The fundamental cycle of concept construction underlying various theoretical frameworks

Dedicated to the memory of Kevin F Collis 1930-2008

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Abstract: In this paper, the development of mathematical concepts over time is considered. Particular reference is given to the shifting of attention from step-by-step procedures that are performed in time, to symbolism that can be manipulated as mental entities on paper and in the mind. The development is analysed using different theoretical perspectives, including the SOLO model of John Biggs and Kevin Collis and various theories of concept construction to reveal a fundamental cycle underlying the building of concepts that features widely in different ways of thinking that occurs throughout mathematical learning.

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Introduction

This paper is a revision and extension of an earlier paper (Pegg & Tall, 2005), written to analyze major theories of cognitive growth with particular reference to local and global issues: the local development of processes and concepts and the global development of mathematical knowledge over the years of individual growth. In these frameworks, the work of Kevin Collis is central. Collis (1975) was the first to place Piaget's *early formal* stage into the earlier group of stages covered by *concrete operations*. He claimed that most children between 13 and 15 years are "concrete generalizers" and not "formal thinkers". This implies that students in this age range are, in general, tied to their own concrete experience where a few specific instances satisfy them of the reliability of a rule. Building on this idea he and John Biggs took earlier global theories of Piaget, Dienes, Bruner and others to formulate the SOLO Taxonomy (now generally referred to as the SOLO model) addressing the global growth of knowledge through successive modes of operation. These modes were formulated as sensori-motor, ikonic, concrete symbolic, formal and post-formal. He also considered local cycles of growth formulated as unistructural, multistructural, relational and extended abstract. By complementing the local and global, he clarified major issues faced in building a comprehensive theory of cognitive development of value both in theory and in practice.

The focus in this paper is to consider various theories that address local and global issues in cognitive growth, to raise the debate beyond simple comparison to move towards identifying deeper underlying themes that enable us to offer insights into issues concerning the learning of mathematics. In particular, a focus of analysis on fundamental learning cycles provides an empirical basis from which important questions concerning the learning of mathematics can and should be addressed.

To assist us with this focus we distinguish two kinds of theory of cognitive growth:

- **global frameworks of long-term growth** of the individual, such as the stage-theory of Piaget (e.g., see the anthology of Piaget's works edited by Gruber & Voneche, 1977), van Hiele's (1986) theory of geometric development, or the long-term development of the enactive-iconic-symbolic modes of Bruner (1966).
- **local frameworks of conceptual growth** such as the action-process-object-schema theory of Dubinsky (Czarnocha, Dubinsky, Prabhu, & Vidakovic, 1999) or the unistructural-multistructural-relational-unistructural sequence of levels in the SOLO Model (**S**tructure of the **O**bserved **L**earning **O**utcome, Biggs & Collis, 1991; Pegg, 2003).

Some theories (such as those of Piaget, van Hiele, and the full SOLO model) incorporate both global and local frameworks. Bruner's enactive-iconic-symbolic theory formulates a sequential development that leads to three different ways of approaching given topics at later stages. Others, such as the embodied theory of Lakoff and Nunez (2000) or the situated learning of Lave and Wenger (1990) paint in broader brush-strokes, featuring the underlying biological or social structures involved.

Global theories address the growth of the individual over the long-term, often starting with the initial physical interaction of the young child with the world through the development of new ways of operation and thinking as the individual matures. Table 1 tabulates four global theoretical frameworks.

Table 1. Global stages of cognitive development

Piaget Stages	van Hiele Levels (Hoffer,1981)	SOLO Modes	Bruner Modes
Sensori Motor	I Recognition	Sensori Motor	Enactive
Pre-operational	II Analysis	Ikonic	Iconic
Concrete Operational	III Ordering	Concrete Symbolic	Symbolic
Formal Operational	IV Deduction	Formal	
	V Rigour	Post-formal	

An example of the type of development that such global perspectives entail can be seen by the meaning associated with the five modes in the SOLO model proposed by Biggs and Collis (1982) and summarised in Table 2 (Pegg, p. 242, 2003).

Table 2. Description of Modes in the SOLO Model

Sensori-motor: (soon after birth)	A person reacts to the physical environment. For the very young child it is the mode in which motor skills are acquired. These play an important part in later life as skills associated with various sports evolve.
Ikonic: (from 2 years)	A person internalises actions in the form of images. It is in this mode that the young child develops words and images that can stand for objects and events. For the adult this mode of functioning assists in the appreciation of art and music and leads to a form of knowledge referred to as intuitive.
Concrete symbolic: (from 6 or 7 years)	A person thinks through use of a symbol system such as written language and number systems. This is the most common mode addressed in learning in the upper primary and secondary school.
Formal: (from 15 or 16 years)	A person considers more abstract concepts. This can be described as working in terms of 'principles' and 'theories'. Students are no longer restricted to a concrete referent. In its more advanced form it involves the development of disciplines.
Post Formal: (possibly at around 22 years)	A person is able to question or challenge the fundamental structure of theories or disciplines.

Underlying these 'global' perspectives is the gradual biological development of the individual. The newborn child is born with a developing complex sensory system and interacts with the world to construct and coordinate increasingly sophisticated links between perception and action. The development of language introduces words and symbols that can be used to focus on different aspects and to classify underlying similarities, to build increasingly sophisticated concepts.

