Making Sense of Mathematical Reasoning and Proof

David Tall
University of Warwick
<david.tall@warwick.ac.uk>

Introduction

This paper is written in honour of Ted Eisenberg who has been a friend and a colleague for more years than I care to tell. Ted is a mathematician who became a mathematics educator, who has always championed the need to get the mathematics right, in particular, that students should develop a true understanding of mathematical proof.

He has also championed an appreciation of the aesthetics of mathematical thought (e.g. Dreyfus & Eisenberg, 1986), yet was concerned to find that few students derive pleasure from the beauty of mathematics and that mathematics educators seemed to be failing to cultivate such a feeling in their students. Twenty years later, Koichu, Berman & Katz (2007) related the "aesthetical blindness" of students to factors such as self-esteem that affect students' aesthetic judgement.

In this paper, I present a global theoretical framework that complements cognitive and affective aspects of the increasing sophistication of mathematical thinking and proof, taking into account the nature of mathematics itself and the way in which learners mature by building on their previous experiences.

The purpose of this framework is to encompass the development of mathematical thinking that embraces the full spectrum of individual success from the discalculic child to the gifted mathematician from the new-born child to the adult who may be an ordinary member of the public using everyday mathematics, or an expert in pure or applied mathematics. It is based on our shared human facilities of perception, action and reason that mature in very different ways, to explain and predict how we may develop mathematical thinking in general and mathematical reasoning and proof in particular.

The framework presented here builds on insights of many individuals who contributed ideas that inspired new trains of thought.[1] However, it is not a compilation of such ideas, nor is it a review of previous theories, or a comprehensive attempt to include all possible sources; it is a deep personal reflection on how I have come to view the development of mathematical reasoning and proof and the blame for any errors or omissions must lie on my own shoulders.

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[1] This article is a product of personal experience, working with colleagues such as Shlomo Vinner who gave me the insight into the notion of concept image, Eddie Gray, whose experience with young children led me to grasp the essential ways in which children develop ideas of arithmetic and to build a theoretical framework for the different ways in which mathematical concepts are conceived, Michael Thomas who helped me understand more about how older children learn algebra, the advanced mathematical thinking group of PME who broadened my ideas about the different ways that undergraduates come to understand more formal mathematics, many colleagues and doctoral students who I celebrate in Tall (2008) and, more recently, the working group of ICMI 19 who focused on the cognitive development of mathematical proof (Tall *et al.*, 2012).

The sensori-motor language of mathematics

The cognitive development of mathematical thinking and proof is based on fundamental human aspects that we all share: human perception, action and the use of language and symbolism that enables us to develop increasingly sophisticated thinkable concepts within increasingly sophisticated knowledge structures. It is based on what I term the *sensori-motor language of mathematics*, as formulated in a book, *How Humans Learn to Think Mathematically* (here termed *HHLTM*, Tall, 2012).

Mathematical thinking develops in the child as perceptions are recognised and described using language and as actions become coherent operations to achieve a specific mathematical purpose. According to Bruner (1966), these may be communicated first through enactive gestures, then iconic images, then the use of symbolism, including not only written and spoken language but also the operational symbolism of arithmetic and the axiomatic formal symbolism of logical deduction.

My theoretical framework follows a similar path enriched by the experience over time, building from *conceptual embodiment* that combines the enactive and iconic modes of human perception and action, developing into the mental world of perceptual and mental thought experiment. Embodied operations, such as counting, adding, sharing, are symbolised as manipulable concepts in arithmetic and algebra in a second mental world of *operational symbolism*. As the child matures, there is a further shift into a focus on the *properties* of mental objects as in Euclidean geometry, or the properties of arithmetic operations that are recast as 'rules' that underlie the generalized operations and expressions in algebra. Each of these leads to different forms of mathematical proof: *Euclidean proof* in geometry and *algebraic proof*, based on the 'rules of arithmetic' in algebra.

The focus on properties in both embodiment and symbolism leads to the development of set-theoretic definitions and formal proof in the *axiomatic formal* world of pure mathematics. This builds naturally from the embodied and symbolic experiences of school mathematics to a new level of mathematical thinking in which formal proof guarantees that any new context in which specific axioms and definitions hold will necessarily have all the properties given by theorems proved from those axioms and definitions.

