THE LONG-TERM COGNITIVE DEVELOPMENT OF DIFFERENT TYPES OF REASONING AND PROOF

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This paper uses the framework of 'three worlds of mathematics' (Tall 2004a, 2004b) to chart the development of mathematical thinking from the thought processes of early childhood to the formal structures of set-theoretic definition and formal proof. It sees the development of mathematical thinking building on experiences that the individual has met before, as the child coordinates perceptions and actions to construct thinkable concepts in two different ways. One focuses on objects, exploring their properties, describing them, using carefully worded descriptions as definitions, inferring that certain properties imply others and on to coherent frameworks such as Euclidean geometry through a developing mental world of ***conceptual embodiment****. The other focuses on actions (such as counting), first as procedures and then compressed into thinkable concepts (such as number) using symbols such as 3+2, ¾, 3a+2b, f (x), dy/dx; these operate dually as computable processes and thinkable concepts, termed* ***procepts****, in a developing mental world of* ***proceptual symbolism****. These may lead later to a third mental world of* ***axiomatic formalism*** *based on set-theoretic definition and formal proof. In addition to charting the development of proof concepts through these three worlds, we use the theory of Toulmin to analyse the processes of reasoning by which proofs are constructed.*

INTRODUCTION

In recent years, a framework of cognitive development from child to mathematician has been developed in the Mathematics Education Research Centre at the University of Warwick, based on the work of Eddie Gray, David Tall, and their research students (Tall, 2006). A paper indicative of the collaborative nature of this effort is presented by Tall, Gray, Bin Ali, Crowley, DeMarois, McGowen, Pitta, Pinto, Thomas, and Yusof (2001) under the title *Symbols and the bifurcation between procedural and conceptual thinking*; the authors address the broader question of why some students succeed in mathematics, yet others fail, based on research studies carried out for doctoral dissertations in mathematics education at the University of Warwick. These papers may be found via the website davidtall.com.

In this presentation we focus specifically on the transition from school mathematics to the formal theory of mathematics as published in journals, crucially taking into account the concepts that undergraduate students have met before their introduction to the mathematics as it is practiced by mathematicians. Technically, a *met-before* is part of the individual's concept image in the form of a mental construct that an individual uses at a given time based on experiences they have met before. Human beings bring their previous experiences to bear on new situations that they meet. As they grow more sophisticated, this prior knowledge is compressed into *thinkable concepts* that, connected together in *knowledge schemas,* frame the way in which

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individuals think. In particular, proof develops initially through practical experiment and then through thought experiment drawing implications from given starting points, through symbolic manipulation of arithmetic and algebraic formulae, and only at a later stage through set-theoretic definition and formal proof.

THEORETICAL FRAMEWORK

The child is born with a genetic structure set-before birth in the genes, but the generic facilities of perception and action need to be coordinated and refined into coherent perceptions of the world and integrated action schemas such as see-grasp-suck. Mathematical procedures are extensions of these natural propensities that may be learnt in a basic procedural sense but are usually better appreciated within a more coherent meaningful framework of connections.

In the final chapter of *Advanced Mathematical Thinking,* Tall (1991) reflected on the nature of mathematical proof and theorized that there were two different sources of meaning prior to the introduction of formal definition and proof. One focused on objects and their properties, classified into categories and leading to a van Hiele type development of increasing sophistication, building from primitive perception, to more refined conceptions, descriptions, then definitions used for making inferences, building a coherent deductive framework characteristic of Euclidean Geometry. In the initial stages of perception and description, properties occur at the same time, a triangle with three equal sides also has three equal angles. Proof begins to arise in this development at the level of definition and deduction where an equilateral triangle defined as having three equal sides, as a consequence, also has three equal angles.

The other source of meaning builds through the compression of a repeatable action as an overall process that can be performed without effort, which enables students to learn procedures to perform routine mathematical algorithms. Some students develop a flexible use of symbolism that can operate both as processes to *do* mathematics and concepts to *think about* it. Gray & Tall (1991) introduced the term *procept* to refer to the dual use of symbolism as process and concept in which a process (such as counting) is compressed into a concept (such as number), and symbols such as 3+2, ¾, *3a+2b, f(x), dy/dx* operate dually as computable processes and thinkable concept. Here proof develops through generalised arithmetic and algebraic manipulation.

In subsequent years, this framework has been developed into what Tall (2006) described as three mental worlds of mathematics:

- the **conceptual-embodied** (based on perception of and reflection on properties of objects);
- the **proceptual-symbolic** that grows out of the embodied world through actions (such as counting) and symbolization into thinkable concepts such as number, developing symbols that function both as processes to *do* and concepts to *think about* (called procepts);

- the **axiomatic-formal** (based on formal definitions and proof) which reverses the sequence of construction of meaning from definitions based on known concepts to formal concepts based on set-theoretic definitions.