Whereas some commentators are interested in how successive modes introduce new ways of operation that *replace* earlier modes, the SOLO model explicitly nests each mode within the next, so that an increasing repertoire of more sophisticated modes of operation become available to the learner. At the same time, all modes attained remain available to be used as appropriate. This is also reflected in the enactive-iconic-symbolic modes of Bruner, which are seen to develop successively in the child, but then remain simultaneously available.

In a discussion of local theories of conceptual learning, it is therefore necessary to take account of the development of qualitatively different ways (or modes) of thinking available to the individual. In particular, in later acquired modes in SOLO, such as the formal or concrete symbolic mode, the student also has available sensori-motor/ikonic modes of thinking to offer an alternative perspective.

Local Cycles

Local cycles of conceptual development relate to a specific conceptual area in which the learner attempts to make sense of the information available and to make connections using the overall cognitive structures available to him/her at the time. Individual theories have their own interpretations of cycles in the learning of specific concepts that clearly relate to the concept in question.

Following Piaget's distinctions between empirical abstraction (of properties of perceived objects) and pseudo-empirical abstraction (of properties of actions on perceived objects), Gray & Tall (2001) suggested that there were (at least) three different ways of constructing mathematical concepts: from a focus on *perception* of objects and their properties, as occurs in geometry, from *actions* on objects which are symbolised and the symbols and their properties are built into an operational schema of activities, as in arithmetic and algebra, and a later focus on the *properties* themselves which leads to formal axiomatic theories. However, these three different ways of concept construction are all built from a point where the learner observes a moderately complicated situation, makes connections, and builds up relationships to produce more sophisticated conceptions. This notion of development leads to an underlying cycle of knowledge construction.

This same cycle is formulated in the SOLO model to include the observed learning outcomes of individuals responding to questions concerning problems in a wide range of contexts. The SOLO framework can be considered under the broad descriptor of neo-Piagetian models. It evolved as a reaction to observed inadequacies in Piaget's framework where the child is observed to operate at different levels on different tasks supposedly at the same level, which Piaget termed 'décalage' (Biggs & Collis, 1982). The model shares much in common with the ideas of such theorists as Case (1992), Fischer (see Fischer & Knight, 1990) and Halford (1993).

To accommodate the décalage issue, SOLO focuses attention upon students' *responses* rather than their level of thinking or stage of development. This represents a critical distinction between SOLO and the work of Piaget and others in that the focus with SOLO is on describing the structure of a response, not on some cognitive developmental stage construct of an individual. A strength of SOLO is that it provides a framework to enable a consistent interpretation of the structure and quality of responses from large numbers of students across a variety of learning environments in a number of subject and topic areas.

The 'local' framework suggested by SOLO comprises a recurring cycle of three levels. In this interpretation, the first level of the cycle is referred to as the unistructural level (U) of response and focuses on the problem or domain, but uses only one piece of relevant data. The multistructural level (M) of response is the second level and focuses on two or more pieces of data where these data are used without any relationships perceived between them; there is no integration among the different pieces of information. The third level, the relational level (R) of response, focuses on all the data available, with each piece woven into an overall mosaic of relationships to give the whole a coherent structure.

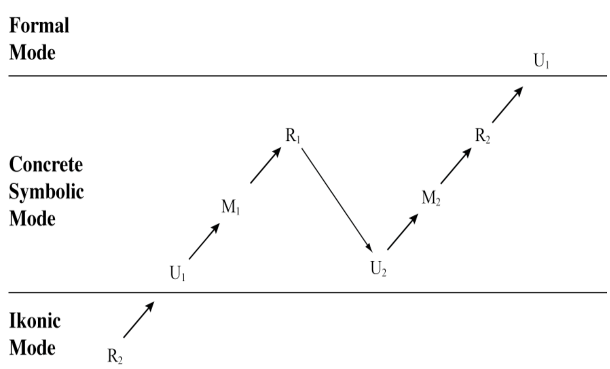
These three levels, *unistructural*, *multistructural*, and *relational*, when taken together, are referred to as a UMR learning cycle. They are framed within a wider context with a preceding *prestructural* level of response to a particular problem that does not reach even a unistructural level and an overall *extended abstract* level where the qualities of the relational level fit within a bigger picture that may become the basis of the next cycle of construction.

In the original description of the SOLO Taxonomy, Biggs and Collis (1982) noted that the UMR cycle may be seen to operate on different levels. For instance, they compared the cycle with the long-term global framework of Piagetian stage theory to suggest that "the levels of prestructural, unistructural, multistructural, relational, extended abstract are isomorphic to, but logically distinct from, the stages of sensori-motor, pre-operational, early concrete, middle concrete, concrete generalization, and formal operational, respectively" (ibid, p. 31). However, they theorized that it was of more practical value to consider the UMR sequence occurring in each of the successive SOLO modes, so that a UMR cycle in one mode could lead to an extended abstract foundation for the next mode (ibid, table 10.1, p.216). This provides a framework to assign responses to a combination of a given level in a given mode.

Subsequently, Pegg (1992) and Pegg and Davey (1998) revealed examples of at least two UMR cycles in the concrete symbolic mode, where the relational level response in one cycle evolves to a new unistructural level response in the next cycle within the same mode. This observation re-focuses the theory to smaller cycles of concept formation within different modes.