Embodiment and symbolism continue to play their part, not only in imagining new possibilities that may be defined and proved formally, but also in an amazing turnaround in which certain theorems (called structure theorems) prove that axiomatic systems have embodied and symbolic structures based on formal proof. This reveals mathematical thinking at the highest level in general, and mathematical proof in particular, as an intimate blend of embodiment, symbolism and formalism.

Aesthetic sense in mathematics may be seen to arise from the surprising manner in which patterns in embodiment and symbolism blend together to give insight, and formal structures – based on subtly worded axioms and definitions – may be proved to have a structural simplicity that allows mathematicians to interpret them in embodied and symbolic ways as part of a coherent, logically connected framework.

The evolution of theories of mathematical thinking and proof

Pierre van Hiele (1986) focused on *structure* and insight, seeing a succession of levels that may be described as *recognition* and *description* of figures, leading to *definition* and *deduction* of properties through Euclidean proof.

Ed Dubinsky and others (Asiala et al, 1996) took an apparently different path, following Piaget's idea of reflective abstraction to focus on operations that are seen first as *actions*, routinized as *processes*, then *encapsulated* as mental *objects* within knowledge *schemas*.

Anna Sfard (1991) proposed a framework that alternated between *operational* and *structural* ways of thinking in which operations are *condensed* as *processes*, and then *reified* as mental *objects* which now have a certain structure. She suggested at the time that an operational approach inevitably precedes structural mathematics. However, the examples given mainly involved operational symbolism becoming reified as mental objects, and for me at the time, this question remained open.

She spent several weeks staying in my home in the autumn 1990. I remember explaining to her the idea that Eddie Gray and I had just formulated by giving the name *procept* to an entity that functioned dually as *pro*cess and con*cept* (Gray & Tall, 1994). She joked with me by suggesting that the idea should relate to the duality of *pro*cess and ob*ject* by calling the idea a *pro-ject*. I smiled and gently suggested that the term project already had a different meaning.

I offered her a new possibility: that her notion of *condensation* of an action into a process could be extended by renaming the shift from process to object as *crystallization*. This would extend the metaphor of condensation of a gas to a liquid, which enabled the liquid to be poured into a container that could be carried around, to the crystallization of a liquid into a solid that enabled the object to be *grasped* and manipulated in the mind.

Neither of us was willing to accept the suggestion proposed by the other. Nor could we resolve our differences relating to the shift from school mathematics to the advanced mathematical thinking of set-theoretic definition and formal proof. I suggested that her term 'structural' was used by Bourbaki to relate to the axiomatic structure of a formal mathematical object, but she insisted that her notion of 'structural' was quite different.

This led eventually led to the distinction between the structural development in geometry following van Hiele (1986) and structure in formal mathematics analysed in Tall, Thomas *et al*. (2000) and later to the three-part analysis of long-term growth through parallel developments of conceptual embodiment (broadly following van Hiele) and operational symbolism (using process-object theories of operational symbolism) in school, leading much later to the axiomatic formal framework of set-theoretic definition and proof in university pure mathematics (Tall, 2004a, 2004b).

Following the recent death of van Hiele in 2011, at the grand old age of one hundred, I revisited his ideas of structure and insight, which he asserted applied to geometry, but not to the symbolism of arithmetic and algebra (van Hiele, 2002). I suddenly realised that the term *operation* should not be restricted to the symbolic operations in arithmetic and algebra. Operations clearly occur in the constructions of Euclidean geometry. For instance, we may operate on an isosceles triangle by joining the vertex to the midpoint of the base to cut the triangle into two parts that are congruent (having three corresponding sides). This proves that the base angles must be equal, and a number of other properties follow, such as the property that the line from the vertex to the midpoint of the base is at right angles to the base. Not only that, these various properties are all *equivalent* in the context of Euclidean geometry: any one of these properties can be taken as the definition of an isosceles triangle and all the others can be deduced from it.

The operations of construction in geometry and the various operations in arithmetic and algebra have a common definition: they consist of 'a coherent sequence of actions and decisions performed to achieve a specific purpose.' A geometric operation is a construction that focuses on the *object* (the figure) and results in enabling us to see relationships concerning *the properties of the object*. A symbolic operation performs a calculation or manipulation focusing more on *the properties of the operations themselves* as the operations lead to a symbolic output.

