TSG 12: **Research and development in the teaching and learning of calculus**

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Aims and focus

In preparing for this working group, the organisers made a call for papers across a wide range of possibilities, including, but not confined to: the role and use of technology in the teaching and learning of calculus; the role and use of history in the teaching and learning of calculus; research about cognitive process in the learning of calculus; pre-calculus; from calculus to mathematical analysis (or) transition between secondary school and university; contextual approaches; graphic approaches; didactical engineering; misconceptions and epistemological obstacles in the learning of calculus.

The papers received covered a wide range of topics from which seventeen were selected for presentation on the ICME website with eight having short presentations in the topic group, preceded by two introductory lectures and a final session ending with a one hour round-table summarizing session.

The papers as a whole covered a range of different approaches to calculus for different target groups. Several papers focused on the nature of the limit concept, which in turn focused on the distinction between calculus and mathematical analysis. Most papers made reference to the use of new technologies, either through their increasing use in teaching or through research on their effects in learning. Several papers formed didactical structures that were used to guide the construction of curriculum goals and also as a basis for analysing the practices of learning and teaching.

To frame the discussions the organisers focused on three sets of questions to be kept in mind during the lectures and to form the basis of the final round-table debate. These questions were as follows:

1) *How can we differentiate the teaching of calculus or mathematical analysis according to the target public?* What means and criteria are important to realize a particular approach adapted for a specific public? How do we address the difficulty of the initial limit concept? Is it helpful to see the first approach to calculus through a blend of embodied meaning and symbolism and to postpone formalism to a study of mathematical analysis? How do we deal with the problematic transition between calculus and mathematical analysis?

2) *What is the role of technology?* How can we characterize and categorize more deeply the ways of using new technologies to teaching calculus and mathematical analysis? How do we take into account its use as a mathematical tool to solve problems, as a means of delivering the curriculum, and as a cognitive environment for learning?

3) *What do the various didactical theories bring to structure the preceding questions?* How do we evaluate the uses of new technologies in the learning of the students? What about the teacher's practices? How can we evaluate effects of these practices?

Summary of presentations

In the opening lecture, *David Tall* presented a framework for mathematical thinking that distinguishes three different modes of operation: the embodied world, "based on our sensory experiences and characterized by thought experiments," the symbolic world, "based on our use of symbolism to carry out calculations and manipulations," and the formal world that "relates to the building of formal theories based on definitions and proofs." This framework was used to formulate the growth of ideas in the calculus, including two significant discontinuities: the shift from finite processes in arithmetic and algebra to the potentially infinite limit concept and the shift from embodied thought experiments and symbolic calculations to quantified definitions and proofs. He suggested that the new technologies benefit the symbolic world by "performing calculations and symbol manipulations at a level of accuracy that would be difficult or impossible by hand." They also benefit the embodied world "in a more subtle way by providing an enactive interface (...) that allows the user to control and experiment with visual representations.. On the other hand, the formal world has the least benefit because of the gap between "the finite machine and the actual infinity of the theoretical limit process."

Isabelle Bloch, France, and *Maggy Schneider* presented a francophone view using the theory of the didactical situations of Guy Brousseau to analyse questions about learning and teaching calculus and analysis. They revealed a progression from situations to help students to contrast a view of mathematics about instantaneous velocities and curvilinear areas to manipulation of quantifiers in the production of graphs of functions that satisfy given conditions. Through the use of Brousseau's three connected types of dialectics of action, of formulation and of validation these situations formulate an analysis that has links with two or more of the three worlds theorized by David Tall. One important criterion which emerges from analysing new trends in teaching is the learning by adaptation created by an appropriate milieu (a term for the context of learning introduced by Brousseau): the students have to construct concepts that are mobilized by necessity in their own problem solving processes, rather than from the usual didactical technique (teachers' explanations) showing the objects of knowledge

Deploring the frequent lack of a proper but necessarily difficult approach to real numbers in high school or in university level calculus and the implicated misconceptions with the students, *Talma Leviathan* from Israel proposed "a project of a transitional course centred around number systems, where the real numbers wouldn't be introduced through an abstract set-theoretic definition but rather through the concrete notion of decimal expansion". This project articulates a geometric approach by "filling the gaps" in the "rational numbers ruler" and a more abstract approach where a real number is defined as a decimal expansion to which one associates its infinite sequence of truncations. On that basis the classical properties of the real numbers can be rigorously proved.