The term ‘world of mathematics’ is used here with special meaning. It has often been suggested to us that these should be simply considered as different ‘modes of thinking’, in particular these ideas may easily be reformulated in what the French school refer to as different ‘registers’, such as verbal, spoken, written, graphic, symbolic, formal, etc. (Duval, 2006), or as different representations in American College Calculus such as verbal, numeric, algebraic, graphic, analytic.

The choice of the word ‘world’ is used here deliberately to represent not a single register or group of registers, but the *development* of distinct ways of thinking that grow more sophisticated as individuals develop new conceptions and compress them into more subtle thinkable concepts. The focus on long-term development involves making new links and suppressing earlier aspects which are no longer relevant to develop an increasingly sophisticated world of mental thought, rather than a cross-sectional study of the use of different registers or representations to focus on different aspects of a particular problem situation.

The conceptual embodied world includes not only perceptions of physical objects, but also (later on) visuo-spatial reasoning using internal conceptions built from external perceptions. It grows from the immediate perception and action of the young child to the focus of attention on aspects such as the idea that a point has location but not size, that a line has no thickness and can be extended as far as desired. In this way the focus of attention moves from the specifics of human perception to the subtle essence of underlying regularities that grow into Platonic conceptions that some experts may see as a separate and ideal world. Others, however, see this greater level of sophistication as a natural product of human mental construction focusing on essentials and suppressing detail that is no longer central to the growing sophisticated thought processes.

The symbolic world grows in quite a different way, encapsulating counting as number, addition as sum, repeated addition as product, sharing as fractions, generalised arithmetic processes as algebraic expressions, infinite approximating sequences as limit. This development is described with its growth and discontinuities in Tall et al. (2001). It relates to the process-object compression that Dubinsky (1991) calls ‘encapsulation’ following Piaget, and Sfard (1991) terms ‘reification’ within her framework in which operational mathematics is recast in a structural form.

The axiomatic formal world develops from the properties arising in embodiment and symbolism, now formulated in terms of set theoretic definitions of mathematical structures with all other properties derived using mathematical proof (Tall, 1991; Tall, 2002).

There is a concern that each of the terms used here is employed with different meanings in the literature. For instance, Lakoff (1987) says that *all* thought is 'embodied', Peirce (1932) and Saussure (1916) use the term 'symbolic' in a wider sense than this, Hilbert (1900/2000) and Piaget (Piaget & Inhelder, 1958) use the term 'formal' in different ways—Hilbert in terms of formal mathematical theory, Piaget in terms of the 'formal' operational stage when teenagers begin to think in logical ways about situations that are not physically present.

It is for this reason that the two-word names are introduced as 'conceptual-embodied' referring to the embodiment of abstract concepts as familiar images (as in 'Mother Theresa is the embodiment of Christian charity'), 'proceptual-symbolic' referring to the particular symbols that are dually processes (such as counting, or evaluation) and concepts (such as number and algebraic expression), 'axiomatic-formal' to refer to Hilbert's notion of formal axiomatic systems. However (and this is a simple but important compression of knowledge), when these terms are used in a context where their meaning is clear, they will be shortened to *embodied*, *symbolic* and *formal*. This will allow the worlds to operate in tandem, such as the embodied-symbolic combination which can operate in both directions, for instance, representing algebraic equations as graphs or projective geometry as homogeneous coordinates. Later the embodied and symbolic worlds may underpin formal thinking as embodied formalism or symbolic formalism or even an integrated combination of all three.

Although there is a hierarchy in the order in which these worlds begin to develop, each new world develops concurrently with older worlds (see Figure 1). As school students enter the symbolic world, their ways of thinking in the embodied world continue to develop, just as mathematics university students continue to operate in the embodied and the symbolic worlds as they begin to develop more formal ways of