Furthermore the compression of operation into mental object in symbolism begins for the child as *embodied operations* on objects such as counting, adding, sharing, and is compressed into *symbolic operations* on whole numbers, fractions, signed numbers and so on. I suddenly realised that there are two distinct forms of compression from operation to mental object that I termed *embodied compression* and *symbolic compression* (see *HHLTM*, chapter 7).

Embodied compression focuses on the effect of the operations on the objects, such as counting a collection to find the number of objects, such as 'six'. The set of six objects can be subdivided, say, into subsets of 'four' and 'two' and, by rearranging the sets, it may be seen that 'two' and 'four' is also 'six'. Reorganizing the subsets as two rows of 'three' allows them to be seen as three columns of 'two' so that 'two threes' is the same as 'three twos'. Embodied compression enables us to see *at a glance* the flexible properties of arithmetic. 'Proof' at this early stage is a form of reasoning based on our interpretation of the coherence of our own perceptions and actions. This form of proof, in which a specific example is seen to be typical of a whole category of examples, is termed *generic proof* (Harel & Tall, 1991).

Symbolic compression involves performing a counting operation to obtain a number concept, for instance, the operation of 'count-on' calculates 'two and eight' as counting on eight to get 'three, four, five, six, seven, eight, nine, *ten*' while 'eight and two' is the short count 'nine, *ten*'. Here the two operations are very different, one is a long count, and the other is short. The general properties of the symbolic compression are therefore not as self-evident as they are with embodied compression.

This leads to two totally different ways of making sense of arithmetic. One involves the recognition of the embodied properties of arithmetic, to see general patterns that enable arithmetic operations to be performed in simpler ways. To calculate $2+8$, just calculate $8+2$. The same insight may be used to generate new facts from those already known. For instance, that '$4+3$' is 'one less than $4+4$' so it is seven, using rich conceptual links within a growing knowledge structure.

A gifted child may simply see these flexible properties of arithmetic as part of a coherent knowledge structure and use this overall coherence as an organising principle to simplify operations in arithmetic. A child who focuses on procedural operations of counting taking place in time will find arithmetic operations to be far more difficult to cope with. Eddie Gray and I called this bifurcation 'the proceptual divide' between those fixed in increasingly complicated counting procedures and those who develop flexible ways to derive new facts from known facts (Gray & Tall, 1994).

This bifurcation between those who find mathematics 'easy' and those who find it impossibly difficult begins at a very early age. In *How Humans Learn to Think Mathematically* (*HHLTM*), I open with an example of two young boys, one of whom struggles to operate by counting on his fingers, while a younger boy in the same school responds to the problem of finding 'a sum whose answer is 8' by responding 'a million take away 999,992'.

This reveals immense differences between children even in the earlier stages of learning. This should be taken into account in seeking to explain and predict how each individual attempts to make sense of mathematics by building on personal ways of knowing and operating.

Long-term pleasure and pain

Emotions play a vital role in mathematical thinking. As my dear supervisor, Richard Skemp used to say: 'pleasure is a signpost, not a destination.' In his goal-oriented theory of learning (Skemp 1979), he saw children starting out with the goal of seeking to make sense of the world. Successfully linking together ideas in coherent ways gives pleasure, success breeds more success, so that a child with a history of success builds up a positive feed-back loop where an encounter with a problematic situation is often met with the determination to conquer the difficulty. However, lack of success leads to an anti-goal, to avoid the feeling of stress. Further encounters with stress may lead to a negative feed-back loop in which the desire to avoid failure leads to less engagement with the mathematics and less technical proficiency that causes even more difficulty and even higher levels of mathematics anxiety (Baroody & Costlick, 1998).

A product of this bifurcation between these two positions often leads to a switch from the anti-goal of avoiding the pain of failure to seek the alternate goal of being able to 'do' mathematics 'to pass the test.' Having seen the difficulties of mathematics teaching and learning in many contexts around the world, I am convinced that the greater majority of mathematics teachers and learners are drawn into the strategy of learning procedurally to obtain a public measure of success in passing examinations rather than to continue to attempt to make sense of mathematics which is becoming, for them, increasingly stressful.

But why is this?

An analysis of the development of mathematical thinking reveals the surprising conclusion that mathematics is not a system that builds logically on previous experience at each stage, even though every mathematics curriculum in the world is intent on presenting topics in a coherent sequence, carefully preparing the necessary pre-requisites at each stage for the more sophisticated stages that follow. On the contrary, an experience that has been 'met before' may be supportive in some new situations yet be problematic in others.

