Making Sense of Mathematical Reasoning and Proof[1]

David Tall
University of Warwick
<david.tall@warwick.ac.uk>

Abstract

This chapter charts the growth of proof from early childhood through practical generic proof based on examples, theoretical proof based on definitions of observed phenomena, and on to formal proof based on set theoretic definitions. It grows from human foundations of perception, operation and reason, based on human embodiment and symbolism that may lead, at the highest level, to formal structure theorems that give new forms of embodiment and symbolism.

Increasing sophistication in mathematical thinking and proof is related to earlier experiences, called 'met-befores' where supportive met-befores encourage generalisation and problematic met-befores impede progress, causing a bifurcation in the perceived nature of mathematics and proof at successive levels of development and in different communities of practice. The general framework of cognitive development is offered here to encourage a sensitive appreciation and communication of the aims and needs of different communities.

Keywords

Mathematical proof, crystalline concepts, met-before, generic proof, van Hiele theory, structure theorems.

Mathematical thinking in terms of human perception, operation and reason

The cognitive development of mathematical thinking and proof is based on fundamental human aspects that we all share: human perception, action and the use of language and symbolism that enables us to develop increasingly sophisticated thinkable concepts within increasingly sophisticated knowledge structures. It is based on what I term the *sensori-motor language of mathematics*, blending together perception, operation and reason (Tall, 2013).

Mathematical thinking develops in the child as perceptions are recognised and described using language and as actions become coherent operations to achieve a specific mathematical purpose. According to Bruner (1966), these may be

[1] This article is a product of personal experience, working with colleagues such as Shlomo Vinner who gave me the insight into the notion of concept image, Eddie Gray, whose experience with young children led me to grasp the essential ways in which children develop ideas of arithmetic and to build a theoretical framework for the different ways in which mathematical concepts are conceived, Michael Thomas who helped me understand more about how older children learn algebra, the advanced mathematical thinking group of PME who broadened my ideas about the different ways that undergraduates come to understand more formal mathematics, many colleagues and doctoral students who I celebrate in Tall (2008) and, more recently, the working group of ICMI 19 who focused on the cognitive development of mathematical proof (Tall, Yevdokimov *et al.*, 2012).

communicated first through enactive gestures, then iconic images, then the use of symbolism, including not only written and spoken language but also the operational symbolism of arithmetic and the axiomatic formal symbolism of logical deduction.

The theoretical framework proposed here follows a similar path enriched by the experience over time, building from *conceptual embodiment* that combines the enactive and iconic modes of human perception and action, developing into the mental world of perceptual and mental thought experiment. Embodied operations, such as counting, adding, sharing, are symbolised as manipulable concepts in arithmetic and algebra in a second mental world of *operational symbolism*. As the individual matures, there is a further shift into a focus on the *properties* of mental objects as in Euclidean geometry, the blending of visual and symbolic modes of thought and the properties of arithmetic operations recast as 'rules' that underlie the generalized operations and expressions in algebra. Each of these leads to different forms of mathematical proof: *Euclidean proof* in geometry, *symbolic proof*, based on the 'rules of arithmetic', and *blending embodied and symbolic reasoning* using language.

Embodiment and symbolism develop alongside each other and interact with each other. The early stages of *practical mathematics* begin with experience of shape and space, and of operations in arithmetic, in which properties of specific examples are seen to offer *generic proof*, such as realising that 2+3=3+2 holds not just for the numbers 2 and 3, but for *any* pair of whole numbers. This develops into the *theoretical mathematics* of definition and deduction in Euclidean and symbolic forms of proof.

Properties in both embodiment and symbolism develop into the *formal mathematics* of set-theoretic definition and proof in the *axiomatic formal* world of pure mathematics. While theoretical mathematics is based on embodied and symbolic experiences, formal mathematics guarantees that all the properties proved from given set-theoretic axioms and definitions will also hold in any new context that satisfies the given axioms and definitions.