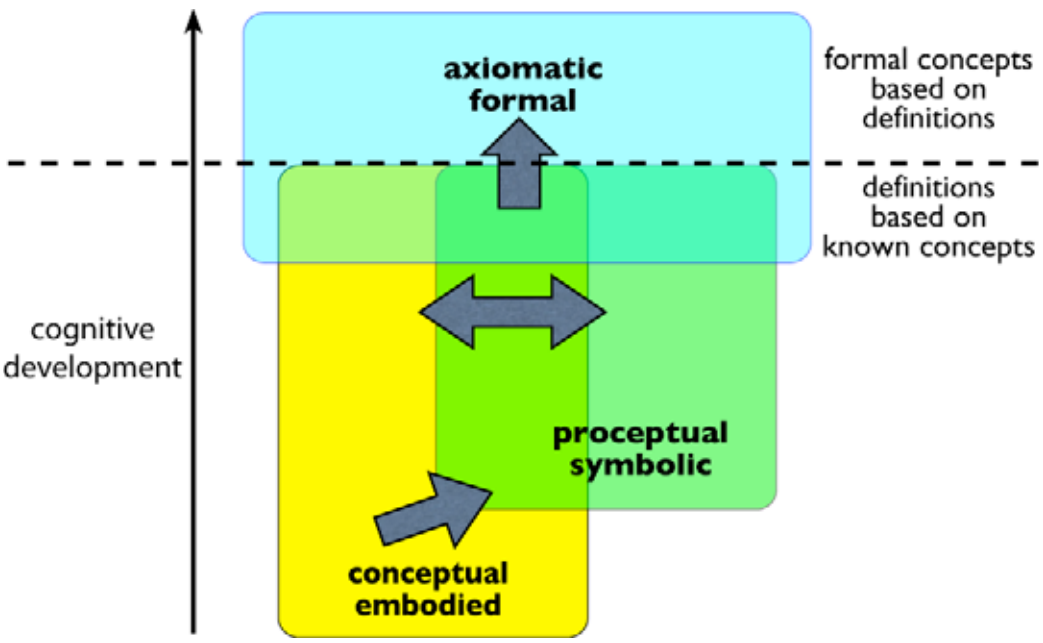


Figure 1: The cognitive growth of three mental worlds of mathematics

thinking. Similarly, professional mathematicians have a variety of working methods; some performing embodied thought experiments to suggest theorems which may then be published in purely formal terms, others basing their mathematical proofs explicitly on powerful computations and symbol manipulations.

DIFFERENT TYPES OF REASONING AND PROOF

Each world carries with it aspects that are more than simply ways of thinking, they also involve ways of perception, action and reflection and the emotions and meanings that accompany that thinking. Tall (2004a,b) suggests that each world of mathematics carries with it different kinds of warrants for truth that grow in sophistication as the individual matures.

For instance:

- In the embodied world, the individual begins with physical experiments to find how things fit together, for example, squares fit together to form a pattern that covers a flat table, so that four corners make a complete turn, and two corners make a straight line. Later verbal descriptions become definitions and are used in Euclidean geometry both to support the visual constructions with verbal proofs and to build a global theory from definitions and proof.
- In the (proceptual) symbolic world, arguments begin with specific numerical calculations and develop into the proof of algebraic identities such as $(a-b)(a+b)=a^2-b^2$ by symbolic manipulation.
- In the formal world, the desired form of proof is by formal deduction, such as the intermediate value theorem proved by using the completeness axiom.

In this way we see that the categorization into three worlds *each of which develops in sophistication* is not simply a question of three different modes of thinking, but of different strands of long-term development that complement and extend each other.

DEGREES OF CONFIDENCE IN PROOF

In addition to these different kinds of justification, there is also considerable variation in the level of confidence that students and mathematicians have in the conclusion of a given mathematical argument. Proof in mathematics requires that each statement must be true or false with no middle ground. But this is only the tip of the iceberg: as a proof is constructed, arguments may be used at various times with varying levels of confidence. Toulmin (1958) put forward a perspective of argumentation that takes into account the

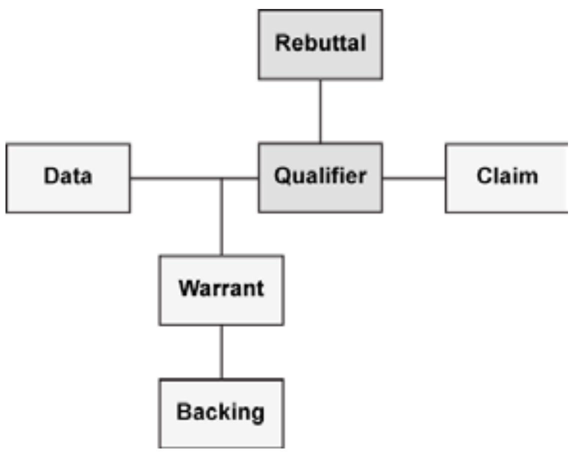


Figure 2: Toulmin's layout

kind of arguments that may be used in proof building, and he introduced a layout for modelling a general argument that differentiates six main types of statements. Starting from a *claim* that one wishes to support with given *data*, some kind of reason is produced to link the data and the claim. This linking statement is called the *warrant* of the argument, which may be supported by some kind of *backing*. Most importantly, a *qualifier* may be used to express the strength with which the claim may be taken, and a *rebuttal* may be used to state the possible limitations in the scope of the argument (Figure 2).