Using this finding, more sophisticated responses building on relational responses can become a new unistructural level representing a first level of a more sophisticated UMR cycle. This new cycle may occur as an additional cycle of growth within the same mode. Alternatively, it may represent a new cycle in a later acquired mode. These two options are illustrated in Figure 1.

Figure 1. Diagrammatic representation of levels associated with the concrete symbolic mode



To unpack this idea further we first need to consider what is meant by thinking within the ikonic mode and the concrete symbolic mode. The ikonic mode is concerned with 'symbolising' the world through oral language. It is associated with imaging of objects and the thinking in this mode can be described as intuitive or relying on perceptually-based judgements.

For the concrete symbolic mode the 'concrete' aspect relates to the need for performance in this mode to be rooted in real-world occurrences. The 'symbolic' aspect relates to where a person thinks through use and manipulation of symbol systems such as written language, number systems and written music notation. This mode can become available to students around about 5-to-6 years of age. The images and words that dominated thinking in the ikonic mode now evolve into concepts related to the real world. The symbols (representing objects or concepts) can be manipulated according to coherent rules without direct recourse to what they represent. Hence, immersion in this mode results in the ability to provide symbolic descriptions of the experienced world that are communicable and understandable by others.

As an example of figure 1 in action, let us focus on the development of number concepts. In the ikonic mode the child is developing verbally, giving names to things and talking about what (s)he sees. Numbers in this mode develop from the action-schema of counting, to the concept of number, independent of how the counting is carried out, to become *adjectives*, such as identifying a set of *three* elephants, and being able to combine this with another set comprising *two* elephants to get *five* elephants.

In the concrete symbolic mode, in the case of the concept of number, the status of numbers shifts from adjectives to *nouns*, i.e., a symbol in its own right that is available to be communicated to others, context free and generalisable. A unistructural level response in the first cycle concerns the ability to use one operation to answer simple written problems such as 2 + 3 without reference to context, by carrying out a suitable arithmetic procedure. A multistructural response would involve a couple of operations involving known numbers that can be carried out in sequence. The final level in the first cycle culminates in students being able to generate numerous responses to the question 'if 5 is the answer to an addition question what are possible questions?'

The second cycle in the concrete symbolic mode for number sees the numbers operated upon move beyond those with which the student has direct experience. At the unistructural level, single operations can be performed on larger numbers; many of the operations become automated, reducing demand on working memory. The multistructural level response concerns students being able to undertake a series of computations. Critical here is the need for the task to have a sequential basis.

Finally, the relational level in this second cycle concerns an overview of the number system. This is evident in students undertaking non-sequential arithmetic tasks successfully and being able to offer generalisations based on experienced arithmetic patterns. The issue here is that the response is tied to the real world and does not include considerations of alternative possibilities, conditions or limitations. In the SOLO model, these considerations only become apparent when the level of response enters the next mode of functioning referred to as the formal mode.

The value of acknowledging earlier UMR cycles enables a wider range of 'credit' to be given to responses of more complex questions. For instance, Biggs and Collis (1982) posed a question that required students to find the value of x in the equation:

$$(72 \div 36)9 = (72 \times 9) \div (x \times 9)\,.$$

Responses to this question which show some appreciation of arithmetic, without grasping the essential qualities of the problem itself can be classified into a first UMR cycle, and recorded as U1, M1 and R1 respectively, which simply involve:

U1: responding to a single feature, e.g., "has it got something to do with the 9s?";

M1: responding to more than one feature, e.g., "It's got 9s and 72s on both sides";

R1: giving an 'educated guess', e.g., "36 – because it needs 36 on both sides".

The second UMR cycle (recorded as U2, M2 and R2) involves engaging with one or more operations towards finding a solution:

U2: One calculation, e.g. '72 ÷ 36= 2';

M2: observing more than one operation, possibly performing them with errors;

R2: seeing patterns and simplifying, e.g., cancelling 9s on the right.

Further, responses that have evolved beyond the concrete symbolic mode and can be categorised as formal mode responses occur when the student has a clear overview of the problem based on the underlying arithmetic patterns, using simplifications, only resorting to arithmetic when it becomes necessary.

In a curriculum that focuses on making sense at one level and building on that sense-making to shift to a higher level, the acknowledgement of two or more cycles of response suggests more than a successive stratification of each mode into several cycles. It suggests the UMR cycle also operates in the construction of new concepts as the individual observes what is initially a new context with disparate aspects that are noted individually, then linked together, then seen as a new mental concept that can be used in more sophisticated thinking.

This view of cycles of cognitive development is consistent with the epistemological tradition of Piaget and with links with working memory capacity in cognitive science. It is also consistent with neuro-physiological evidence in which the biological brain builds connections between neurons. Such connections enables neuronal groups to operate in consort, forming a complex mental structure conceived as a single sophisticated entity that may in turn be an object of reflection to be operated on at a higher level (Crick, 1994; Edelman & Tononi, 2000).

Process-Object Encapsulation

A major instance of concept construction, which occurs throughout the development of arithmetic and the manipulation of symbols in algebra, trigonometry, and calculus, is the symbolizing of actions as 'do-able' procedures and to use the symbols to focus on them mentally as 'think-able' concepts. This involves a shift in focus from *actions* on already known objects to thinking of those actions as manipulable mental *objects*.