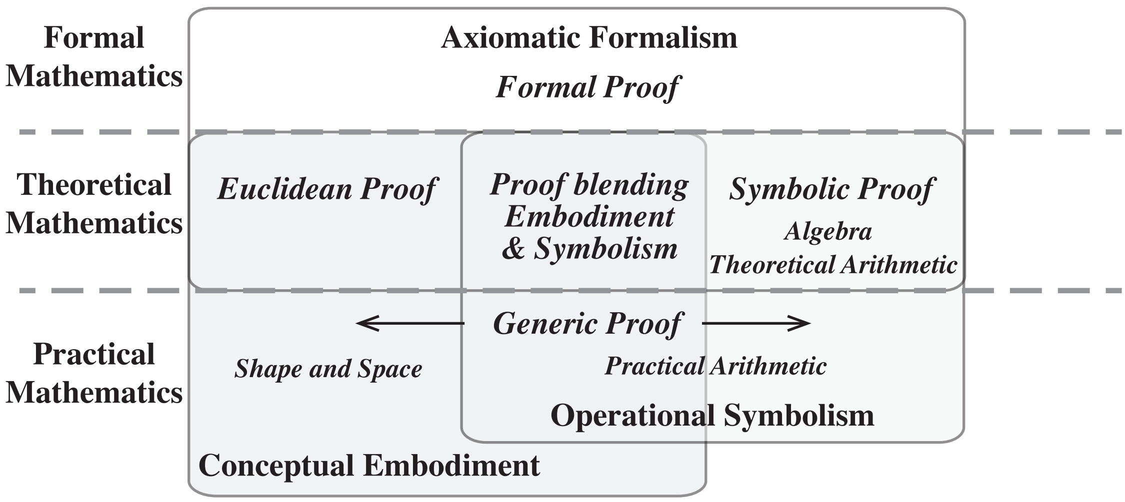

Figure 1: Outline of long-term development of proof

Embodiment and symbolism continue to play their part in axiomatic formalism, not only in imagining new possibilities that may be defined and proved formally, but also in an amazing turnaround in which certain theorems (called structure theorems) prove that axiomatic systems have embodied and symbolic structures established by formal proof. This reveals mathematical thinking at the highest level, and mathematical proof in particular, as an intimate blend of embodiment, symbolism and formalism where individual mathematicians develop a preference for different aspects.

The evolution of theories of mathematical thinking and proof

Pierre van Hiele (1986) focused on *structure* and insight, seeing a succession of levels that may be described as *recognition* and *description* of figures, leading to *definition* and *deduction* of properties through Euclidean proof.

Ed Dubinsky and others (Asiala et al., 1996) took an apparently different path, following Piaget's idea of reflective abstraction to focus on operations that are seen first as *actions*, routinized as *processes*, then *encapsulated* as mental *objects* within knowledge *schemas*.

Anna Sfard (1991) proposed a framework that alternated between *operational* and *structural* ways of thinking in which operations are *condensed* as *processes*, and then *reified* as mental *objects* that now have a certain structure. She suggested at the time that an operational approach inevitably precedes structural mathematics. However, her examples involve operational symbolism being reified as mental objects, without any reference to the van Hiele development of the properties of objects.

This led to a three-part analysis in Tall, Thomas *et al*. (2000) through parallel developments of conceptual embodiment (broadly following van Hiele) and operational symbolism (using process-object theories) in school, leading much later to the axiomatic formal framework of set-theoretic definition and proof in university pure mathematics (Tall, 2004a, 2004b).

Following the recent death of van Hiele in 2011 at the grand old age of one hundred, I revisited his ideas of structure and insight, which he applied to geometry, but not to the symbolism of arithmetic and algebra (van Hiele, 2002). I realised that the term *operation* should not be restricted to the symbolic operations in arithmetic and algebra. Operations occur in the constructions of Euclidean geometry. For instance, we may operate on an isosceles triangle by joining the vertex to the midpoint of the base to cut the triangle into two parts that are congruent (with three corresponding sides). This proves that the base angles must be equal, and various other properties follow, such as the property that the line from the vertex to the midpoint of the base is at right angles to the base.

The operations of construction in geometry and the various operations in arithmetic and algebra have a common definition: they consist of 'a coherent sequence of actions and decisions performed to achieve a specific purpose.' A geometric operation is a construction that focuses on the *object* (the figure) and results in enabling us to see relationships concerning *the properties of the object*. A symbolic operation performs a calculation or manipulation, focusing more on *the properties of the operations themselves* as the operations lead to a symbolic output.