Although Toulmin (1958) did not address the modelling of mathematical argumentation and proof, in a later work Toulmin, Rieke and Janik (1979) suggested that this layout could indeed be useful to model the procedure of proving in mathematics, and illustrated this in the context of Euclidean geometry (p.89). Furthermore, Toulmin's layout has been used in mathematics education to analyse the collective argumentation of students and teachers in the mathematics classroom (Krummheuer, 1995; Forman, Larreameny-Joerns, Stein & Brown, 1998; Yackel, 2001; Stephan & Rasmussen, 2002; Rasmussen, Stephan & Allen, 2004; Knipping, 2003), students' written and verbalised arguments in task-based interviews (Hoyles & Küchemann, 2002; Evens & Houssart, 2004; Weber & Alcock, 2005; Alcock & Weber, 2005; Pedemonte, 2007; Inglis, Mejia-Ramos & Simpson, 2007), and by philosophers of mathematics to analyse mathematical proofs (Alcolea Banegas, 1998; Aberdein, 2005, 2006a, 2006b).

The following example from Inglis & Mejia-Ramos (2008), illustrates how a student uses a non-absolutely-qualified embodied argument to gain insight into a possible proof. Linvoy is a 2nd year maths undergraduate in a top ranked U.K. university. In an interview, he was asked to work on the following task (based on a problem by Raman, 2002):

> Determine whether the next statement is true or false (explain your answer by proving or disproving the statement): *The derivative of a differentiable even function is odd.*

After working unsuccessfully for a couple of minutes with the definitions of even/odd function and that of the derivative of a function, Linvoy said:

> "Perhaps if I think of it in a bit of a less formal way, if I just think of it as the derivative of a function being the gradient at a particular point… and… um… [draws the graph of an even sinusoidal function] I think of some graph like this which happens to be [inaudible] because it's an even function, and then… yes, I suppose one way of looking at this is that at any point here, like say you take this point [picks a point of the function in the first quadrant], you've got this gradient going like that, if you compare the exact other part, you've got the gradient going in the opposite direction because it's exactly, ummm, it's like a mirror image, so... and that is, that is odd, because that gradient would be exactly the negative of that gradient.
> So, yeah, I suppose, just from that basic example I suppose that intuitively does, does seem like it would make sense, but what about… maybe it's just the example of the function I've chosen, but that can't be right, because, what I'm thinking is… if you take, I mean, any [draws another set of axis]… this can do whatever it likes, but say we're interested at some

point where it's doing that [draws a small portion of the graph of a generic function in the first quadrant], then it's going to have that gradient and then if we transfer it it's going to be like that, so it's going to have that gradient, which would be the exact opposite of that… yeah, thinking of it like this, it does seem true, just thinking of it in those terms, ummm… like before I'd be happier if I could think of some way to prove it…"

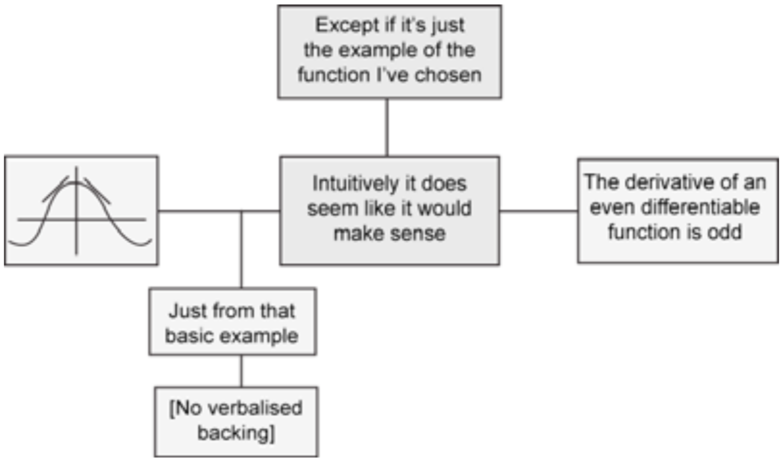


Figure 3: Linvoy's response

Linvoy uses particular and generic examples as warrants to reach conclusions paired with non-absolute qualifiers such as "[it] does seem like it would make sense", and "it does seem true" (Figure 3). This kind of argumentation proves to be common not only in the work of undergraduate students, but also successful mathematicians (Inglis, Mejia-Ramos & Simpson, 2007). This suggests that in considering proof in mathematics we need to take into account not only the final form of proof, but the nature of the argumentation that leads to the proof which may carry with it different types of warrant and degrees of confidence. During the *building* of a proof, and even at the stage of presenting a proof, warrants need not be absolute, but may be accompanied by qualifiers which may be different for different individuals depending on their experience.