The concept of 'met-before' was introduced by Lima & Tall, (2008) and McGowen & Tall (2010) to describe 'a structure we have in our brains *now* as a result of experiences we have met before'. Some ideas that work in one situation such as 'addition makes bigger' or 'take away makes smaller' in whole number arithmetic are supportive in the context of fractions yet problematic in the context of signed numbers. Of course, we have known this kind of thing for ages, including the concept of 'epistemological obstacle' developed by Bachelard (1938) and Brousseau (1983) and the need for accommodation by Piaget (see, for example, Baron et al, 1995) or reconstruction by Skemp (1971).

However, the notion of met-before refers to the effect of previous experience on new learning. A particular met-before is not in itself supportive or problematic, it *becomes* supportive or problematic in a new situation where the learner may find it supportive or problematic in attempting to make sense of the new ideas. For instance, 'take away leaves less' is supportive in some contexts (e.g. everyday situations where something is removed, in the postulates of Euclidean geometry, or taking one whole number from another) but it is problematic in others (such as taking away a negative number or in the theory of infinite cardinals).

As I contemplated the phenomena more closely, I suddenly realized that the problem lies not only in the individual learner, *it is a widespread feature of the nature of mathematics itself*. In shifting to a new context, say from whole numbers to fractions, or from positive numbers to signed numbers, or from arithmetic to algebra, generalization is encouraged by supportive met-befores (ideas that worked in a previous context and continue to work in the new one) and impeded by problematic met-befores (that made sense before but do not work in the new context).

This explains and predicts known phenomena such as 'the didactic cut' where linear equations of the form 'expression = number' can be 'undone' by arithmetic operations which becomes problematic for equations of the form 'expression = expression' that require algebraic manipulation. A similar analysis applies to the embodied idea of an 'equation as a balance' which is supportive when the two sides involve addition of positive quantities but becomes problematic when it involves subtraction or negative quantities. Even the principle of 'doing the same thing to both sides', which is supportive for some who are able to use it to generate the techniques for solving equations can be problematic for many who require to work with specific operations and re-interpret 'add the same thing to both sides' as an embodied shift of the term to the other side with a rote-learnt principle to 'change signs' (Tall, Lima & Healy, under review).

Problematic aspects also arise in the introduction of set-theoretic definitions to prove theorems formally in situations where students' prior experience of embodiment and symbolism may be problematic. (For instance, they may sense that a theorem is clearly true, and so it does not require a complicated technical proof, or they may have beliefs that conflict with the formalism and impede their progress.)

In each of these cases, the student is taken out of his or her comfort zone, built on previous experience, and is faced with problematic met-befores. The result, for a confident and successful student, may be the desire to struggle with the new problem, to find a way to conquer it and to experience once more the ultimate pleasure of success.

However, students who have succumbed to the earlier need to learn procedures 'to get by' are more likely to avoid the difficulty by learning to 'do' whatever is necessary to pass the test. For those who are already suffering from mathematics anxiety, the problem may be too great for them to attempt to make much sense at all.

This is often seen as a difference between students of various abilities. However, it does not mean that we cannot seek to encourage each individual student by building confidence in what they know and seeking to make sense of problematic aspects that arise in new contexts. In particular, we should seek to understand the structures that make mathematical thinking flexible and powerful, yet simple.

Crystalline concepts

In Tall (2011), I returned to the idea of crystallizing ideas into entities that can be grasped as thinkable concepts and manipulated fluently in the mind. But now I was no longer restricted to thinking of compressing/encapsulating/reifying a process into a concept in the operational symbolic world. Suddenly I felt that I grasped the whole game. I formulated a working definition of a *crystalline concept* as 'a concept that has an internal structure of constrained relationships that cause it to have necessary properties as a consequence of its context.'

Crystalline concepts occur throughout every area of mathematics, including;

- platonic objects in geometry, such as points, lines, triangles, circles, congruent triangles, parallel lines;
- operational symbols as flexible procepts in arithmetic, algebra and symbolic calculus;
- set-theoretically defined concepts in axiomatic formal mathematics.

Not only do crystalline concepts occur at the highest levels of mathematical thinking, they emerge in the thinking of a young child who sees the flexible proceptual structure of arithmetic through embodied compression rather than the procedural step-by-step counting procedures of arithmetic that operate in time.

