around the world are concerned with maintaining and raising standards in mathematics as measured in international tests. This is leading to curriculum design formulated in terms of successive levels of subject development that are tested to judge the apparent level of achievement of children at various stages.

In the USA the Executive Summary of the NCTM Standards (NCTM, 2004) presents a desired positive view of development through successive stages of learning. However, there is no mention of the problems that students face as they encounter new ideas in mathematics that may cause them difficulties, even in more recent versions (e.g. NCTM, 2013). The reason for this is that the summary is written to specify desired objectives, written by a group of experts working for the government (NGA, 2009), who are mainly involved in the testing industry (Schneider, 2014). The implementation of the standards is left to the professionals in mathematics education.

The NCTM (2013) issued a statement supporting the standards while acknowledging that they are "not sufficient to produce the mathematical achievement that our country needs to be competitive in the 21st century." To accommodate the standards within a wider framework, an additional list of positive initiatives were proposed to achieve that success. However, these initiatives again make no explicit mention of the problematic feelings towards mathematics that are said to affect around two thirds of the adult population (Burns, 1998). It is as if speaking only of the positive while remaining silent on the negative aspects that affect the majority of the population can lead to success.

History does not support this view. In recent times there have been resounding calls to change everything for the good: the 'new math' of the 1960s promised that if only we got the mathematics right, everything would be well. But the mathematics, as seen by experts, did not make sense to the wider population of learners. Then the constructivism of the 1980s turned attention to the learners to encourage them to make sense of mathematics in their own way but this did not produce the desired results required in universities and in the work place. In the new millennium — despite repeated calls to 'raise standards' — the level of performance in the USA (and also in the UK and other western countries) has failed to improve in international comparisons (Pisa, 2012).

Drawing together the lessons of the past may offer a way forward to enhance the future evolution of the teaching and learning of mathematics. While the 'new math' focused on a modern approach to mathematics, the constructivist approach focused on the development of the individual. Now it is time to blend both together to develop an integrated approach by combining three aspects mentioned earlier:

- the increasing sophistication of mathematics,
- the personal interpretations of the individual,
- the long-term effects of sense making in increasingly sophisticated contexts.

In the remainder of this chapter, we look successively at these three aspects. First we consider the coherent structure of mathematical concepts that evolves as mathematics increases in sophistication in successive contexts by using the notion of 'crystalline concept' (Tall, 2010). This offers a positive overall picture of development that will enable teachers to be aware of how their current teaching can help the student focus on long-term learning.

Second, we consider the personal interpretations of the individual where an awareness of supportive and problematic met-befores can assist the teacher and learner to focus on ideas that improve insight and guard against negative effects that impede long-term learning.

Finally these two aspects will be blended together to suggest strategies for encouraging long-term sense making.

The increasing crystalline sophistication of mathematics

We have seen how arithmetic develops in sophistication and generalizes into the study of algebra. At each stage operations are introduced and symbolized so that the symbols themselves can be manipulated as mental entities at a higher level. For instance the process of counting leads to the concept of number and then operations can be performed on number such as addition, subtraction, multiplication, division to lead to new concepts of sum, difference, product, and in the final case, sharing in the context of whole numbers and fractions in the context of dividing objects into smaller parts.

Gray and Tall (1994) noted that the same symbol could be used to represent both a process and a concept: 2+3 as the process of addition and the concept of sum, 2 × 3 as multiplication and product, $^2/_3$ as sharing and fraction, $2 + 3x$ as a general operation of adding 2 to 3 times x and as an expression which could itself be manipulated. At the same time, different processes could give rise to the same concept, for instance 2 + 3 is the same as 3 + 2, or 1+4 or 6 –1. The term 'procept' was used to speak of the underlying entity that could be manipulated flexibly in many ways.

As the curriculum progresses, procedures that are seen initially as being different are seen as a rich blending of different processes giving the same procept. Counting a particular set that can be performed in different ways but always gives rise to the same number. A whole number is a procept that has a rich internal structure where a number 5 may be seen as 3+2, 2+3, 6–1 and so on. Algebraic expressions are also procepts. The expressions $2(x + 3)$ and $2x + 6$ involve different processes of calculation but they later give rise to the same function $f(x) = 2(x + 3) = 2x + 6$.

The steady progress through the curriculum in which different procedures are seen to be 'essentially the same' is dealt with mathematically by introducing the notion of 'equivalence'. Early in the curriculum fractions that involve different procedures giving the same final quantity such as $^2/_4$ and $^3/_6$ are said to be equivalent, later they are conceived as being the same rational number.