NATURAL AND FORMAL THINKING

All individuals build on their met-befores. Pinto and Tall (1999, 2002) expressed this succinctly by distinguishing between *formal thinking* that builds on set-theoretic definitions to construct formal proofs and *natural thinking* that uses thought experiments based on embodiment and symbolism to give meaning to the definition and suggests possible theorems to translate into formal proof.

The met-befores evoked in the building of proof include not only conceptual embodiments, as in Linvoy's case, but also proceptual symbolic calculations, for instance, in group theory developing from permutations, in vector space theory handling matrices, or in analysis performing calculations in specific cases to provide a warrant for the truth of a possibly more general statement.

A *natural* approach can be based on embodiment, symbolism or a combination of both and may continue to link to embodied mental imagery while translating the imagery into a written proof. A *formal* approach, on the other hand focuses on the statement of the theorem and the necessary logical steps to reach the desired conclusion. These distinctions may be seen in the work of famous mathematicians, with Polya, Poincaré, Einstein and Atiyah talking about natural thinking in terms of examples and visualisations while Weierstrass, Dieudonné and MacLane speak of formal thinking based explicitly on well-formulated definitions.

In the undergraduate classroom, Weber (2004) added to this framework a *procedural* approach that simply involves learning the proof by rote. This fits into our framework with a procedural approach corresponding to a more primitive action-schema form of learning while natural and formal thinkers attempting to build up knowledge schemas based on concept image and/or concept definition.

FROM FORMAL PROOF BACK TO EMBODIMENT AND SYMBOLISM

A major goal in building axiomatic theories is to build a *structure theorem*, which essentially reveals aspects of the mathematical structure in embodied and symbolic ways. Typical examples of such structure theorems are:

- An equivalence relation on a set *A* corresponds to a partition of *A*;
- A finite dimensional vector space over a field *F* is isomorphic to F^n;
- Every finite group is isomorphic to a group of permutations;
- Any complete ordered field is isomorphic to the real numbers.

In every case, the structure theorem tells us that the formally defined axiomatic structure can be conceived an embodied way and in many cases there is a corresponding manipulable symbolism. For instance, an equivalence relation on a set *A*—axiomatized as reflexive, symmetric and transitive—corresponds to an embodiment that partitions the set. Any (finite dimensional) vector space is essentially a space of *n*-tuples that can (in dimensions 2 and 3) be given an embodiment and (in all dimensions) can be handled using manipulable symbolism. Any group can be manipulated symbolically as permutations and embodied as a group of permutations on a set. A complete ordered field specified as a formal axiomatic system corresponds precisely to the symbolic system of infinite decimals and to the embodied visualisation of the number line.

Thus, not only do embodiment and symbolism act as a foundation for ideas that are formalized in the formal-axiomatic world, structure theorems can also lead back from the formal world to the worlds of embodiment and symbolism (see Figure 4). These new embodiments are fundamentally different with their structure built using concept definitions and formal deduction. Furthermore, the structure theorems have a life of their own which may go beyond and extend human imagination, as for instance with vector space theory where two dimensional space can be embodied in a plane and three-dimensional space in the human world we live in, yet higher dimensions require conceptual embodiments that are only obtained by deep introspection, as in the case of Zeeman (1960) visualising how to unknot spheres in five dimensions.

