EMBODIED ACTION, EFFECT AND SYMBOL IN MATHEMATICAL GROWTH

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This study of the learning of vectors is situated in the intersection of embodied theory relating to physical phenomena, and process-object encapsulation of actions as mathematical concepts. We consider the subtle effects of different contexts such as vector as displacement or force, and focus on the need to create a concept of vector that has greater flexibility. Our approach refocuses the development from 'action to process' as a shift of attention from 'action to effect' in a way that we hypothesise is more meaningful to students. At a general level we embed this development from enactive action to mental concept within a broad theoretical perspective and at the specific level of vector we report initial experimental data.

INTRODUCTION

This study is part of an ongoing enterprise to build a practical cognitive theory embracing human learning and the powerful use of symbolism in mathematics (Gray & Tall, 2001). As such, it stands at the conjunction of two major theories of cognitive development: the embodied cognition of Lakoff and others (Lakoff & Johnson, 1999, Lakoff & Nunez, 2000) and the development of symbolic mathematics through process-object encapsulation (Dubinsky, 1999, Tall *et al*, 2001). The current three-year study focuses on the concept of vector. This is particularly appropriate for the wider development of the theory as it encompasses embodied aspects in Physics—such displacement, force, velocity, acceleration—and a complementary approach in Mathematics that is based on a text with an implicit process-object approach (Pledger *et al*, 1996).

In the first year, (September 2000 – July 2001), the first author followed a strategy using full class plenaries to encourage students to construct their own coherent conceptions. As we shall see, this had positive effects in improving the flexibility of the students' concept of vector. In the second year, which is now in progress, a new initial emphasis focuses on embodied activities in which the new students enact vectors as transformations moving a shape on a flat surface. The movement of any point on the shape can be represented by an arrow from start to finish, and all the arrows have the 'same effect'. This idea (formulated by a student, Joshua) has proved to be a helpful bridge in translating the sophisticated theory of process-object construction to a practical idea of action-effect. In this article we discuss the findings of the exploratory study and data analysis of the first stages of implementation.

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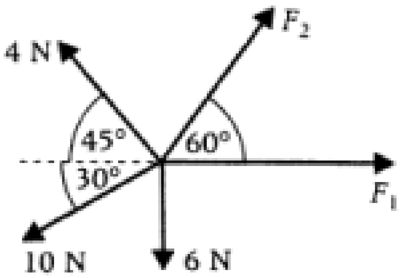
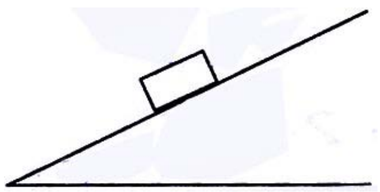
The name of the second author was deleted in the Proceedings to comply with a rule limiting the number of papers an individual could co-author. The rule was rescinded after this one year and is no longer in force.

THE COMPLEXITY OF THE LEARNING SITUATION

Students meet the notion of vector in different contexts with subtle differences in embodiment. For instance vectors may be encountered as displacements sensed as physical journeys from one place to another, or as forces acting at particular points. In the addition of displacements, one journey followed by another is naturally interpreted using the triangle law, but the addition of forces operating at a point is more naturally represented by the parallelogram rule. In the curriculum we are considering, the notion of vector is first introduced as a translation in the plane and dealt with as a column matrix in mathematics, or as the separate horizontal and vertical components in physics. Both versions are linked to a picture of the vector as the hypotenuse of a right-angled triangle with components as horizontal and vertical sides. In turn this links more easily to the triangle law than to the parallelogram law. Given a problem solvable by horizontal and vertical components such as figure 1a, 25 out of 26 students were able to solve it. However, given a more complex physical problem such as that in figure 1b, asking the student to mark the forces involved with an object on a rough sloping plane, only 4 out of 26 students were successful. In interviews, it transpired that several students, who used the triangle law to draw a picture as in figure 1c, used the triangle of forces to mark the components; because the force parallel to the plane is drawn well below the object, it did not seem to be acting *on* it and was ignored.