Furthermore the compression of operation into mental object in symbolism begins for the child as *embodied operations* on objects such as counting, adding, sharing, and is compressed into *symbolic operations* on whole numbers, fractions, signed numbers and so on. This reveals two distinct forms of compression from operation to mental object that I termed *embodied compression* and *symbolic compression* (Tall 2013, chapter 7).

Embodied compression focuses on the effect of the operations on the objects, such as counting a collection to find the number of objects, such as ‘six’. Focusing on the way that the objects are placed leads to a realisation of the fundamental properties of whole number arithmetic. For instance, the set of six objects may be subdivided, say, into subsets of ‘four’ and ‘two’ and, by rearranging the sets, it may be seen that ‘two’ and ‘four’ is also ‘six’. Reorganizing the subsets as two rows of ‘three’ allows them to be seen as three columns of ‘two’ so that ‘two threes’ is the same as ‘three twos’. Embodied compression enables us to see *at a glance* the flexible properties of arithmetic. ‘Proof’ at this early stage is a form of reasoning based on our interpretation of the coherence of our own perceptions and actions. This form of proof, in which a specific example is seen to be typical of a whole category of examples, is termed *generic proof* (Mason & Pimm, 1986; Harel & Tall, 1991).

Symbolic compression involves performing a counting operation to obtain a number concept, for instance, the operation of ‘count-on’ calculates ‘two and eight’ as counting on eight to get ‘three, four, five, six, seven, eight, nine, *ten*’ while ‘eight and two’ is the short count ‘nine, *ten*’. Here the two operations are very different, one is a long count, and the other is short. The general properties of the symbolic compression are therefore not as self-evident as they are with embodied compression.

A gifted child may grasp the flexible properties of arithmetic as part of a coherent knowledge structure in which symbols operate dually as process or concept (which we termed a ‘procept’) that may be used as an organising principle to simplify operations. A child who focuses on procedural operations of counting taking place in time will find arithmetic operations to be far more difficult to cope with. Eddie Gray and I called this bifurcation ‘the proceptual divide’ between those fixed in increasingly complicated counting procedures and those who develop flexible ways to derive new facts from known facts (Gray & Tall, 1994).

This bifurcation between those who find mathematics ‘easy’ and those who find it impossibly difficult begins at a very early age. It should be taken into account in seeking to explain and predict how each individual attempts to make sense of mathematics, building on increasingly sophisticated perception, operation and reason.

Long-term pleasure and pain

Emotions play a vital role in mathematical thinking and have a profound effect on how individuals make sense of mathematical proof. As my supervisor, Richard Skemp used to say: ‘pleasure is a signpost, not a destination.’ His goal-oriented theory of learning (Skemp 1979) saw children starting out with the goal of seeking to make sense of the world. Successfully linking together ideas in coherent ways gives pleasure, success breeds more success, so that a child with a history of success builds up a positive feed-back loop where an encounter with a problematic situation is often met with the determination to conquer the difficulty. However, lack of success leads to an anti-goal, to avoid a sense of stress. Further encounters with stress may lead to a negative feed-back loop in which the desire to avoid failure leads to less engagement with the mathematics and less technical proficiency that causes even more difficulty and greater mathematical anxiety (Baroody & Costlick, 1998). As a result of the negative feedback, students may seek the comfort of learning procedures by rote to succeed in examinations and prefer to learn proofs procedurally rather than seek to grasp deeper meanings that do not seem to make sense.

An analysis of the development of mathematical thinking reveals the surprising conclusion that mathematics is not a system that builds logically on previous experience at each stage, even though every mathematics curriculum in the world is intent on presenting topics in a coherent sequence, carefully preparing the necessary pre-requisites at each stage for the more sophisticated stages that follow. On the contrary, an experience that has been 'met before' may be supportive in some new situations yet problematic in others.

The concept of 'met-before' was introduced by Lima & Tall, (2008) and McGowen & Tall (2010) to describe 'a structure we have in our brains *now* as a result of experiences we have met before.' Some ideas that work in one situation such as 'addition makes bigger' or 'take away makes smaller' in whole number arithmetic are supportive in the context of fractions yet problematic in the context of signed numbers. This recalls the concept of 'epistemological obstacle' developed by Bachelard (1938) and Brousseau (1983) and the need for accommodation by Piaget (see, for example, Baron et al., 1995) or reconstruction by Skemp (1971).