They enable flexible thinkers to see mathematical ideas in astonishingly simple ways. It is not that the fractions $4/8$, $7/14$, $101/202$ are all *equivalent* to each other and to the simplest possible canonical form $\frac{1}{2}$, it is that these are all manifestations of a single crystalline concept: the rational number as a unique point on the number line.

It is not that the expressions $2(x+7)$ and $2x+14$ are equivalent but different, where the first can be turned into the second by 'multiplying out the brackets' and the second can be turned into the first by 'factorization', it is that *both expressions are different ways of writing the same crystalline concept* as an algebraic expression. Students who think flexibly in terms of crystalline concepts have much more powerful means of relating mathematical ideas than those who see equivalent ideas that are changed from one form to another by carrying out procedures.

Likewise, in axiomatic formal mathematics, an axiomatic system such as 'a group' is a crystalline concept with rich interconnections between its properties. We may not know what specific group we are dealing with, but we *do* know that it has an identity that we may denote by e, and that if x is any element, we can define the power x^n for any positive or negative integer and prove that $x^{m+n} = x^m x^n$ for any integers m, n. A crystalline concept may be defined formally and then its properties may be deduced as theorems to build up a knowledge structure where relationships are tightly interconnected by formal proof.

For example, we can prove that if we begin with the axiomatic definition of an ordered field F, then in this context we may formulate any of the equivalent definitions for completeness, to prove that a complete ordered field is not only unique up to isomorphism, it is also unique as a crystalline concept.

At the highest level of pure mathematical research, it is the compression of structural properties of defined formal concepts into crystalline concepts that gives gifted mathematicians a simplicity of thought that is beyond the mere proving of theorems of equivalence. An ordered field not only contains a subfield *isomorphic* to the rational numbers, it can be conceived as a crystalline concept that *contains* the crystalline concept of the rational numbers.

I recall the ideas that I encountered as a graduate student when theoreticians spoke of the identification of one structure with another structure as 'an abuse of notation'. On the contrary, it is the very vision that gives the biological brain of the mathematician a level of flexibility to conceive mathematical ideas mentally in more simple and insightful ways.

The transition from proof in embodiment and symbolism to formal proof

Having developed an overall framework for cognitive development from the new-born child to the frontiers of mathematical research, it is now appropriate to consider the cognitive development of proof, as outlined in the ICMI Project on Proof and Proving (Tall, Yevdokimov *et al*, 2012), described in greater detail in *HHLTM*.

The van Hiele levels (1986) have been variously reconsidered by a range of authors, may be seen in as four successive levels which I term

- **Recognition** of basic concepts such as points, lines, and various shapes;
- **Description** of observed properties;

- **Definition** of concepts to test new examples to see if they satisfy the definition and to use the definitions to formulate geometric constructions;
- **Deduction** in the form of Euclidean proof in plane geometry.

Each of these is a form of *structural abstraction* in which the structure of the objects under consideration and their relationships shift to successive new levels of sophistication. This begins first with observations of geometric objects whose structures are recognised and described. At this point the foundations of Euclidean proof are laid down by formulating definitions for figures that not only allow them to be categorised and constructed but also to use ideas such as congruent triangles and parallel lines to construct Euclidean proof.

Van Hiele also described a fifth level of *rigour* that I see as shifting in two directions, the first to different embodied contexts such as projective geometry or spherical geometry, the second in terms of the more sophisticated world of *axiomatic formalism* as prescribed by Hilbert.

While van Hiele (2002) saw these levels apply to geometry and not to the symbolic development from arithmetic to algebra, I now see structural abstraction of properties as an integral part of operational symbolism. In geometry the objects under consideration are shapes that are recognised, described, defined and properties are deduced by Euclidean proof. In operational symbolism, the objects are manipulable symbols that are formed from the encapsulation of symbolic operations and the properties of arithmetic arise from the regularities of these operations that are *recognised*, then *defined* as 'rules of arithmetic' to be used in algebraic proofs to *deduce* theorems.

Anna Sfard (1991) suggested that operational abstraction invariably occurs before structural abstraction. APOS theory (Asiala et al. 1996) also begins with actions, not objects. However, operations such as counting, measuring, adding, sharing all operate on physical objects at the outset. Furthermore, the longer-term development is far more complicated as the learner encounters successive systems of numbers — whole numbers, fractions, finite and infinite decimals, signed numbers, rational numbers, real numbers, complex numbers. Each change involves supportive aspects that generalize problematic aspects that impede progress. At the same time, structural abstraction of the properties of number systems are being developed and generalized into algebraic manipulation based on the observed rules of arithmetic.