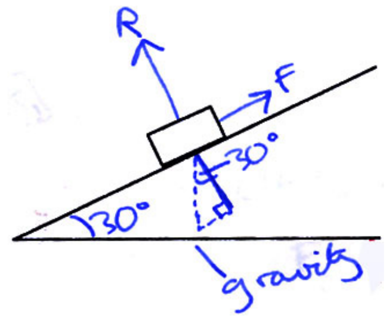


Fig 1a: find F_1, F_2 **Fig 1b: describe & mark forces** **Fig 1c: forces as marked**

The research project therefore focuses on two main objectives, the first is to analyse the cognitive development of the notion of vector in the curriculum, the second is to help students develop the notion of vector as a flexible cognitive unit that can be applied transparently in its various incarnations.

METHODOLOGY

The research method draws upon qualitative and quantitative data and includes lesson observations, standard class tests to assess progress, and a specially designed conceptual questionnaire coordinated with clinical interviews with students and Mathematics and Science teachers (Ginsburg, 1981; Swanson *et al.*, 1981). The data was triangulated, by analysing the books used by students and teachers, by videoing and observing classes, by interviewing teachers on their preferences and their expectations of the students' knowledge about vectors, with particular emphasis on the questions used in the conceptual questionnaire.

The research is conducted at a Comprehensive School with a good academic reputation (for example, in 2001, 63% attained a grade C or above in mathematics in the GCSE examination taken at age 16, as compared with a national average of 54%). The research involved 23 Lower Sixth students (aged 16-17), 26 Upper Sixth students, (aged 17-18), 2 teachers of Physics, and 4 teachers of Mathematics (two covering the preliminary work on vectors at GCSE level, and two teaching the two-year 'A' level course in the Sixth Form).

THEORETICAL FRAMEWORK

The topic of vectors spans both mathematics education and science education. In science the ideas often begin from what we would now call a real-world embodied viewpoint. For instance, in dealing with vectors, Aguirre and Erikson write:

> Teachers could [...] build upon students intuitions (developed through experience in everyday settings) by relating these intuitions to the more formal problem settings in the scientific domain. (Aguirre & Erikson, 1984, p 440.)

They proceed by detailing networks of vector concepts to support this approach, however, their network shows no indication of the mathematical concept of vector. After many years of such developments, Rowlands, Graham and Berry observe:

> … various attempts at classifying student conceptions has been by and large unsuccessful [...]. A taxonomy of students conceptions may be impossible because the considerations of 'misconceptions' require a specific regard for the framework from which the 'misconception' occur [...] and how misconception is linked to the other forms of reasoning. (Rowlands, Graham & Berry, 1999, p 247.)

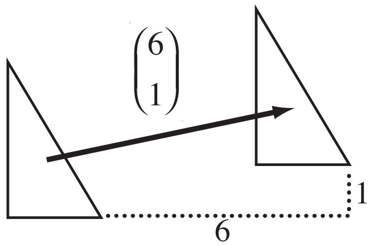
The recent development of embodied cognition, particularly in the formulation of Lakoff (eg. Lakoff & Johnson, 1999, Lakoff & Nunez, 2000) even formulate the idea that *all* thought is embodied. Here we are faced with a genuine quandary. If the attempts to classify the development of vector through intuitive student concepts is, by and large, unsuccessful, and the evidence is that the notion of vector has subtly different meanings in different contexts, how are we to progress in teaching the subject? One strategy that seems evident is to encourage students to reflect on the different aspects of different embodiments and to help them rationalise the various contexts to give an entry into the mathematical notion of vector wherein the ideas of the triangle law and parallelogram law are different aspects of the same theoretical idea, not distinct rules that each have their separate domain of meaning. To gain insight into this possibility, the first author began a programme of plenary discussions with the Upper Sixth based on the underlying unity of the idea of vector.