However, the notion of met-before refers to the effect of previous experience on new learning. A particular met-before is not in itself supportive or problematic, it *becomes* supportive or problematic in a new situation when the learner attempts to make sense of the new ideas. For instance, 'take away leaves less' is supportive in some contexts (e.g. everyday situations where something is removed, in the postulates of Euclidean geometry, or taking one whole number from another) but it is problematic in others (such as taking away a negative number or in the theory of infinite cardinals).

A problematic met-before arises not only in the individual learner, *it is a widespread feature of the nature of mathematics itself*. In shifting to a new context, say from whole numbers to fractions, or from positive numbers to signed numbers, or from arithmetic to algebra, *generalization is encouraged by supportive met-befores* (ideas that worked in a previous context and continue to work in the new one) *and impeded by problematic met-befores* (that made sense before but do not work in the new context).

For instance, properties such as commutativity, associativity, distributivity are supportive as number systems are broadened through whole numbers, integers, real numbers, complex numbers, but other aspects such as 'take away gives less' or 'the square of a non-zero number is positive' become problematic.

Crystalline concepts

Given this increasing difficulty of problematic aspects that occur in generalization, I sought a unifying principle that is supportive in mathematical thinking and binds mathematical ideas together in any given context. In Tall (2011), I formulated a working definition of a *crystalline concept* as 'a mathematical concept that has an internal structure of relationships that cause it to have specific properties in the given mathematical context.' Such concepts include:

- **platonic objects in geometry**, such as points, lines, triangles, circles, congruent triangles, parallel lines that have properties arising through Euclidean proof;
- **operational symbols as flexible procepts** in arithmetic, algebra and symbolic calculus that have necessary properties through calculation and manipulation;
- **set-theoretic concepts** in axiomatic formal mathematics whose properties are deduced by formal proof.

Not only do crystalline concepts occur at the highest levels of mathematical thinking, they emerge in the thinking of a young child who sees the flexible proceptual structure of arithmetic through embodied compression rather than the procedural step-by-step counting procedures of arithmetic that operate in time.

They enable flexible thinkers to see mathematical ideas in astonishingly simple ways. It is not that the fractions $4/8$, $7/14$, $101/202$ are all *equivalent* to each other and to the simplest possible canonical form $1/2$, it is that they are all manifestations of a single crystalline concept – the rational number one half – also represented as a unique point on the number line.

It is not that the expressions $2(x+7)$ and $2x+14$ are equivalent but different, where the first can be turned into the second by 'multiplying out the brackets' and the second can be turned into the first by 'factorization', it is that *both expressions are different ways of writing the same crystalline concept* as an algebraic expression. Indeed, the functions $f(x)=2(x+7)$ and $g(x)=2x+14$ are not simply equivalent, t*hey are precisely the same function*. Students who think flexibly in terms of crystalline concepts have much more powerful means of relating mathematical ideas than those who see equivalent ideas that are changed from one form to another by carrying out procedures.

Likewise, in axiomatic formal mathematics, an axiomatic system such as 'a group' is a crystalline concept with rich interconnections between its properties. We may not know what specific group we are dealing with, but we *do* know that it has an identity that we may denote by *e*, and that if *x* is any element, we can define the power x^n for any positive or negative integer and prove that $x^{m+n}=x^m x^n$ for any integers *m*, *n*.

A crystalline concept may be defined formally and then its properties may be deduced as theorems to build up a knowledge structure where relationships are tightly interconnected by formal proof. For example, we can prove that if we begin with the axiomatic definition of an ordered field F, then in this context we may formulate any of the equivalent definitions for completeness, to prove that a complete ordered field is not only unique up to isomorphism, it is also unique as a crystalline concept.

At the highest level of pure mathematical research, it is the compression of structural properties of defined formal concepts into crystalline concepts that gives gifted mathematicians a simplicity of thought that is beyond the mere proving of theorems of equivalence. An ordered field not only contains a subfield *isomorphic* to the rational numbers, it can be conceived as a crystalline concept that *contains* the crystalline concept of the rational numbers.