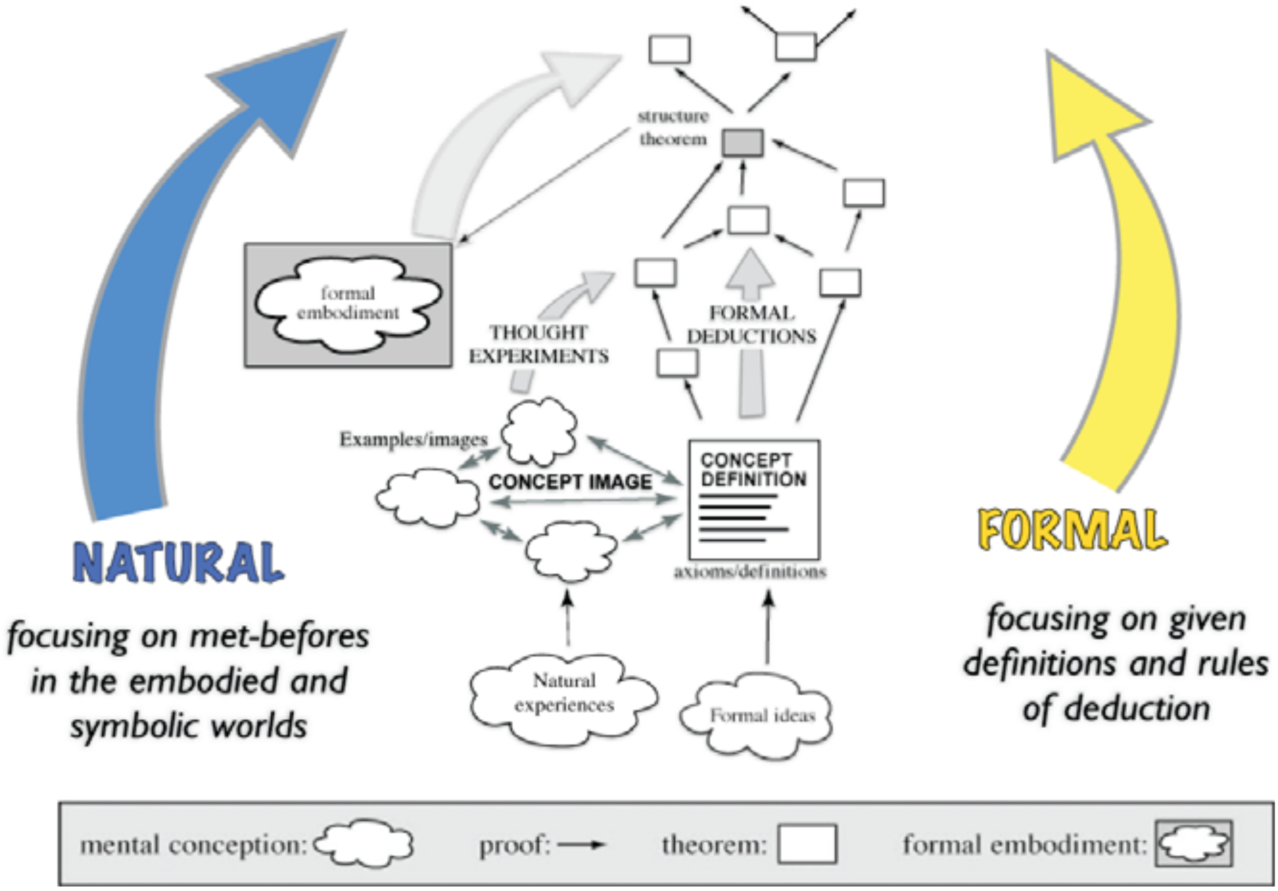


Figure 4: From embodiment and symbolism to formalism and back again (Tall, 2002)

New embodiment and symbolism may be a springboard for imagining new developments and new theorems; it may not. For instance, the embodied interpretation that a complete ordered field is the real line gave generations of mathematicians the belief that including the irrationals completed the real line geometrically by ‘filling in all the gaps between rationals’. This is not true, for it is possible to imagine (as did earlier generations) that the embodied number line has yet more elements that are infinitesimally close, but not equal, to real numbers (Tall, 2005). Thus the embodiment of structure theorems proved formally still need to be considered as warrants for truth that may *suggest* possible new theorems that may in fact be flawed.

Even well-accepted theorems may later prove to have ‘gaps’ in their proof that are not justified by their assumptions that may be based not on logic, but on embodied conceptions of the mathematics. For instance, after 2000 years of belief in the logic of Euclidean proof, Hilbert found a subtle flaw in the proof that the diagonals of a rhombus meet inside the figure at right angles. The Euclidean theory had not defined the notion of ‘inside’ and so new axioms were added to specify when a point C on a line AB was ‘between’ A and B.

STUDENTS AND EMBODIMENT IN PROOF

The role of embodiment proves (☺) to be a two-edged sword in the learning of students for it can mislead as well as inspire. For instance, Chin (2002) found that students learning about equivalence relations may embody not the whole definition, but subtly embody *individual axioms*. Thus the transitive axiom

if $a \sim b$ and $b \sim c$ then $a \sim c$ for all a, b, c

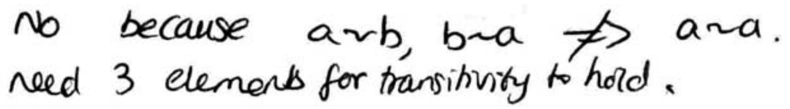
may be interpreted like the transitive law in a strong order relation, so that a, b, c are seen to be *different.*

In his famous lecture given at the turn of the twentieth century, Hilbert (1900/2000) referred to embodiment of the transitive law in the following terms:

> To new concepts correspond, necessarily, new signs. These we choose in such a way that they remind us of the phenomena which were the occasion for the formation of the new concepts. So the geometrical figures are signs or mnemonic symbols of space intuition and are used as such by all mathematicians. Who does not always use along with the double inequality $a > b > c$ the picture of three points following one another on a straight line as the geometrical picture of the idea "between"? (p. 410)

Even Hilbert, the architect of the formalist viewpoint, took inspiration from embodiment.