In linking various embodied ideas to the mathematical notion of vector, we were conscious of the large number of theories relating embodied experiences with mathematical symbolism. These include Piagetian stage theory developing through sensori-motor via concrete operational and formal operational, the Bruner (1966) theory of enactive-iconic-symbolic representations, the work of Krutetskii (1976) on geometric, harmonic and symbolic styles of thought, the van Hiele (1986) development in geometry, and the SOLO taxonomy (Structure of Observed Learning Outcomes) of Biggs & Collis (1991), not to mention the wide range of work in using visual and symbolic interfaces with computers. Amongst all of these we found most empathetic was the notion of how successive modes of thought arise in the SOLO taxonomy whereby each broadens to be included in the next and how physical action (in the sensori-motor mode) is broadened to the ikonic mode, then, through the introduction of symbols, to the symbolic mode and on to successive formal modes of operation. As each of these becomes available and is added to previous modes, we found ourselves dealing with students in a situation where the two main modes of operation are a combined sensori-motor/visual embodied mode of thought and a fundamentally concrete-symbolic mode.

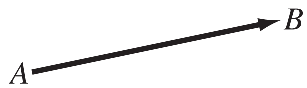
ANALYSIS OF THE SCHOOL APPROACH TO VECTORS

The text-book (Pledger *et al*, 1996) used in the school for introducing vectors to the students in the previous year followed a pattern that is reminiscent of the encapsulation of processes as objects. In this approach, the processes are translations of objects in the plane and these lead to vector concepts, as follows:

1. translations are described using column vectors, $\begin{pmatrix} x \\ y \end{pmatrix}$ […] with the column vector $\begin{pmatrix} 6 \\ 1 \end{pmatrix}$ meaning 6 units in the positive x direction and 1 unit in the positive y direction.

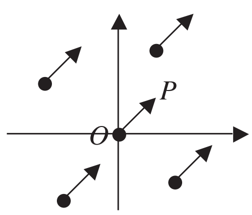


2. an alternative notation which can be used to describe the translation is $\overrightarrow{AB}$ representing where A is the starting point and B is the finishing point. […] The lines with arrows are called **directed line segments** and show a unique **length** and **direction**



3. a third way to way to describe a translation is to use single letters such as **a** …. Translations are referred to simply as **vectors**. […] [Each vector] has a unique **length** and **direction** ...

4. **Position Vectors**. The column vector $\begin{pmatrix} x \\ y \end{pmatrix}$ denotes a translation. There are an infinite number of points which are related by such a translation. … The diagram shows several pairs of points linked by the same vector. The vector which translates O to P, $\overrightarrow{OP}$, is a special vector, the *position vector of P.*



This pragmatic approach has some of the aspects of process-object encapsulation. Stage 1 sees a vector as an action on a physical object. The translation is already represented as an arrow. In stage 2 the object is omitted and the focus of attention is on the line segment as a journey from a point A to a point B. Stage 3 shifts the focus to the vector as a single entity drawn as an arrow and labelled with the single symbol **a**. This entity has both an enactive aspect (the movement from tail to nose of the arrow) and an embodied aspect (as the arrow itself). In stage 4 the column vector is used to denote an infinite family of arrows with the same length and direction, with one specific vector starting at the origin singled out as a position vector as a special representative of the whole family.

In practice, the Physics teachers preferred to 'simplify' the ideas by referring separately to horizontal and vertical components of vectors. For example, to add two vectors, they would consider each vector separately, calculate its horizontal and vertical components and add them together to get the components of the sum. In parallel, the students would often use the equivalent matrix method to add vectors in pure mathematics. Thus, although they had been taken through the spectrum of development in stages 1 to 4, to make any computations, they were encouraged to fall back to level 1.

ACTIONS AND EFFECTS – THE INSIGHT OF A SPECIAL STUDENT

In attempting to build a more flexible conception of the notion of vector that encapsulates the whole structure of embodiment and process-object encapsulation, we were struck by the interpretation formulated by one particular student whom we will call Joshua. He explained that different actions can have the same '*effect'*. For example, he saw the combination of one translation followed by another as having the same effect as the single translation corresponding to the sum of the two vectors.