Reflecting on this phenomenon that arises throughout arithmetic and algebra I realized that even though we may distinguish between objects that are equivalent, mentally we operate more efficiently by thinking of them as being fundamentally the same object.

In Tall (2011), I extended the idea of flexible meaning of symbolism to the full range of mathematics. A 'crystalline concept' was given a working definition as 'a concept that has an internal structure of constrained relationships that cause it to have a necessary property as a consequence of its context.'

In geometry, crystalline concepts include objects such as a circle which is defined to be the locus of a point that remains a fixed distance from its center, but has many constrained relationships (such as 'the angle in a semicircle is a right angle'). In arithmetic, procepts are special cases of crystalline concepts, including various kinds of number such as whole numbers, fractions, signed numbers, real numbers, complex numbers, vectors and expressions in algebra. In axiomatic mathematics, formal concepts defined set-theoretically are crystalline concepts whose properties are deduced from the definitions by mathematical proof.

Mathematicians use the notion of 'equivalence relation' to deal with this idea in a technical way. For example, a fraction m/n may be considered as an ordered pair (m, n) of whole numbers under the equivalence relation

$(m, n) \sim (p, q)$ if and only if $mq = np$.

Then a rational number can be defined as an equivalence class of such pairs. At this stage it is customary to agree that we can think of the equivalence class as a rational number. The notion of crystalline concept represents how we think about such mathematical ideas in practice, not as 'equivalent' objects or as being 'essentially' the same, but as *a single idea* that can be imagined in different ways.

The notion of a crystalline concept enables us to link together different kinds of representation in a single entity. For example, the real numbers may be defined as a complete ordered field, but it is possible to prove that two systems satisfying the axioms for a complete ordered field must be isomorphic and that they may be embodied (as a number line) and symbolized (as decimals). This gives the real numbers a rich structure as a crystalline concept which can be defined formally, visualized as points on a line and operations can be performed using decimal arithmetic.

In other cases, a list of axioms, such as those for a group, may apply to many different examples, yet they all have a common structure given by the group axioms. We can think of a group as a crystalline concept. It is possible to show that the group structure may be embodied (as permutations of a set) or that groups may be classified in ways that may be proved from the axioms.

More generally, formal axiomatic systems often can be proved to satisfy 'structure theorems' that classify its structure in a way that may often be embodied or symbolized in a manner that is more easily handled by the human mind. The notion of 'crystalline concept' proves to be of value throughout the whole of mathematics, linking formal, embodied and symbolic ways of thinking within a overall framework.

In school mathematics, these higher-level structures are not an explicit part of the curriculum. However, it is possible for learners to sense these structures at successive stages of development and for the teacher as mentor to encourage the learner to become aware of the underlying flexible structure. This includes an awareness of the ways in which concepts that are different at one stage may be classified at a later stage as one and the same.

In geometry this development is formulated in terms of successive van Hiele levels. For instance, at the first visual level, squares and rectangles are seen as being different, but at the next level a square is a special case of a rectangle. Likewise in arithmetic and algebra, expressions that represent different processes are later seen as representing the same crystalline concept. As the number systems become more sophisticated through whole numbers, fractions, signed numbers, infinite decimals, real numbers, complex numbers, the crystalline structure subtly changes.

In practice, new ideas are often introduced by learning how to carry out procedures. For example, in the United States, the acronym '**FOIL**' is introduced to calculate $(a+b)(c+d)$ by multiplying the **F**irst elements in the brackets $a \times c$, then the **O**utside elements $a \times d$, then the **I**nside elements $b \times c$, then the **L**ast elements $b \times d$. We also teach a more subtle technique to factorize an expression such as $x^2 + 5x + 6$ by seeking two numbers whose product is 6 and sum is 5.

This may have the unintended consequence that what is happening is the translation of one expression into a *different* expression, without realizing that these are just different ways of representing the same underlying crystalline concept. In this case, level reduction has occurred in which the student has learnt to carry out the procedures without grasping the rich flexibility of the mathematical structure.

In the USA there are many examples of level reduction in teaching college algebra where text books are laid out using various devices such as color coding text or placing significant statements in boxes to remind the learner what should be remembered to be able to pass the test. As a result there are more and more disconnected ideas to be remembered that increase the longer-term likelihood of overload and error.

A positive strategy is therefore to seek to make sense of the mathematics not only in terms of specific examples in practical contexts, but also to draw out the pure thought that underlies the fundamental crystalline concepts.

This need not be a highly sophisticated activity. It simply means that the learner is encouraged to develop a sense of the underlying structures that cause mathematics to fit together in a coherent manner.