This cycle of mental construction has been variously described as: *action*, *process*, *object* (Dubinsky, 1991); *interiorization*, *condensation*, *reification* (Sfard, 1991); or *procedure*, *process*, *procept*—where a procept involves a symbol such as 3+2 which can operate dually as *pro*cess or *concept* (Gray & Tall, 1991, 1994). Each of these theories of 'process-object encapsulation' is founded essentially on Piaget's notion of 'reflective abstraction', in which actions on existing or known objects become interiorized as processes and are then encapsulated as mental objects of thought.

Over the years, successive researchers, such as Dienes (1960), Davis (1984), and Greeno (1983) theorized about the mechanism by which actions are transformed into mental objects. Dienes used a linguistic analogy, seeing the predicate in one sentence becoming the subject in another. Davis saw mathematical procedures growing from sequences of actions, termed 'visually moderated sequences' (VMS) in which each step prompted the next, until familiarity allowed it to be conceived as a total process, and thought of as a mental entity. Greeno used an information-processing approach focusing on the manner in which a procedure may become the input to another procedure, and hence be conceived as a 'conceptual entity'.

Dubinsky described the transformation of action to mental objects as part of his APOS theory (Action-Process-Object-Schema) in which actions are interiorised as processes, then thought of as objects within a wider schema (Dubinsky, 1991). He later asserted that objects could also be formed by encapsulation of schemas as well as encapsulation of processes (Czarnocha et al., 1999). Sfard (1991) proposed an 'operational' growth through a cycle she termed interiorization-condensation-reification, which produced reified objects whose structure gave a complementary 'structural growth' focusing on the properties of the objects.

There are differences in detail between the two theories of Dubinsky and Sfard. For instance, Sfard's first stage is referred to as an 'interiorized process', which is the same name given in Dubinsky's second stage. Nevertheless, the broad sweep of both theories is similar. They begin with actions on known objects (which may be physical or mental) which are practised to become routinized step-by-step procedures, seen as a whole as processes, then conceived as entities in themselves that can be operated on at a higher level to give a further cycle of construction.

This analysis can be applied, for example, to the increasing sophistication of an algebraic expression. An expression $x^2 - 3x$ may be viewed as a command to carry out a sequences of actions: start with some number x (say x = 4), square it to get x^2 (in the particular case, 16), now multiply 3 times x (12) and subtract it from x^2 to get the value of $x^2 - 3x$ (in this case, 16–12, which is 4). We can also think of the sequence of actions as a sequential procedure to take a particular value of x and compute $x^2 - 3x$. An alternative procedure that produces the same result is to calculate $x - 3$ and multiply this x times to give the result represented by the expression $x(x-3)$. Now we have two different step-by-step procedures that give the same output for given input. Are they ‘the same’ or are they ‘different’? As procedures, carried out in time, they are certainly different but in terms of the overall process, for a given input, they *always* give the same output. In this sense *they are ‘the same’.* It is this sameness that Gray & Tall (1994) call a ‘process’. We can write the process as a function $f(x) = x^2 - 3x$ or as $f(x) = x(x-3)$ and these are just different ways of specifying the same function.

In this case, we can say that the expressions $x^2 - 3x$ and $x(x-3)$ may be conceived at different levels: as procedures representing different sequences of evaluation, as processes giving rise to the same input-output, as expressions that may themselves be manipulated and seen to be ‘equivalent’, and as functions where they are fundamentally the same entity.

Gray and Tall (1994) focused on the increasing sophistication of the role of symbols, such as 3+4. For some younger children it is an instruction to carry out the operation of addition, more mature thinkers may see it as the concept of sum, giving 7. Others may see the symbol as an alternative to 4+3, 5+2, 1+6, all of which are different ways of seeing the same concept 7. Gray and Tall used this increasing compression of knowledge, from a procedure carried out in time, to a process giving a result, and on to different processes giving the same result to define the notion of *procept*. (Technically, an *elementary procept* has a single symbol, say 3+4, which can be seen dually as a *pro*cedure to be carried out or a con*cept* that is produced by it, and a *procept* consists of a collection of elementary procepts, such as 4+3, 5+2, 1+6, which give rise to the same output.)

Such cycles of construction occur again and again in the development of mathematical thinking, from the compression of the action-schema of counting into the concept of number, and on through arithmetic of addition of whole numbers, multiplication, powers, fractions, integers, decimals, through symbol manipulation in arithmetic, algebra, trigonometry, calculus and on to more advanced mathematical thinking. In each case there is a local cycle of concept formation to build the particular mathematical concepts. At one level actions are performed on one or more known objects, which Gray & Tall (2001) called the *base object(s)* of this cycle, with the operations themselves becoming the focus of attention as procedures, condensed into overall processes, and conceived as mental objects in themselves to become base objects in a further cycle.

Table 3 shows three theoretical frameworks for local cycles of construction (Davis, 1984; Dubinsky (Czarnocha et al, 1999); Gray & Tall, 1994, 2001) laid alongside the SOLO UMR sequence for assessing responses at successive levels.