Laure Barthel, Israel, presented a computerized interactive module that familiarizes the students with decimal expansions of real numbers. With the help of this module, students are encouraged to investigate the properties of decimal expansions (e.g., periodic ones) to have a concrete introduction to several issues concerning the notion of limit.

Michel Helfgott, USA, who teaches calculus at college level in the USA, presented and illustrated five guidelines which to him appear to be important in the teaching of first-year calculus:

- Try to strike a balance with regard to what to prove and, what to accept without proof. For example, we can rely on geometric intuition in the case of the mean value theorem and prove an unexpected result such as the derivative of the product of two functions.
- Convey the idea that sometimes there is more than one way to solve a problem; for instance, to calculate some integrals by substitution or integration by parts.
- Discuss significant applications in the classroom, not relegating them to the end as optional materials; for example, problems about kinematics, optimization, physical work and so on.
- Place the subject in a historical perspective whenever possible, like summations linked to problems of integration across history.
- Use technology to supplement mathematical learning, not to supplant it; for example, having students learn how to build programs related to Newton's and Euler's method of approximation.

Victor Giraldo and *Luiz Mariano Carvalho* from Brazil described a qualitative study of the effects on learning caused by an approach to differentiation based on the embodied idea that a differentiable function, when magnified locally, 'looks straight'. Their approach uses computer technology including a software called *Best Line* and the symbolic manipulation software *Maple* to study both simple cases in which the graphic picture is a good representation of the numeric processes and also cases where finite computer arithmetic is compromised and produces pictures that fail to fit the expected theory of limits. The conflict between the finite world of computers and the perfection of human thought experiments is used to enrich formal meaning with a suitable pedagogical approach.

Erhan Bingolbali from Turkey showed the influence of the departmental affiliation of students (first-year undergraduate mechanical engineering and mathematics) on their developing conceptions of the derivative, based on several kinds of data: quantitative (pre-, post- and delayed post-tests), qualitative (questionnaires and interviews) and ethnographic (observations of calculus courses and `coffee-house´ talk). The findings reveal that mechanical engineering students develop a tendency to focus on rate of change while mathematics students develop an inclination towards tangent-oriented aspects. He argued that this difference cannot be solely attributed to the practice of the courses that the students had followed. He further suggested that departmental affiliation appears to have an influence on cognition, and plays a crucial role in the emergence of different tendencies between the two groups.

Yury Shestopalov and *Igor Gachkov* from Sweden described their method for using mathematical software in courses based on the "real-time-mode" for teaching university students in university. Demonstrations and computations are performed directly in the classes using calculators, PCs, desktops or workstations. The authors focused on an example concerning interpolations with natural splines. The main argument which supports this approach is as follows: the use of CAS programs is often reduced to pure illustrations of computing processes and is unsuitable for non-conventional mathema-

tical courses which gain increasing popularity and which become indispensible for the modern engineer, such as coding theory, discrete mathematics and scientific computing.

Isabelle Bloch, France, and *Imene Ghedamsi*, Tunisia, brought to the fore factors of rupture between the secondary mathematical organisation of (pre-)calculus teaching and the university approach to the case of the concept of limit. They invoked various theoretical references: tools introduced by Aline Robert to distinguish between different functionalities of the limit and several sorts of knowledge (technical, or summonable, or available); the anthropological theory of Yves Chevallard who modelled mathematical activities in terms of tasks, techniques, technologies and theories; the distinction made by Anna Sfard between procedural and structural approaches; and Raymond Duval's notion of semiotic representational settings. These theoretical frames allowed the authors to identify the main didactical variables which measure the very important rupture between the two levels: the degree of formalisation, the setting of validation, the degree of generalisation, the number of new notions introduced in the limit environment, the type of tasks, the choice of techniques, the degree of autonomy of the students, the process or object status of the concepts and the nature of the transitions between the semiotic representation settings at issue.