This may be one explanation of the following statement where a student was unable to deduce that if $a \sim b$ and $b \sim a$ then $a \sim a$:



An alternative explanation put forward by Asghari (2005) noted that the Greek notion of equivalence (in terms of lines being parallel or triangles being congruent) was always conceived in terms of a relation between two *different* things. According to this explanation, an element cannot be equivalent to itself, just as a line fails to be parallel to itself, for it meets itself and two parallel lines do not meet.

Thus it is always necessary to look at the interpretations that individuals place on concepts to find the more subtle sources of their beliefs. As their cognitive structure is built genetically on structures set-before birth and experiences met-before throughout their lives, previous conceptual embodiment and proceptual symbolism will colour their thinking in subtle ways.

CONCLUSION

Our analysis of how the mathematical thinking is built up by individuals over their life-time from child to mathematician reveals a combination of various kinds of conceptual embodiment and proceptual symbolism leading on to axiomatic formal proof and how concepts that have been met-before affect new thinking. Proof as practiced by mathematicians builds on the experiences that they have integrated into their thinking. Even though proof as an ideal may be considered to be absolute, proof as practiced by human beings, even mathematicians, is a human construct with human strengths of insight and human weaknesses of construction. In practice, it is not ‘all or nothing’, but is based on implicit or explicit ‘warrants for truth’ that carry with them a measure of uncertainty that varies between individuals and between the ways in which their proofs are framed.

In this paper we have put forward a framework based on conceptual embodiment leading to proceptual symbolism, combining to underpin the axiomatic-formal world of mathematical proof. We have given examples of how mathematicians and students think about proof and how not only does embodiment and symbolism lead into formal proof, but how structure theorems return us to more powerful forms of embodiment and symbolism that can support the quest for further development of ideas. We have also cautioned how proofs presented by students (and also mathematicians) can contain subtle meanings that are at variance with the formalism. Mathematical proof may indeed be the summit of mathematical thinking but it is just the top of one mountain and requires human ingenuity, with all its strengths and flaws, to attempt to reach for the peak of ultimate perfection.

REFERENCES

Aberdein, A. (2005). The uses of argument in mathematics. *Argumentation, 19,* 287–301.

Aberdein, A. (2006a). The informal logic of mathematical proof. In R. Hersh (Ed.), *Eighteen Unconventional Essays on the Nature of Mathematics* (pp. 56–70). New York: Springer.

Aberdein, A. (2006b). Managing informal mathematical knowledge: Techniques from informal logic. In J. M. Borwein & W. M. Farmer (Eds.), *Lecture notes in computer science: Vol. 4108, Mathematical Knowledge Management* (pp. 208–221). New York: Springer.

Alcock, L., & Weber, K. (2005). Proof validation in real analysis: Inferring and checking warrants. *Journal of Mathematical Behavior, 24,* 125–134.

Alcolea Banegas, J. (1998). L’Argumentació en matemàtiques. In E. C. Moya (Ed.), *XII Congrés Valencià de Filosofia* (pp. 135–147). Valencia: Diputació de Valencià.

Asghari, A. (2005). *Equivalence*. PhD thesis, University of Warwick.

Chin, Ehr-Tsung, (2002). *Building and using concepts of equivalence class and partition*. PhD, University of Warwick.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. O. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 95–123). Dordrecht: Kluwer.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics, 61,* 103–131.

Evens, H., & Houssart, J. (2004). Categorizing pupils’ written answers to a mathematics test question: ‘I know but I can’t explain’. *Educational Research, 46,* 269–282.

Forman, E., Larreameny-Joerns, J., Stein, M. K., & Brown, C. A. (1998). “You’re going to want to find out which and prove it”: Collective argumentation in a mathematics classroom. *Learning and Instruction, 8* (6), 527–548.

Gray, E., & Tall, D. O. (1994). Duality, Ambiguity and Flexibility: A Proceptual View of Simple Arithmetic. *Journal for Research in Mathematics Education, 26* (2), 115–141.

Hilbert, D. (2000). Mathematical problems. *Bulletin of the American Mathematical Society,* 37 (4), 407-436. (Original work published 1900)

Hoyles, C., & Küchemann, D. (2002). Students’ understandings of logical implication. *Educational Studies in Mathematics, 51* (3), 193–223.

Inglis, M., & Mejia-Ramos, J. P. (2008). Theoretical and methodological implications of a broader perspective on mathematical argumentation. *Mediterranean Journal for Research in Mathematics Education, 7* (2), 107–119.

Inglis, M., Mejia-Ramos, J. P., & Simpson, A. (2007). Modelling mathematical argumentation: The importance of qualification. *Educational Studies in Mathematics, 66* (1), 3–21.