Table 3: Local cycles of cognitive development

SOLO Model	Davis	APOS of Dubinsky	Gray & Tall
			[Base Objects]
Unistructural Multistructural	Procedure (VMS)	Action	Procedure
Relational	Integrated Process	Process	Process
Unistructural (in a new cycle)	Entity	Object	Procept
		Schema	

In each framework, it is possible to apply a SOLO analysis to the cycle as a whole. The initial action or procedure is at a unistructural level of operation, in which a single procedure is used for a specific problem. The multistructural level would suggest the possibility of alternative procedures without them being seen as interconnected, and hence remains at an action level in APOS theory; the relational level would suggest that different procedures with the same effect are now seen as essentially the same process. This leads to the encapsulation of process as object (a new unistructural level) and its use as an entity in a wider schema of knowledge.

If one so desired, a finer grain SOLO analysis could be applied to responses to given problems, for instance the initial action level may involve a number of steps and learners may be able to cope initially only with isolated steps, then with more than one step, then with the procedure as a whole. Once more this gives a preliminary cycle within the larger cycle and both have their importance. The first enables the learner to interpret symbols as procedures to be carried out in time, but the larger cycle enables the symbols themselves to become objects of thought that can be manipulated at increasingly sophisticated levels of thinking.

Similar Cycles in Different Modes

Now we move on to the idea that different modes are available to individuals as they grow more sophisticated, so that not only can students in, say, the concrete symbolic mode operate within this mode, they also have available knowledge structures in earlier modes, such as sensori-motor or ikonic. The question arises, therefore, how does knowledge in these earlier modes relate to the more sophisticated modes of operation. For example, in what way might the development of conceptions in the symbolic mode be supported by physical action and perception in more sophisticated aspects of the sensori-motor and ikonic modes of operation?

In the case of the concept of vector, Poynter (2004) began by considering the physical transformation of an object on a flat surface while encouraging students to switch their focus of attention from the specific actions they performed to the *effect* of those actions. The action could be quite complicated: push the object from position *A* to position *B* in one direction and then to position *C* in another direction. The action is quite different from the direct translation from position *A* to position *C*, however, the *effect* of both actions are the same: they all start at *A*, end at *C*, without being concerned about what happens in between. The perception of actions as being different may be considered a multistructural response, while the focus on the same effect shifts to a relational perspective.

The effect of the translation can be represented by an arrow from any start point on the object to the same point on the translated object; all such arrows have the same magnitude and direction. This can be represented as a *single* arrow that may be shifted around, as long as it maintains the same magnitude and direction. This moveable arrow gives a new embodiment of the effect of the translation as a *free vector*. It is now an entity that can be operated on at a higher level. The sum of two free vectors is simply the single free vector that has the same effect as the two combined, one after the other. The movable free vector is an enactive-ikonic entity that encapsulates the process of translation as a mental object that can itself be operated upon.

In this example, the shifting of the arrow is both a physical action (sensori-motor) and also an ikonic representation (as an arrow described as a free vector). Taking the hint from the view of the SOLO model, that each mode remains part of a later mode, Tall (2004) put together sensori-motor and ikonic aspects—or, in Brunerian terms, a combination of enactive and ikonic—into one single corporate mode of operation which he named 'conceptual-embodied' (to distinguish it from Lakoff's broader use of the term 'embodied') but shortened to 'embodied' mode where there was no possibility of confusion. Embodiment is a combination of action and perception and, over the years, it becomes more sophisticated through the use of language.

The embodied mode of operation is complemented by the use of symbols in arithmetic, algebra, trigonometry, calculus, and so on, which have a proceptual structure. Tall (2004) calls this mode of operation 'proceptual--symbolic' or 'symbolic' for short. Studying these complementary modes of operation, he found that they offer two quite different worlds of mathematics, one based on physical action and perception becoming more conceptual through reflection, the other becoming more sophisticated and powerful through the encapsulation of processes as mental objects that can be manipulated as symbols.

He saw a third 'formal-axiomatic' world of mental operations where the properties were described using set-theory and became part of a formal system of definitions and formal proof. Here whole schemas, such as the arithmetic of decimal numbers, or the manipulation of vectors in space, were generalised and encapsulated as single entities defined axiomatically as 'a complete ordered field' or 'a vector space over a field of scalars'.

This framework has a similar origin to that of the SOLO model, but is different in detail, for whereas SOLO looks at the processing of information in successive modes of development and analyses the observed structure of responses, the three worlds of mathematics offer a framework for cognitive development from the action and perception of the child through many mental constructions in embodiment and symbolism to the higher levels of formal axiomatic mathematics. Over the years, Tall and Gray and their doctoral students have mapped out some of the ways in which compression of knowledge from process to mental object occur in arithmetic, algebra, trigonometry, calculus, and on to formal mathematics, not only observing the overall process of compression in each context, but the way in which the different contexts bring different conceptual challenges that face the learner (Gray, Pitta, Pinto, & Tall, 1999; Tall, Gray, Ali, Crowley, DeMarois, McGowen, Pitta, Pinto, Thomas, & Yusof, 2000).

In the school context, just as the target SOLO mode is the concrete symbolic mode, with sensori-motor and ikonic support, this framework categorises modes of operation into just two complementary worlds of mathematics: the embodied and the symbolic.

The question arises: can this formulation offer ways of conceptualising parallel local cycles of construction in mathematics? The example of vector shows one case in which the embodied world enables a shift in focus of attention from action to effect to be embodied as a free vector. In parallel, the symbolic world allows translations represented by column matrices to be reconceptualized as vectors. Later, focus on the properties involved can lead to the selected properties for operations on vectors being used as a formal basis for the definition of a vector space.