I recall the ideas that I encountered as a graduate student when theoreticians spoke of the identification of one structure with another structure as 'an abuse of notation'. On the contrary, it is this way of thinking that gives the biological brain of the mathematician a level of flexibility to conceive mathematical ideas in more simple and insightful ways.

Formal constructions building up more general systems – for example, from natural numbers, to integers, to rational numbers, to real numbers, and beyond – all involve equivalence relations of ordered pairs in one structure to construct the next. At each stage we get an isomorphism between equivalence classes of ordered pairs and a substructure of the larger system. This development involves supportive met-befores that encourage generalization and problematic met-befores that impede progress. Yet once we have the larger system, we no longer need to speak of isomorphisms, we can simply refer to the subsystem as a subset given by specified properties. Being able to move flexibly between seeing subsystems as subsets or as isomorphic copies leads naturally to the cognitive notion of crystalline concept. It offers the human brain a simpler way to think of strictly formulated isomorphic systems as a single underlying crystalline concept that can occur in different contexts yet operate in the same coherent way in every representation.

The transition from proof in embodiment and symbolism to formal proof

The overall framework for cognitive development from the newborn child to the frontiers of mathematical research was further developed in the *ICMI Study 19 on Proof and Proving* (Tall, Yevdokimov *et al*, 2012), and has been extended in *How Humans Learn to Think Mathematically* (Tall, 2013).

The van Hiele levels (1986) have been variously reconsidered by a range of authors, may now be seen in as four successive levels which I term

- **Recognition** of basic concepts such as points, lines, and various shapes;
- **Description** of observed properties;
- **Definition** of concepts to test new examples to see if they satisfy the definition and to use the definitions to formulate geometric constructions;

- **Deduction** in the form of Euclidean proof in plane geometry.

Each of these is a form of *structural abstraction* in which the structure of the objects under consideration and their relationships shift to successive new levels of sophistication. This begins first with observations of geometric objects whose structures are recognised and described. At this point the foundations of Euclidean proof are laid down by formulating definitions for figures that not only allow them to be categorised and constructed but also to use ideas such as congruent triangles and parallel lines to construct Euclidean proof.

Van Hiele also described a fifth level of *rigour* that may be seen as shifting in two directions, the first is to different embodied contexts such as projective geometry or spherical geometry, the second is in terms of the more sophisticated world of *axiomatic formalism* as prescribed by Hilbert.

Van Hiele (2002) saw these levels apply to geometry and not to the symbolic development from arithmetic to algebra. The calculation with numbers and manipulation of algebraic symbols involve quite different mental activities from those in Euclidean proof. However, once operations are encapsulated as number concepts and generalized as algebraic expressions, these too have properties that can be *recognised* and *described*, then *defined* as 'rules of arithmetic' to be used in algebraic proofs to *deduce* theorems. Thus the sequence of structural abstraction also occurs in the higher levels of operational symbolism to provide definitions of whole numbers, such as even, odd, prime and to deduce theorems such as the uniqueness of factorization into primes.

Exactly the same structural abstraction arises in the axiomatic formal world of set-theoretic definition and formal proof. This builds on our experience of conceptual embodiment and operational symbolism, beginning with the *recognition* and *description* of mathematical situations and then the *definition* of axiomatic systems and of defined concepts within those systems, and *deduction* of properties of systems and defined concepts using formal proof.

Experienced mathematicians have flexible knowledge structures that they wish to pass on to their students. However, by the time students pass through school to enter university, they will have already developed in very different ways based on how they have managed to make sense of previous experiences.

Krutetskii (1976) produced significant evidence that the most gifted children are more likely to develop a strong verbal-logical basis to mathematical thinking than a visual-pictorial foundation. Out of over a thousand students, the most gifted nine were classified with five analytic (verbal logical), one geometric (visual-pictorial), two combining both (one more visual, the other more verbal) and one who was not classified. Presmeg (1986) found that the most outstanding senior school mathematics students in her study (7 pupils out of 277) were almost always non-visualizers. Of 27 'very good' students (10% of the sample), eighteen were non-visualizers and five were visualizers.

This suggests that a small number of those students who enter university are powerful verbal-analytic thinkers who may benefit from making sense of set-theoretic definitions, an even smaller number base their thinking on visual-pictorial representations, and others who may have a blend of visual embodied thinking and operational symbolism or who prefer to learn procedurally by rote.