Experience with the properties of embodiment and symbolism may eventually lead to a transition to the axiomatic formal world of definition and deduction. This builds on our experience of conceptual embodiment and operational symbolism, beginning with the *recognition* and *description* of mathematical situations and then the *definition* of axiomatic systems and of defined concepts within those systems, and *deducing* properties of systems and defined concepts using formal proof.

Experienced mathematicians have flexible knowledge structures that they wish to pass on to their students. However, by the time students pass through school to enter university, they will have already developed in very different ways based on how they have managed to make sense of previous experiences. The successive encounters with new ideas may be grasped by some students who sense the overall patterns of mathematics, as happens for example with the flexible patterns that may be apparent in the embodied compression of operations into symbolic concepts. Others may struggle with the procedural aspects of arithmetic and seek the alternative goal of being able to carry out the procedures to pass the tests. Others may be suffering increasingly from mathematical anxiety.

Krutetskii (1976) produced significant evidence that gifted children are much more likely to develop a strong verbal-logical basis to mathematical thinking than a visual-pictorial foundation. The nine most gifted students selected from a population of over a thousand contained five analytic thinkers (verbal-logical), one geometric thinker (visual-pictorial), two harmonic thinkers combining the two (one more verbal, the other more visual) and one who was not classified. Norma Presmeg (1986) found that the most outstanding senior school mathematics students (7 pupils, out of 277) were almost always non-visualizers. Of 27 'very good' students (10% of the sample), eighteen were non-visualizers and five were visualizers.

This suggests that a small number of those students who enter university are powerful verbal-analytic thinkers who may benefit from making sense of set-theoretic definitions, an even smaller number who base their thinking on visual-pictorial representations, and many others with a blend of visual embodied thinking and operational symbolism. often succeeding in examinations through rote learning.

Some students seek a *natural* approach based on previous experiences of embodiment and symbolism from school mathematics. Some seek to use a *formal* approach based on set-theoretic definitions and the deduction of properties using formal proof. Some seek to learn proofs procedurally to reproduce in examinations. All of these approaches may involve supportive and problematic aspects, some of which have been detailed in the literature (e.g. Pinto & Tall, 1999, Weber, 2004).

As students become more experienced and shift to graduate studies, Weber (2001) produced evidence that when research graduates are presented with problems, they are more likely to reply flexibly making links between mathematical concepts in a sophisticated knowledge structure while undergraduates in their early studies, have yet to develop such flexibility.

This is consistent with the observations about the lack of aesthetic appreciation of mathematical ideas noted by Dreyfus and Eisenberg (1986). It is also consistent with the relationship noted by Koichu, Berman & Katz (2007) between "aesthetical blindness" of students and factors, such as self-esteem, that affect students' aesthetic judgement. However, now, we have a theoretical framework that traces the lack of self-esteem in terms of the impediments caused by problematic met-befores that occur throughout the curriculum. The problems faced by students are based on deeply embedded experiences that they have met before since their early childhood!

A few students, characterized as being 'gifted' develop verbal-analytic skills that enable them to build formally from set-theoretic definitions to construct highly connected crystalline concepts that may have embodiments and operations linked to underlying formally proved structure theorems. But many others, who focus on 'maximising their mark on the exam' to 'get a good degree' to move on in their lives, have good reasons for doing so. The mathematics is *problematic* for them and *it doesn't make sense.*

The framework just described covers the full spectrum of student thinking within the long-term cognitive and affective development of mathematical thinking from the earliest years to university mathematics. It also takes us on towards the frontiers of mathematics as researchers build on their experience in conceptual embodiment, operational symbolism and axiomatic formalism to *recognise* new problems, to *describe* possible solutions and propose conjectures, to *define* appropriate axiomatic systems and definitions and seek to *deduce* new theorems using formal proof.

We now have a broad framework for the cognitive and affective development of mathematical thinking which may be used to seek to explain and predict how mathematicians and their students make sense of mathematical reasoning and proof.