By focusing on the effect, rather than the specific actions involved, we realised that it proves possible to get to the heart of several highly sophisticated concepts. For instance, in fractions, 'divide into three equal parts and take two' is a different action from 'divide into six equal parts and take four' but they have the same *effect,* giving rise to the central idea of equivalent fractions. The same idea occurs in algebra where $2(x + 4)$ and $2x + 8$ involve different sequences of actions with the same effect, leading to the notion of equivalent expressions. We hypothesize that the notion of action-effect is a more approachable way of describing the theory of action-process (Dubinsky, 1991) or procedure-process (Gray & Tall, 1994).

In dealing with this approach, we encouraged students to participate actively by shifting a triangle placed on the table. Figures 2a and 2b have the same start and endpoint (and therefore the same effect, even though the journeys they take in between are different.) The arrows in figure 2b represent equivalent vectors (free vectors having the same effect) and figure 2c represents Joshua's idea that the sum of two vectors has the same effect as the two vectors applied one after the other.

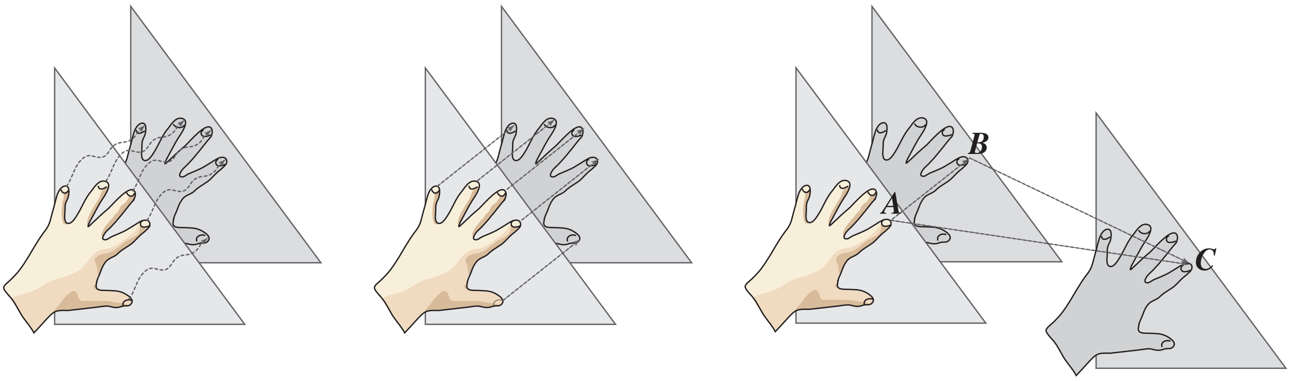


Fig 2a: a translation **Fig 2b: a translation** **Fig 2c: The sum of two vectors**

Figures 2a, 2b 'have the same effect' **Two translations and their total effect**

This approach implicitly encapsulates the notion of equivalent free vectors 'having the same effect' and encourages students to feel able to shift free vectors around in any appropriate manner. We would therefore expect students following this approach to be more flexible in handling free vectors.

EMPIRICAL DATA ANALYSIS

To investigate the reasons underlying the original problem in figure 1c, a question was given showing a body on an inclined plane, as in figure 3. Figures 3b and 3c represent the ways in which two students James and Chris split the weight W into components W_1 and W_2. Are either or both of James and Chris right?