Elfrida Ralha, Portugal, *Keith Hirst*, United Kingdom, and *Olga Vaz*, Portugal, presented a form of cooperative learning using Mathematica in the teaching of multi-variable calculus for first-year university informatics' students in Portugal. The size of the traditional classes (200 in lectures and 65 in tutorials) was not changed, but a new methodology was introduced in tutorial classes, partitioning the allocated time into "students working on their own", "students sharing solutions" and "the lecturer summarizing the fundamental ideas for the session". Some of the examples treated offered the students the opportunity to realize misconceptions that can arise from graphics and to express their conceptual doubts. The aim is to engage them in mathematical/productive reasoning and to motivate them for the necessity of algebraic/analytical justification. A qualitative study as well as statistics show, among other things, an improvement in the pass rate for the course. There were also indications of a more positive attitude towards the course among students.

Salahattin Arslan, France, and *Hamid Chaachoua*, France, talked about the dominance of an algebraic approach to teaching differential equations in the upper secondary school in France and the lack of numeric and qualitative study of ODEs. They formulated the hypothesis that the limitation to the algebraic frame only for the treatment of differential equations can be the origin of difficulties and habits which students face with qualitative interpretation tasks, for example:

- difficulty in recognizing that the slope of the tangent can be calculated from the differential equation;
- difficulty in distinguishing a differential equation, in particular a non-linear one, from other types of equations.

The authors described a module on differential equations set up within a framework of training teacher-trainees to test this hypothesis, among other things. They explained reasons why one can hope to gain insight from the use of software, in particular dynamical software such as *Cabri*, because it can provide students with opportunities to discover new resolution tools.

In the final discussion session, Michael Thomas David Smith and De Ting Wu in their responses framed their presentations with regard to the initial three questions: relating to the variations in approach that may be appropriate for different target groups, the role of technology and the role of theoretical frameworks.

Michael Thomas, United Kingdom, focused on the role of technology and various theoretical frameworks, asking whether technology should be used to teach familiar ideas better or to teach new ideas appropriate to the new situations made possible by technology. He suggested that research and development to date has been either driven by practice or by theory but that there are signs that the two approaches are converging.

David Smith, USA, focused initially on the question of different approaches for different target groups and suggested that in his own university the decision was taken that there were only three major concepts to be studied: rate of change, accumulation, and the relationship between the two, so that all students should attend the same course, with the possible exception of mathematics majors who may require a more comprehensive understanding of analysis. He suggested to adopt a new theoretical framework by Kolb that emphasized learning activities that start with "concrete experience", then proceed through stages of "reflective observation", "abstract conceptualization", and "active experimentation". He related this to a neurophysiological theory of Zull that linked the way in which the Kolb cycle of learning relates to the use of different parts of the brain. He concluded by speaking about his own ways of using computer technology to teach the calculus.

De Ting Wu, USA, presented his professional opinion based on teaching central parts of the traditional college calculus and using new technology where this was appropriate. He acknowledged that technology is a powerful tool and a helpful aid in teaching and learning but affirmed that it would not replace the study of mathematics or the value of good teaching.

In the closing discussion, *Bronislaw Czarnocha*, Poland, questioned the strong emphasis in the discussion on the cognitive difficulties in understanding calculus and emphasized instead the power and beauty of the subject. Maggy Schneider argued that it was necessary to subordinate the analysis of teaching using new technologies to more general theoretical frameworks about learning and teaching relating to the nature of the educational institutions and teachers' practices.

The TSG provided a fruitful platform for sharing many ideas, with anglophone and francophone theories meeting in a constructive manner and research from many parts of the globe showing that the study of calculus remains vigorous and creative in many different ways.

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