This enables us to consider the action-effect-embodiment cycle in the embodied world to be mirrored by an action-process-procept cycle in the symbolic world. This link between compression from 'do-able' action to thinkable concept in the embodied and symbolic worlds arises naturally in other formations of symbolic concepts in mathematics.

In the case of fractions, for example, the action of dividing an object or a set of objects into an equal number of parts and selecting a certain number of them (for instance, take a quantity and divide into 6 equal parts and select three, or divide it into 4 equal parts and select two) can lead to different actions having the same effect. In this case three sixths and two fourths have the same effect in terms of quantity (though not, of course, in terms of the number of pieces produced). The subtle shift from the *action* of sharing to the *effect* of that sharing leads to the fractions $3/6$ and $2/4$ representing the same effect. This parallels the equivalence of fractions in the symbolic world and is an example of the concept of equivalence relation defined, initially in the form of manipulation of symbols in the symbolic world and later in terms of the set-theoretic definition of equivalence relation in the formal-axiomatic world of mathematical thinking.

In this way we see corresponding cycles giving increasingly sophisticated conceptions in successive modes of cognitive growth. Although there are individual differences in various theories of concept construction through reflective abstraction on actions, this fundamental cycle of concept construction from 'do-able' action to 'think-able' concept underlies them all.

Table 4: The fundamental cycle of conceptual construction from action to object

Constructing a Concept via Reflective Abstraction on Actions					
SOLO [Structure of Observed Learning Outcome]	Davis	APOS	Gray & Tall	Fundamental Cycle of Concept Construction	
Unistructural	Visually Moderated Sequence as Procedure	Action	Base Object(s)	Known objects	SCHEMA ↓
			Procedure [as Action on Base Object(s)]	*Procedure as Action* on Known Objects	
Multistructural			[Alternative Procedures]	[*Alternative Procedures*]	
Relational	Process	Process	Process	*Process* [*as Effect* of actions]	
Unistructural [new cycle]	Entity	Object	Procept	Entity as *Procept*	
		Schema			

SOLO and local cycles of development

Building on SOLO Taxonomy, Pegg and his colleagues have been involved in two large-scale longitudinal projects funded by the Australian Research Council with support from two education jurisdictions, the NSW Catholic Education Office and the NSW Department of Education. The projects while different, share the common aim of exploring the impact on, and implications for, immersing practising teachers in an environment where they were supported in learning about and applying the SOLO framework. The two research studies involved groups of teachers over two and three years, respectively. A significant theme within the research was helping teachers to unpack the assessment *for* learning agenda as a complement to the more traditional assessment *of* learning. In particular, there was a specific focus on local cycles of development in terms of unistructural, multistructural and relational responses (UMR).

In the first of these studies (Pegg & Panizzon, 2003; Pegg & Panizzon, 2007; Panizzon & Pegg, 2008) primary and secondary teachers were asked to explore the changing emphasis of assessment and how they reconceptualized these changes in practice by working with SOLO as the underpinning theoretical framework. Using a grounded theory approach, questioning in the classroom was identified as the core component with six contributing categories. These linked SOLO to identified changes in teachers' practice in:

- types and varieties of questions used;
- references to cognition in explaining the development of higher-order skills;
- framing teacher thoughts about their pedagogical practices;
- influencing techniques used in the classroom;
- identifying current student understanding so as to more explicitly drive the focus of lessons;
- developing positive changes in classroom interactions among students; and
- creating positive changes in classroom interactions between teachers and students.

The main finding of the study was that teachers reported a fundamental shift in their perception of learning and this was reflected in their teaching and assessment practices; their colleagues and students noticed and reported changes in their classroom practices and procedures. Understanding and applying the SOLO model was seen as both a catalyst for action and a framework to guide teacher's thinking.

The second study (Pegg, J., Baxter, D., Callingham, R., Panizzon, D., Bruniges, M. & Brock, P., 2004) provided evidence to school systems, subject departments and teachers as to how different forms of assessment and assessment information can improve the learning environment for students. Outcomes include details on how to utilise qualitative and quantitative assessment practices, and detailed longitudinal analyses of teacher growth and perceptions as a result of using the SOLO model within the social context of classrooms (Panizzon et al, 2007).

Emerging from this work and to be reported in (Pegg et al, under preparation) is the observation that while the lower levels of UMR can be taught in the traditional sense, the shift to a relational level response requires a quality in the thinking of the learner, and this cannot be guaranteed by teaching alone. There appear to be certain teaching approaches that might be better applied when students are identified as responding at one level than when at another. Knowledge of this pattern can better help teachers develop a rationale for their actions and help inform the nature of their instruction at that time.

Let us first consider the case of students who, during an activity, respond at the unistructural level. The implication here is that students provide a single relevant feature/aspect as an answer. In terms of cognitive capacity, the students' role is first to separate the cue (question) and the response. In doing this students need to hold the question in their mind while answering the question and then be able to relate the question and answer with one relevant aspect. The teaching implications for these students include numerous experiences (to practise) in coming to understand this single idea. As this approach proceeds, the single idea takes up less cognitive capacity and this allows the student to respond at the multistructural level.