Some students seek a *natural* approach based on a blend of previous experiences of embodiment and symbolism from school mathematics. Some with a more verbal-logical basis may seek to use a *formal* approach based on set-theoretic definitions and the deduction of properties using formal proof. Others seek to learn proofs procedurally to reproduce in examinations. All of these approaches may involve supportive and problematic aspects, which have been detailed in the literature (e.g. Pinto & Tall, 1999; Weber, 2004; Tall, 2013).

As students become more experienced and shift to graduate studies, Weber (2001) produced evidence that research graduates are more likely to respond flexibly to problems by making links between concepts in a sophisticated knowledge structure while undergraduates in their early studies, have yet to develop such flexibility.

This is consistent with the lack of aesthetic appreciation of mathematical ideas noted by Dreyfus and Eisenberg (1986) and also with the relationship noted by Koichu, Berman & Katz (2007) between "aesthetical blindness" of students and factors such as self-esteem that affect their aesthetic judgement.

The theoretical framework presented here traces the development of cognitive and emotional aspects throughout the lifetime of the individual. A few students, characterized as being 'gifted' develop verbal-analytic skills that enable them to build formally from set-theoretic definitions to construct highly connected crystalline concepts that may have embodiments and operations linked to underlying formally proved structure theorems. But many others, who focus on 'maximising their mark on the exam' to 'get a good degree' to move on in their lives, have good reasons for doing so. The mathematics is *problematic* for them and *it doesn't make sense.*

Structure Theorems

Some theorems based on formal axioms and definitions prove formal structures that enable the ideas to be reconsidered in embodied and symbolic terms. For example, a finite dimensional vector space over a field F is isomorphic to F^n, so that its elements may be represented symbolically as n-tuples and its linear maps as matrices, and in the case where F is the field of real numbers and $n = 2$ or 3, it may be embodied in two or three dimensional space. In the same way a finite group is isomorphic to a subgroup of a group of permutations, which allows it to be operated on symbolically and embodied as the transformations of a geometric object.

Structure theorems enrich formal mathematics with new forms of embodiment and symbolism, to enable mathematicians to recognise problems, imagine possibilities, to formulate conjectures and to prove new theorems. Mathematicians of different persuasions see proof as their main research goal, but achieve it in different ways, as the algebraist Saunders MacLane observed when comparing his approach with that of the geometer Michael Atiyah:

> For MacLane it meant getting and understanding the needed definitions, working with them to see what could be calculated and what might be true, to finally come up with new 'structure' theorems. For Atiyah, it meant thinking hard about a somewhat vague and uncertain situation, trying to guess what might be found out, and only then finally reaching definitions and the definitive theorems and proofs. (MacLane (1994), p. 190–191.)

Both strategies follow the same format – becoming aware of a problem, considering possibilities, formulating conjectures and seeking proof – and this follows the broad van Hiele format of recognition, description, definition and deduction:

	Problems (recognition)	Possibilities (description)	Conjectures (definition)	Proof (deduction)
Atiyah	thinking about a vague and uncertain situation	trying to guess what may be found out	reaching definitions	and definitive theorems and proofs
McLane	getting and understanding needed theorems	working with them to see what could be calculated	what might be true	come up with new theorems

Figure 2: van Hiele-like developments in mathematical research

The overall development of proof

The long-term growth of mathematical thinking of proof begins with the perceptions and actions of young children, and develops through three successive levels:

- **practical mathematics** exploring shape and space and developing experience of the operations of arithmetic. This involves the *recognition* and *description* of properties, such as the observation that the sum of numbers is not affected by the order of operation and proof is often formulated as *generic proof*.
- **theoretical mathematics** of *definition* and *deduction*, as exhibited by Euclidean proof in geometry, and of the definition of the 'rules of arithmetic' and properties such as even, odd, prime, composite, and the theoretical deduction of theorems such as uniqueness of factorization into primes.

Theoretical mathematics is appropriate for most applications of mathematics, while those who go on to study pure mathematics change meaning once more to

- **formal mathematics** based on set-theoretic *definition* and *deduction*.