This requires more than a simple constructivist approach that encourages children to construct their own methods, for this may lead the learner investing effort into a particular way of working that becomes problematic at a later stage. Instead it requires the teacher, as mentor, to encourage the learner to seek to develop more powerful techniques that support longer-term learning while at the same time sensing the underlying crystalline structures that make mathematical thinking both more powerful and at the same time more flexible.

Supportive and problematic aspects of individual growth

Grasping the essential underlying ideas is not as simple as it sounds. While the teacher may be focusing on supportive aspects that generalize to new situations, the learner may be sensing problematic aspects that impede progress. The problem is that different students are affected in different ways by their own personal met-befores. This requires the teacher to be sensitive to problematic met-befores that may occur in some students but not in others so that different difficulties may be addressed in individual students or in groups of students that share a common conception.

For instance, in learning to solve equations, more successful students may realize that 'doing the same thing to both sides' maintains the equality and so this proves to be a successful strategy. However, we have seen that many students are affected by problematic aspects that impede their progress. These may include ideas that have accumulated over many years and become more difficult to address as the ideas become deeply ingrained.

For a few students, algebra is essentially simple, even trivial. All one does is to use letters to represent numbers and manipulate them by the same methods with the same general properties noted in arithmetic. Others who are fixed in procedures without a flexible sense of relationships in arithmetic are more likely to find algebra becoming successively more complicated and even impossible, leading to a sense of anxiety that paralyses thought and prevents future development.

The long-term effects of sense making in mathematical thinking

Algebraic thinking is part of a long development from practical arithmetic with whole numbers, through more sophisticated number systems where general properties of arithmetic may be sensed and form a basis for the manipulation of symbols in algebra. As new ideas are encountered in fractions, signed numbers, generalized arithmetic, algebra, the crystalline structure of the systems changes in ways that involve supportive aspects that generalize and problematic aspects that impede generalization.

Teaching that emphasizes the supportive ideas of generalization, as specified in many curricula around the world, will work with learners who grasp the flexible structure of successive contexts. These are the lucky ones who find mathematics pleasurable and powerful. But children who are impeded by problematic met-befores are less fortunate. If they cannot grasp the flexible structure, they may resort to learning procedurally to gain the pleasure of passing tests. In an era where testing at successive stages is prized, teachers and learners together may use a form of level reduction to know how to complete a specific task without setting it in a context that is appropriate for building meaningfully at the next stage. The consequence is that much teaching in our mathematics classrooms may be aimed at procedural competence rather than mathematical flexibility. In the longer term, successive level reduction impedes the development of flexible mathematical thinking.

However, not all individuals are the same or require the same level of mathematical sophistication. Is it sensible to make all students aspire to the same successive list of targets? If some students are finding mathematics difficult, repeating the same materials a second time may not necessarily help them think in the same way as those who regard mathematical ideas in more flexible ways. Different forms of employment in society require different forms of mathematical competence and it is surely appropriate to teach children according to their needs and to help them learn mathematics in a way that makes sense to them.

Current curriculum development in many countries around the world is focusing on the introduction of problem solving and other aspects of mathematical communication, in Lesson Study in Japan, in realistic mathematics in the Netherlands, in cooperative learning in the USA. All of these encourage positive achievement while often remaining relatively silent on the negative.

Will the current drive to 'raise standards' be successful? Time will tell. However, it is clear that teachers who have emotional difficulties with mathematics are likely to pass on their feelings to many of their students, so the desire to improve the positive aspects of learning for the students must be preceded by improvement of security and confidence of teachers and this in turn requires the understanding of the broader aspects of mathematical thinking to be grasped by those who prepare the curriculum and set the tests.

The analysis of this chapter suggests that there are subtle underlying factors in the development of mathematical thinking that promote flexible thinking on the one hand and impede long-term learning on the other. This suggests a significant re-think in how we view the development of mathematical thinking. While an understanding of the increasingly sophisticated nature of crystalline concepts will help us to see the positive growth of supportive aspects of mathematical knowledge, an awareness of the problematic aspects carried forward from previous experience will help us assist individuals to deal with their difficulties that may become mathematical anxiety.

This suggests the need for mathematicians, curriculum designers, teachers and learners to become explicitly aware of the underlying supportive and problematic aspects of long-term learning. It requires a global rethinking of the whole development of mathematical sense making that balances the subtle changes of meaning of mathematics over the long term with the developing needs of different learners, building from early sense making in the perception and operation of the child and developing increasing sophistication of mathematical reasoning appropriate for differing roles in wider society. In particular, this suggests the development of a whole new course in teacher preparation that addresses not only the supportive aspects to deliver the positive objectives in long-term mathematical sense making while becoming aware of the specific problematic met-befores that impede student learning.

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