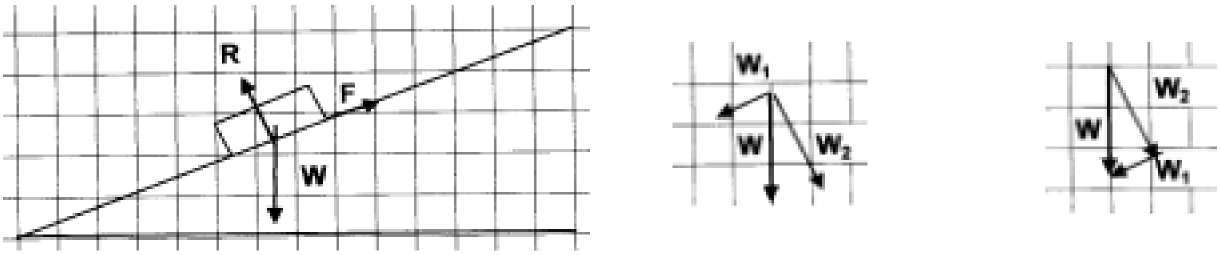
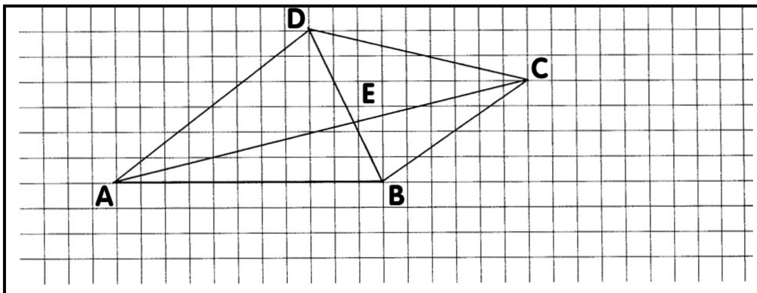


Fig 3a: a body on an inclined plane **Fig 3b: Jame's picture** **Fig 3c: Chris's picture**

The 23 students beginning the course in the lower 6th gave a variety of responses, 11 said both were right, 4 chose fig 3b, 1 chose fig 3c and 6 said neither. All of the students in the sixth form who had taken part in the reflective plenaries found the question trivial and saw the triangle and parallelogram as equivalent.

To test the student's ability to deal with vectors graphically, we gave the question in figure 4. The first part is a natural triangle problem with the vector AB followed by BC, the other two benefit from being able to see the vectors as free vectors to be able to move them so that they follow on end to end.

In the test, *all* the students were easily able to cope with the sum $\overrightarrow{AB} + \overrightarrow{BC}$. However, parts (ii) and (iii) were more problematic. When we consider those students who were able to solve all three problems, we get the data in tables 1a and 1b.



Show clearly a vector equivalent to:

(i) $\overrightarrow{AB} + \overrightarrow{BC}$

(ii) $\overrightarrow{AB} + \overrightarrow{AD}$

(iii) $\overrightarrow{AB} + \overrightarrow{CA}$

Fig. 4: Testing the visual sum of two vectors

In table 1a those upper sixth students who participated in reflective plenaries were more successful than those following standard class lessons and in table 1b, those following an embodied approach also had more success.

Upper 6th	Reflective	Standard
All 3 correct	7	1
Other	8	10

Table 1a: effect of reflective plenaries

Lower 6th	Embodied	Standard
All 3 correct	5	1
Other	2	15

Table 1b: effect of embodied approach in reflective plenaries

Interviews with six selected students in the lower sixth confirmed that students following a standard course had problems adding two vectors that did not follow on one after the other, especially in cases where they were joined head to head. In the latter case, two out of three students thought that two vectors pointing to the same point would have resultant zero, because they would cancel out.

REFLECTION

The study so far has revealed the complexity of the meaning of vectors as forces and as displacements and the subtle meanings that are inferred in differing contexts. Studies in science education have attempted to build a classification of misconceptions without clearly identifying the underlying problems. Our approach is to develop a pragmatic method that will work in the classroom. One aspect is the use of conceptual plenaries, which are already becoming part of the formally defined curriculum in England. The other is to continue to develop a theory that links physical embodiments to mathematical concepts via a strategy that focuses on the effects of actions. Our experience shows that such an approach can be beneficial in the short-term and we are continuing our practical and theoretical developments over the longer term.

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