In mathematical research, mathematicians use various combinations of embodiment, symbolism and formalism to imagine possible theorems and to formulate conjectures to seek proof and to shift to ever more sophisticated levels using structure theorems.

This framework offers mathematicians, mathematics educators, teachers and learners the opportunity to share an overall development of proof based on the fundamental sensori-motor bases of human thinking that becomes increasingly sophisticated through the use of language and symbolism. It offers an integration of the cognitive and affective development of mathematical knowledge and mathematical proof.

To enable different communities of practice to come together for mutual benefit, it is essential to develop a common context of discourse that enables different communities to speak meaningfully to each other. In the final chapter of *How Humans Learn to Think Mathematically* (Tall, 2013), I consider the problems encountered in communication between different communities. It becomes clear that each community has its own ways of working that may be highly appropriate in its own context but that the shift to another context involves met-befores that may impede the possibility of an expert in one community making sense of the needs of another community. This suggests the need for a sense of openness and willingness to listen to other points of view and to see the relevance of various viewpoints in different contexts. It should be possible for a community to realise that viewpoints that may be essential in their own context may not be appropriate in others. For instance, a formal mathematician could become more sensitive to the practical needs of mathematics in the everyday community, or recognise the theoretical requirements of applied mathematicians, who build on natural modelling of real situations rather than formal set-theoretic definitions and proof. In the other direction, it should be possible for those involved with practical mathematics to develop some insight into more technical requirements, or for technical mathematicians to have a sense of the power of the greater generality of axiomatic mathematics. The goal should surely be a more respectful understanding between various communities of practice involved in mathematics, including pure and applied mathematicians, mathematics educators and a range of other communities of practice in science, sociology, psychology, philosophy, history, cognitive science, constructivism and so on.

The theory presented here focuses on the fundamental ideas of proof that occur as humans use their perception, operation and reason to build increasingly sophisticated mathematical knowledge. It begins with practical experiences in which specific examples may be seen as *generic* examples of proof. Then these experiences lead to *theoretical* proof based on Euclidean definition and proof in geometry, definitions based on the symbolic 'rules of arithmetic' in arithmetic and algebra, or a blending of embodied thought experiment and symbolic proof. At a formal level, definitions are given as quantified set-theoretic definitions and *formal* proof that apply in any context where the axioms and definitions are satisfied.

The long-term development is affected by supportive and problematic met-befores that apply not only to developing students, but also to the historical evolution of mathematics and to the competing views of differing communities of practice. Experts with sophisticated knowledge structures are subject to personal conceptions of mathematics that they may share with other experts in their community but perhaps not with other communities. The framework given here offers an opportunity to evolve theoretical ideas into the future by blending differing viewpoints to grasp the fundamental basis of the long-term development of mathematical thinking and proof by building on the fundamental ideas of perception, operation and reason.

References

Asiala, M., Brown, A., DeVries, D., Dubinsky, E., Mathews, D., & Thomas, K. (1996). A framework for research and curriculum development in undergraduate mathematics education. Research in Collegiate Mathematics Education II: CBMS Issues in Mathematics Education, 6, 1–32.

Bachelard, G. (1938), (reprinted 1983), La formation de l'esprit scientifique, Paris: J. Vrin.

Baron, R., Earhard, B., & Ozier, M. (1995). Psychology (Canadian Edition), Scarborough, ON: Allyn & Bacon.

Baroody, A.J. & Costlick, R.T. (1998). Fostering children's mathematical power. An investigative approach to K–8 mathematics instruction. New Jersey: Lawrence Erlbaum Associates, Inc.

Brousseau, G. (1983). Les obstacles épistémologiques et les problèmes en mathématiques, Recherches en didactique des mathématiques, 4(2), 164–198.

Bruner, J. S. (1966). Towards a Theory of Instruction, New York: Norton.

Dreyfus, T. & Eisenberg, T. (1986). On the aesthetics of mathematical thought. For the Learning of Mathematics, 6(1), 2–10.

Gray, E. M. & Tall, D. O. (1994). Duality, Ambiguity and Flexibility: A Proceptual View of Simple Arithmetic. Journal for Research in Mathematics Education, 26(2), 115–141.

Harel, G. & Tall, D. O. (1991). The General, the Abstract, and the Generic in Advanced Mathematics. For the Learning of Mathematics, 11(1), 38–42.