Knipping, C. (2003). Argumentation structures in classroom proving situations. In M. A. Mariotti (Ed.), *Proceedings of the 3rd Congress of the European Society for Research in Mathematics Education*. Bellaria, Italy: ERME.

Krummheuer, G. (1995). The ethnography of argumentation. In P. Cobb & H. Bauersfeld (Eds.), *The Emergence of Mathematical Meaning: Interaction in Classroom Cultures* (pp. 229–269). Hillsdale: Erlbaum.

Lakoff, G. (1987). *Women, Fire and Dangerous Things*. Chicago: Chicago University Press.

Pedemonte, B. (2007). How can the relationship between argumentation and proof be analysed? *Educational Studies in Mathematics, 66* (1), 23–41.

Peirce, C.S. (1932) *Collected Papers of Charles Sanders Peirce, Volume 2: Elements of Logic*. C. Hartshorne and P. Weiss (eds.), Cambridge, Mass., Harvard University Press.

Piaget, J. & Inhelder, B. (1958). *Growth of logical thinking*, London: Routledge & Kegan Paul.

Pinto, M., & Tall D. O. (1999). Student constructions of formal theory: giving and extracting meaning. In O. Zaslavsky (Ed.), *Proceedings of the 23rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 65–73). Haifa, Israel: PME.

Pinto, M., & Tall, D. O. (2002). Building formal mathematics on visual imagery: a case study and a theory. *For the Learning of Mathematics, 22* (1), 2–10.

Raman, M. (2002). *Proof and justification in collegiate calculus.* Unpublished doctoral dissertation, University of California, Berkeley.

Rasmussen, C., Stephan, M., & Allen, K. (2004). Classroom mathematical practices and gesturing. *Journal of Mathematical Behavior, 23*, 301–323.

Saussure (compiled by Charles Bally and Albert Sechehaye), (1916). *Course in General Linguistics* (*Cours de linguistique générale*). Paris, Payot et Cie.

Sfard, A. (1991). On the Dual Nature of Mathematical Conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*, 1–36.

Stephan, M., & Rasmussen, C. (2002). Classroom mathematical practices in differential equations. *Journal of Mathematical Behavior, 21*, 459-490.

Tall D. O., Gray, E., Bin Ali, M., Crowley, L., DeMarois, P., McGowen, M., Pitta, D., Pinto, M., Thomas, M., & Yusof, Y. (2001). Symbols and the Bifurcation between Procedural and Conceptual Thinking, *Canadian Journal of Science, Mathematics and Technology Education, 1*, 81–104.

Tall, D. O. (Ed.). (1991). *Advanced Mathematical Thinking*. Dordrecht, Holland: Kluwer.

Tall, D. O. (2002). Differing Modes of Proof and Belief in Mathematics. *International Conference on Mathematics: Understanding Proving and Proving to Understand* (pp. 91–107). National Taiwan Normal University, Taipei, Taiwan.

Tall, D. O. (2004a). The three worlds of mathematics. *For the Learning of Mathematics*, *23* (3), 29–33.

Tall, D. O. (2004b). Thinking through three worlds of mathematics. *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (Vol.4, pp.281– 288). Bergen, Norway:PME.

Tall, D. O. (2005). The transition from embodied thought experiment and symbolic manipulation to formal proof. In M. Bulmer, H. MacGillivray & C. Varsavsky (Eds.), *Proceedings of Kingfisher Delta'05: Fifth Southern Hemisphere Symposium on Undergraduate Mathematics and Statistics Teaching and Learning* (pp. 23–35). Fraser Island, Australia.

Tall, D. O. (2006). A life-time's journey from definition and deduction to ambiguity and insight. *Retirement as Process and Concept: A Festschrift for Eddie Gray and David Tall* (pp. 275-288). Prague. ISBN 80-7290-255-5.

Toulmin, S. E. (1958). *The Uses of Argument*. Cambridge University Press.

Toulmin, S. E., Rieke, R., & Janik, A. (1979). *An Introduction to Reasoning* (2nd ed.). New York: Macmillan.

Weber, K. (2004). Traditional instruction in advanced mathematics courses: a case study of one professor's lectures and proofs in an introductory real analysis course, *Journal of Mathematical Behavior, 23*, 115–133.

Weber, K., & Alcock, L. (2005). Using warranted implications to understand and validate proofs. *For the Learning of Mathematics, 25* (1), 34–38.

Yackel, E. (2001). Explanation, justification and argumentation in mathematics classrooms. In M. van den Heuval-Panhuizen (Ed.), *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 9–23). Utrecht, Holland: PME.

Zeeman, E. C. (1960). Unknotting spheres, *Annals of Mathematics, 72* (2), 350–361.