Discussion

How can the given framework be used to encourage undergraduate students to make sense of mathematical proof? The analysis of the cognitive and affective aspects experienced by students learning mathematics requires more than an expert knowledge of mathematics. The situation may be improved if both professors and students are more explicitly aware of their various ways of thinking. Mathematics educators, teachers and learners at every level may benefit by becoming more explicitly aware of the joy of supportive met-befores that can be used to boost confidence and the fear of problematic met-befores that causes increasing anxiety.

As Cassius observed to Brutus in Shakespeare's Julius Caesar, 'the fault lies not in our stars but in ourselves.' We mathematicians and mathematics educators develop theoretical frameworks that are based on our *own* previous experience. The theory of supportive and problematic met-befores applies not just to our students but also to ourselves as mathematicians and educational theorists. It requires us all to develop the confidence to build theories to improve mathematical learning while being strong enough to reflect on aspects that do not fit our frameworks and to seek deeper understanding.

This applies to us all. In my own case, as I contemplated the full framework for the long-term growth of mathematical thinking, I immodestly believed that I had a powerful overarching theory. It is a delusion of course. Any theoretical framework will be limited by the experiences of the theoretician who formulates it and I am definitely limited as a competent mathematician with little to show in terms of original mathematical research and limited experience of teaching younger children. As a human being with a biological brain, I am limited in the number of things that I can contemplate at a given time and I focus on the development of a wide spectrum of individuals as they grow in sophistication in mathematical thinking. Others focus on other vital aspects, such as how individuals in society cooperate for the greater good, or how various communities of practice view their activities.

That being said, the framework, while clearly not a 'final solution', does seek to base its development on the foundational human attributes of a sensori-motor brain with verbal and symbolic modes of representation, communication and reason.

It links together cognitive growth with its emotional consequences that encourage or impede sophisticated developments in new situations and attempts to consider the whole spectrum of individual development, from the discalculic child with difficulties in processing information to the gifted child who senses the generic crystalline structure of mathematics and uses them to build essentially simpler ways of powerful mathematical thinking.

It begins with the new-born child and carries through the development to adult thinking, be it in the everyday blending of embodiment and practical arithmetic, the growing modes of reasoning in geometry and operational symbolism, the applications of mathematics using embodied modelling and symbolic solution processes, or the formal axiomatic developments in pure mathematics research.

Personal reflections

As I wrote this (in December 2011), I had been intrigued by the reactions that my ideas have received in recent years. My own goal is to construct a framework that attempts to go beyond the development of ever more sophisticated theories to find essential simplicities that can be communicated not only to experts in mathematics, mathematics education, and other related disciplines such as psychology, philosophy, cognitive science or in applications of mathematics, but also to those involved with teaching and learning, be they teachers, teacher trainers, curriculum designers, young children, students, parents, or even, dare I say it, politicians. Each community of practice has its own beliefs and ways of making sense and efforts should be made in bridging these differences by acknowledging the supportive aspects that hold a given community together yet become problematic when communicating between different theoretical standpoints.

However, to achieve such a huge goal requires the development of modes of communication between differing communities of practice. Each community shares its own ways of thinking that may become problematic when attempting to make sense of new situations. As I submit papers for review, each time I have some hugely positive responses from some reviewers who warm to my ideas. I also have some that vehemently declare that my ideas are unacceptable, invariably because they go beyond the reviewer's own particular specialism, be it constructivism, cognitive science, philosophy, history of mathematics, formal research mathematics, or various theories of mathematical education.

My own view is that we need to go beyond these parochial specialisms to reflect deeply on more fundamental issues that affect so many billions of individuals around the world—the children who struggle with equations, the students who cannot cope with the limit concept in the calculus, the undergraduates who find difficulty with axiomatic theories and all who are forced to endure a life of anxiety in mathematics separate from their everyday lives. We need to understand how we can improve the experiences of regular students while also nurturing giftedness, not only in supporting those who are manifestly gifted in mathematics, but also those who may be nurtured into a greater sense of mathematical understanding that eventually makes mathematics more sophisticated in ways that are at the same time, essentially simple.

Over the years I have seen the opposition of conflicting theories, but now I often see these conflicts arising from theories that are coherent in their own context where they should be honoured for their insight. I have often struggled to make sense of insights of others that at first I did not grasp, but then, in the longer term, I realized their valuable contribution to a bigger picture as ideas evolve.

I honour my school teacher John Butler who always made me aware that the journey in mathematical understanding has new ideas around every corner.