Koichu, B. (2008). On considerations of parsimony in mathematical problem solving. In O. Figueras, J.L. Cortina, S. Alatorre, T. Rojano, & A. Sepulova (Eds.), Proceedings of the 32nd Conference of the International Group for the Psychology of Mathematics Education, Morelia, Mexico, 3, 273–280.

Koichu, B., Berman, A. & Katz, E. (2007). What is a beautiful problem? An undergraduate students' perspective. In J. H. Woo, H. C. Lew, K. S. Park, & D. Y. Seo (Eds.), Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education, Seoul: Korea, 3, 113–120.

Krutetskii, V. A., (1976). The Psychology of Mathematical Abilities in Schoolchildren. Chicago: University of Chicago Press.

Lima, R. N. de, & Tall, D. O. (2008). Procedural embodiment and magic in linear equations. Educational Studies in Mathematics, 67(1), 3–18.

Mason, J. H. & Pimm, D. (1984). Seeing the General in the Particular. Educational Studies in Mathematics, 15, 277–289.

MacLane, S. (1994). Responses to "Theoretical Mathematics". Bulletin of the American Mathematical Society, 30(2), 190–191.

McGowen, M. C., & Tall, D. O. (2010). Metaphor or Met-before? The effects of previous experience on the practice and theory of learning mathematics. Journal of Mathematical Behavior 29, 169–179.

Pinto, M. M. F., & Tall, D. O. (1999), Student constructions of formal theory: giving and extracting meaning. In O. Zaslavsky (Ed.), Proceedings of the 23rd Conference of the International Group for the Psychology of Mathematics Education, Haifa, Israel, 4, 65–73.

Presmeg, N. C. (1986). Visualisation and Mathematical Giftedness. Educational Studies in Mathematics, 17(3), 297–311.

Sfard, A. (1991). On the Dual Nature of Mathematical Conceptions: Reflections on

processes and objects as different sides of the same coin, Educational Studies in Mathematics, 22, 1–36.
Skemp, R. R. (1971). The Psychology of Learning Mathematics. London: Penguin.
Skemp, R. R. (1979). Intelligence, Learning, and Action. London: Wiley.
Tall, D. O. (2004a). The three worlds of mathematics. For the Learning of Mathematics, 23(3). 29–33.
Tall, D. O. (2004b). Thinking through three worlds of mathematics. In M. J. Joines, M. Johnsen, & A. E. Fuglestadt (Eds.), Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education, Bergen, Norway, 4, 281–288.
Tall, D. O. (2008). A life-time's journey from definition and deduction to ambiguity and insight. Mediterranean Journal for Research in Mathematics Education, 7(2), 183–196.
Tall, D. O. (2011) Crystalline concepts in long-term mathematical invention and discovery. For the Learning of Mathematics, 31(1), 3–8.
Tall, D. O. (2013). How Humans Learn to Think Mathematically. New York: Cambridge University Press.
Tall, D. O., Thomas, M. O. J., Davis, G., Gray, E. M., & Simpson, A. P. (2000). What is the object of the encapsulation of a process? Journal of Mathematical Behavior, 18(2), 1–19.
Tall, D. O, Yevdokimov, O., Koichu, B., Whiteley, W., Kondratieva, M., & Cheng, Y. H. (2012). Cognitive Development of Proof. In G. Hanna & M. de Villiers (Eds.), Proof and Proving in Mathematics Education: The 19th ICMI Study (pp. 14–39), New York: Springer.
Van Hiele, P. (1986). Structure and Insight. Orlando: Academic Press.
Van Hiele, P. (2002). Similarities and Differences between the Theory of Learning and Teaching of Skemp and the Van Hiele Levels of Thinking. In Tall, D. O. & Thomas, M. O. J. (Eds), Intelligence, Learning and Understanding – A Tribute to Richard Skemp, (pp. 27–47). Flaxton, Australia: Post Pressed.
Weber, K. (2001). Student difficulty in constructing proofs: the need for strategic knowledge. Educational Studies in Mathematics, 48(1), 101–119.
Weber, K. (2004). Traditional instruction in advanced mathematics courses: a case study of one professor's lectures and proofs in an introductory real analysis course. Journal of Mathematical Behavior, 23, 115–133.