With responses at the multistructural level, students must again separate the question (cue) and the response, but the cognitive capacity of the student now allows for additional aspects/concepts/features to be reported in a serial fashion. The key feature here is that the individual aspects are seen as independent of one another. Here further practise of the individual elements need to be pursued as well as activities that draw on the use of many elements. Formal language of the discipline has an important focus here as while the appropriate words were developed by practicing single focus questions, students are now better placed to begin to talk more openly about a variety of elements.

In both of the preceding cases, explicit teaching had an important place in the process in helping the student to identify the critical aspects of the work being undertaken and to reduce cognitive demand. Such teaching is able to encourage students to see the benefits of a multistructural response over a unistructural one in the improvement in consistency and in undertaking more advanced tasks. However, the key importance on the multistructural level is the accumulation of numbers of relevant elements by the student.

In facilitating a relational response the students are expected to interrelate the elements identified as isolated aspects at the multistructural level. The characteristics of a relational response include students seeing connections among the elements, and an overriding rule or pattern among the data that are identified. Of course students responding at this level are limited by inductive processes associated with moving from unistructural to multistructural and are not in a position to move beyond this context. This movement may occur as they access a new unistructural level in the next cycle.

For teachers who wish to move their students from the multistructural level to relational, the emphasis must move beyond a focus on explicit teaching to one of creating an environment in which students can find their own way, and develop their own connections. The result in teachers explicitly teaching connections at the relational level has two problems. First the number of connections (implicit and explicit) among the multistructural elements can be very large and hence it can become impossible to cover them all. Second, an emphasis on teaching the relationships among the elements can easily become a new multistructural element and hence not serve the integrative function a particular relationship among elements can achieve.

Developments in global and local theory

Tall (2008) has continued to reflect on both global and local issues of the development of mathematical thinking, seeing the whole long-term development from pre-school through primary and secondary school and on to tertiary education and beyond to mathematical research. This has involved attending more closely to the global framework of development which was formulated earlier in terms of three worlds of mathematics: the (conceptual) embodied, the (procedural-proceptual) symbolic and the (axiomatic) formal, henceforth shortened to embodied, symbolic and formal.

He saw this framework based on perception, action and reflection, where perception and action give two differing ways of making sense of the world and reflection enables increasing sophistication of thought powered by language and symbolism.

He realised, to his astonishment, that just three underlying abilities set before our birth in our genes are the basis for the human activity of mathematical thinking. He called these 'set-befores'. They are *recognition* (the cluster of abilities to recognise similarities, differences, and patterns), *repetition* (the ability to learn to perform a sequence of actions automatically) and *language* (which distinguishes homo sapiens from all other species in being able to *name* phenomena and talk about them to refine meaning).

Perception (supported by action) and language enable us to *categorise* concepts. Actions allow us to *perform* procedures and, using symbolism, language enables us to *encapsulate* procedures as *procepts* that operate dually as processes to perform and concepts to think about. Language also allows us to *define* concepts, related both to concepts perceived and actions performed, leading to a more cerebral sphere of set-theoretic definition and formal proof that gives a new world of axiomatic formal mathematical thinking.

This reveals the global theory of three worlds of mathematics each having local ways of forming mathematical concepts: categorization, encapsulation and definition-deduction. Each world of mathematics uses all of these but has a preference for one of them: categorisation in the embodied world, encapsulation (and categorisation) in the symbolic world and set-theoretic definition in the axiomatic formal world.

Local cycles enable the thinker to *compress* information into *thinkable concepts* specified by words and symbols, and linked together into *knowledge structures*. Not only that, a thinkable concept is in detail a knowledge structure (called the *concept image*) and if a knowledge structure is coherent enough to be conceived as a whole, it can be named and become a thinkable concept. This compression takes us one step beyond the UMR cycle to the next level where the relational structure is named and compressed into a thinkable concept operating at a higher level.

The UMR cycle in categorization involves the individual responding at the initial stage in terms of single pieces of information, then handling multiple pieces, then combining them in a relational manner. It is only when these relational properties are seen to refer to a single overall concept that it can become the unistructural concept at the next level.

With encapsulation of procedures to processes to objects, we have a second type of UMR cycle: a single procedure, several different procedures to achieve the same result, seen as *equivalent* procedures at the relational level before compression into a *procept* which acts as the unistructural concept at the next level.

However, as has been suggested earlier, the UMR cycles in embodiment and symbolism may happen in subtly different ways. It may be possible to perform actions to see the *effect* of those actions in a way that embodies the desired object at the next level, which may then shift the use of symbolism to perform operations that give accurate calculations and precise symbolic representations. For instance, the calculus benefits from an embodied approach in terms of the 'local straightness' of graphs that look essentially straight under high magnification, to see their changing slope. The graph of this 'slope function' may then be translated to a symbolic approach using arithmetic approximations that give a 'good enough' numerical approximation at any given point and algebra to give a precise symbolic formula for the whole global derivative. At the formal level, the set-theoretic definition of limit can be introduced to give a formal axiomatic approach to mathematical analysis.

The global framework also formulates the way in which individuals build knowledge structures on basic set-befores that we all share and personal met-befores that consist of structures we have in our brains *now* as a result of experiences we have met before.