I honour my doctoral supervisor in mathematics, Michael Atiyah, who gave me sophisticated insights into the relationship between geometrical and algebraic thinking at an aesthetic level that was far higher than I could grasp at the time. I sense myself to be the black sheep of his mathematical family as I spread my interests widely and moved from thinking about mathematical research to seek a new life in understanding how others learn to think about mathematics.

I honour my doctoral supervisor in mathematics education, Richard Skemp, who taught me that 'there is nothing so practical as a good theory'. His ideas—of instrumental and relational understanding, of different modes of building and testing mathematics, and theory of how emotional effects of goals and anti-goals affect long-term development—are the foundation of my whole development. He once said to me, 'David, you will be Elisha to my Elijah', a comment that I found puzzling at the time, but now I realise that he could see across the Jordan and did not live to fully realise his vision of the promised land.

I honour my friend and inspiration, Shlomo Vinner, who taught me about concept image and concept definition, yet warned me that I should talk philosophically about the mind, and not speak about the physical brain, because future research would undoubtedly show the errors in my arguments.

I honour Efraim Fischbein and his student Dina Tirosh, whose vision, building from intuition through algorithmic thinking on to formal thinking, deeply influenced me, not only the development of a three-world framework, but also in my long-term understanding of infinitesimals and the completion of potentially infinite processes.

I honour Ed Dubinsky, who taught me so much about the encapsulation of process into object, yet I took years to assimilate these ideas in ways that would make sense compatible with my own search for the links between visualisation and symbolism.

I honour Eddie Gray who taught me how little children think about arithmetic that led to the subtle notion of 'procept' and who inspired me when I needed support.

I honour Anna Sfard, whose ideas about 'operational' and 'structural' mathematics later developed into her theory of 'focal analysis of ideas' and metaphor that I found intellectually stimulating, yet I chose a different path focusing on what individuals had 'met-before' to see mathematical development from the viewpoint of the learner.

I took great delight in the embodiment of Lakoff (1987) and his later work with Rafael Núñez (2000), who moved into neurophysiology to reveal ideas beyond what I expected. Yet I found their criticism of 'the romance of mathematics' did not grasp certain vital aspects of the manner in which research mathematicians construct crystalline concepts. Mathematicians construct formal theories that include structure theorems that offer new forms of conceptual embodiment and operational symbolism. The romance of mathematics has not only formal proof but also opens up new possibilities using human embodiment and symbolism to extend the boundaries of mathematical thinking.

I relished my friendship with Jim Kaput who inspired me with his vision of a democratic approach to mathematics using technology to make ideas more widely available. We differed in our technological approach as he used a mouse to underpin 'point and click' ways of building up piecewise straight approximations and I saw a graphic approach dynamically visualizing 'local straightness' and linking this directly to operational symbolism. However, we both sought to make sense of mathematical ideas for a wider community.

I remember with affections the afternoons I spent with Guershon Harel as I learned his way of mulling over an idea for a long time as it slowly crystallized into a reasoned theoretical perspective.

I honour the insights of all my doctoral students (recorded in Tall, 2008), particularly the insight of Anna Poynter whose work on teaching vectors in school was instrumental (and also relational) in developing the three worlds of conceptual embodiment, operational symbolism and axiomatic formalism (Watson, 2002).

I say 'thank you' to my colleagues in the ICMI study group working on the cognitive growth of mathematical proof (Tall, Yevdokimov *et al.*, 2012), particularly to Walter Whiteley, whose incisive comments on proof in Euclidean geometry encouraged the notion of 'crystalline concept' and Boris Koichu whose insight into the principle of parsimony (Koichu 2008) enhanced the relationship between crystalline concepts and aesthetic insight.

I continue to work with colleagues around the world, on Lesson Study in Japan and other APEC countries and its introduction in Europe, with algebra and calculus in several different communities in the Americas, with the needs for service mathematics and higher level mathematical analysis and proof in the UK. Many of my ideas, perhaps all, originate in collaboration and insights working with others; my own role lies in the alchemy I use to blend these essential ideas together.

Finally, and most importantly on this celebratory occasion, I pay tribute to Ted Eisenberg, who has played a subtle role in this long-term development with his desire to encourage students to grasp the aesthetic values of mathematical proof and his criticism of a behaviourist approach to learning. Out of respect for the thinking of others, even when they are different, indeed *because* they are different, we may come to a greater insight into how we can make sense of mathematics in general and, in particular, how we make sense of mathematical reasoning and proof.

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