There is more to learning than simply putting elements together in a relational way. The learner must make sense of the world through forming knowledge structures that are built on met-befores. Many met-befores are *supportive*. For instance two and two makes four in whole number terms and it continues to make four whether we are speaking of whole numbers, fractions, real numbers, complex numbers or cardinal numbers. Other met-befores that are quite satisfactory in the given context become *problematic* in a later development. For instance, addition of whole numbers makes bigger, take-away makes smaller, but neither are true in the arithmetic of signed numbers or the arithmetic of cardinal numbers.

Learners who face situations that are too complicated for them to make sense with their current knowledge structures, or where confusion is caused by problematic met-befores, are likely to feel anxious and may resort to the solace of rote-learning to have a facility for repeating procedures but without the compression of knowledge that gives long-term development of flexible mathematical thinking.

Indeed flexible mathematical thinking blends together different knowledge structures, for instance, the embodied number line drawn on paper by a stroke of a pen, the symbolic number system of (infinite) decimals for powerful calculation, and the formal structure of a complete ordered field for logical coherence.

Blends give mathematics its power. The number system we use is a blend of discrete counting which has properties where every counting number has a next with none in between and continuous measurement in which any interval can be subdivided as often as desired. While the blend works well in elementary mathematics, a schism appears in infinite mathematics where counting leads to infinite cardinals that can be added and multiplied but not subtracted or divided and measuring leads to non-standard analysis where infinite elements have inverses that are infinitesimal. (Tall, 2002).

The theoretical framework of ‘three worlds of mathematics’ provides a global framework for mathematical thinking with only two categories in early mathematics broadening to three later on. It has local frameworks of compression of knowledge through categorization, encapsulation and definition that take into account the met-befores that are problematic in learning in addition to successive UMR style compressions.

Successive levels of sophistication are addressed with the construction and blending of knowledge structures and their compression into thinkable concepts at higher levels continuing right through to the subtle knowledge structures used in research mathematics. This framework places local UMR cycles of construction within a global framework that allows embodied meaning (such as the changing slope of a graph) to be translated into a symbolic meaning (the function that specifies the changing slope function). Here it is possible for embodiment of the slope function to enable the learner to ‘see’ a higher-level concept before being able to pass through the cycles of symbolic development required to calculate them.

Discussion

This paper has considered several different theoretical frameworks at both a global and local level, with particular reference to the underlying local cycle of conceptual development from actions in time to concepts that can be manipulated as mental entities. This cycle occurs not only in different mathematical concepts, but in different modes of operation in long-term cognitive growth. In the development of symbolic arithmetic and algebra, the heart of the process is the switching focus of attention from the specific sequence of steps of an action to the corresponding symbolism that not only evokes the process to be carried out but also represents the concept that is constructed.

The compression of knowledge to thinkable concepts occurs in different ways, including constructions from *perceptions of* objects, *actions on* objects and *properties of* objects. The first construction leads to a van Hiele type development in which objects are recognized, and various properties discerned and described. This knowing is then used to formulate definitions that are in turn used in Euclidean proof. The second construction uses symbols to represent the actions that become mental objects that can be manipulated at successively sophisticated levels. The third construction leads to the creation of axiomatic structures through formal definition and proof, in which a whole schema, such as the arithmetic of decimals can be reconstructed as a mental object, in this case, a complete ordered field.

In this paper we have focused more specifically on the second case in which concepts are constructed by compressing action-schemas into manipulable concepts by using symbols. This is the major cycle of concept construction in arithmetic, algebra, symbolic calculus, and other contexts where procedures are symbolised and the symbols themselves become objects of thought. It includes the action-schema of counting and the concept of number, the operation of sharing and the concept of fraction, general arithmetic operations as templates for manipulable algebraic expressions, ratios in trigonometry that become trigonometric functions, rates of change that become derivatives, and so on.

In all of these topics there is an underlying local cycle of concept construction from action-schema to mental object. All these operations can be carried out as embodied activities, either as physical operations or thought experiments, and may then be symbolised to give greater flexibility of calculation and manipulation. The local cycle of construction in the embodied world occurs through a shift of attention from the doing of the action to an embodiment of the *effect* of the action. This supports the parallel symbolic activity in which an action is symbolized as a procedure to be carried out, and then the symbols take on a new meaning as mental objects that can be manipulated in higher-level calculations and symbolic manipulations.

In addition, all of these topics share an underlying local cycles of construction that begin with a situation that presents complications to the learner, who may focus at first on single aspects, but then sees other aspects and makes links between them to build not just a more complex conception, but also a richer compressed conception that can be operated as a single entity at a higher level. Such a development is described in the SOLO model to analyse the observed learning outcomes, but also features as a local cycle of learning in a wide range of other local theoretical frameworks.

In the case of compression of knowledge from doing mathematics by performing actions, to symbolising those actions as thinkable concepts, all these theoretical frameworks share the same underlying local cycle of learning. Significantly, they all be categorised so that the learning outcomes can be analysed in terms of the SOLO UMR cycle. More than this, the global theory of three worlds of mathematics fits with development of the SOLO modes of operation. The SOLO sensori-motor and ikonic modes together are the basis for conceptual embodiment, the concrete symbolic mode relates to the procedural-proceptual symbolic world, the higher levels of formal and post-formal can also be seen to relate to the later development of formal axiomatic mathematics and later to mathematical research. A fitting point to end our discussion in tribute to the life and work of Kevin Collis.

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