What Mathematics is Needed by Teachers of Young Children?

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There has long been a dichotomy between the mathematics that is learned by trainee teachers ‘for their own personal development’ and the mathematics they will need to teach to young children. My viewpoint is that a student should be encouraged to study whatever it is that interests them, but when it comes to pre-requisites for teaching mathematics to young children, different criteria apply. I shall provide an analysis of how mathematics grows cognitively in different ways in different individuals. This will provide a basis to formulate what is needed for teachers to participate actively in the long-term mathematical development of their children.

Introduction

The general theme of SEMT 01 is:

> *“What is meant by the competence and confidence of people involved in the teaching of elementary mathematics?”*

In this presentation I focus not only on the kind of knowledge that teachers need for their own competence and confidence in mathematics but on how mathematics develops in the individual so that the teacher may be supportive in the long-term development of the child. I argue that this is a pre-requisite, not just for exceptional teachers talented in mathematics, but also for all teachers of mathematics to young children. I shall consider how this depends on fundamental understanding of mathematics and its cognitive growth. This involves not only the logic and coherence of mathematics, but the reconstructions and discontinuities inherent in its learning.

Mathematics has never been regarded as a subject that is easy to teach or to learn. As I write this presentation, today’s Times newspaper (5th May, 2001), carries yet another article on the problems of provision for teaching mathematics in England. At a pragmatic level, it is difficult to fill posts in schools with qualified mathematicians when there is the lure of greater salaries in banking, commerce and computing. At a deeper level, there is a cumulative problem in which teachers with insufficient competence and confidence fail to inspire the children, producing a downward spiral of interest and achievement in successive generations.

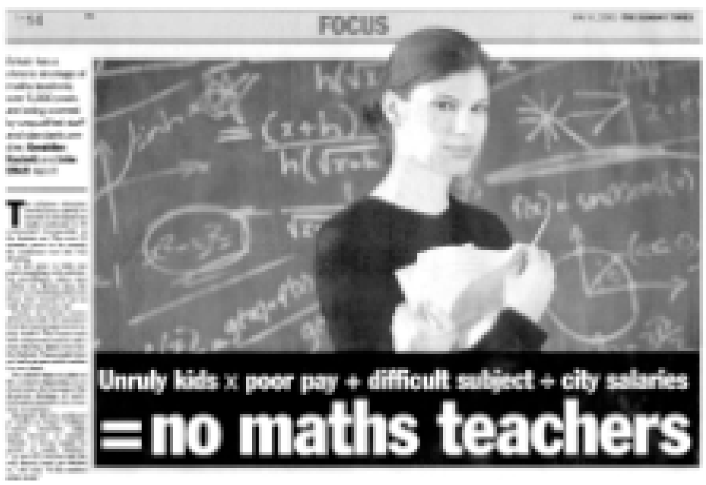


Figure 1: Teacher shortage

What does it mean for teachers to be competent in mathematics?

Given the possibility of teachers without adequate competence producing a downward spiral in successive generations, we turn to the positive side. What qualities are needed for teachers to be insightful in mathematics to create an upward spiral of performance in successive generations?

It is evident that confidence and competence in mathematics is an essential ingredient of being a good mathematics teacher. We must ask, however, what kind of mathematics are we talking about?

In general, the main criterion for judging the competence of teachers in mathematics has focused largely on *the teacher's knowledge of mathematics itself.* A teacher is usually expected to have knowledge of mathematics beyond that which they need in the classroom so that they are working well within their area of competence. Teachers of older children are normally expected to have a degree qualification including mathematics as a major component, while teachers of younger children responsible for a broader range of learning are expected to have a minimum level of mathematical competence. In England, this minimum level is equated with a pass at examinations at age 16. Recently the government has introduced a computerised test for all student teachers based on this level (which they had already passed when they were 16). If the student does not pass within a maximum of four attempts, they must withdraw from teaching. This year around 23% of student teachers are failing at the first attempt. Being 'competent' at 16 does not guarantee competence later.

Learning 'more' mathematics does not guarantee competence either. A range of research studies in the USA revealed little correlation between the number of higher mathematics courses studied and subsequent effectiveness in teaching (Begle, 1979). There was even a small negative correlation between the number of mathematics college courses taken by mathematics teachers and the mathematical achievement of their students. This questions the appealingly intuitive idea that a better grasp of mathematics itself leads necessarily to a better quality of teaching. It does not imply that knowing less means teaching better. What matters is the *quality* of understanding that the teachers have for the task of teaching mathematics to children.

A Profound Understanding of Fundamental Mathematics

Li Ping Ma (1999) revealed a fundamental insight in the teaching of arithmetic by contrasting the contexts in the United States and China. All the American teachers had college degrees and several had MAs, whilst the Chinese students had nine years of school and three years of normal school to prepare them as teachers—the equivalent in time of the years in American school without any time at College. Yet, when it came to explaining elementary arithmetic problems, the Chinese teachers were able to give much more meaningful explanations of the basic processes while the American teachers revealed disturbing deficiencies.

For instance, when asked to explain the processes involved in subtraction of one decimal number from another, all the teachers could do the problems given correctly, however, fewer than 20% of the US teachers had a conceptual grasp of regrouping process decomposing a ten into ten ones. In contrast, 86% of Chinese teachers were able to explain the decomposition procedure. On a second problem involving multi-digit multiplication, only 40% of the US teachers could explain the correct method of aligning the partial products while over 90% of the Chinese teachers showed a firm grasp of the place value ideas that prescribe the alignment procedure. This shows that it is not just a question of a teacher being able to *do* the processes in the mathematics they teach, they also need to be able to describe the underlying principles.

Howe (1999) summarizes three major areas where Ma's observations are critical. The first is that Chinese teachers receive good teacher training that produces good learning—the virtuous cycle that we earlier suggested should be sought. The second is that mathematics teaching in China is a specialism, focusing on the subtleties of mathematics itself. Thirdly, while US teachers spend virtually the whole day in front of a class, Chinese teachers have time within their school day away from their classes to share their difficulties and insights with others, creating a culture for improving their own expertise and the quality of mathematics teaching.

The cognitive development of mathematics as a whole

The work of Li Ping Ma shows how a fundamental understanding in basic principles can pay off in the teaching of arithmetic to young children. However, if we consider wider issues such as economic prosperity, the current positions of China and the USA show that simply being good at arithmetic is not the whole story. We need to look beyond the development in arithmetic to consider a wider picture. In this presentation, I restrict my response to mathematics, but broaden the perspective to look at the longer-term development of the child.

In Tall (1995), I formulated a theory of how the individual builds up mathematical ideas based on perception, action and reflection. Figure 2 (taken from Tall et al., 2001) suggests how this leads to three essentially different kinds of mathematics:

- *space and shape* (geometry) based on theorizing about the (geometric) objects we perceive and construct at increasing levels of sophistication,
- *symbolic mathematics* where actions on objects (such as counting) are symbolised giving new mathematical concepts (number)
- *axiomatic mathematics* (built by reflection on the properties of the first two forms of mathematics in terms of formal definitions and logical deductions).

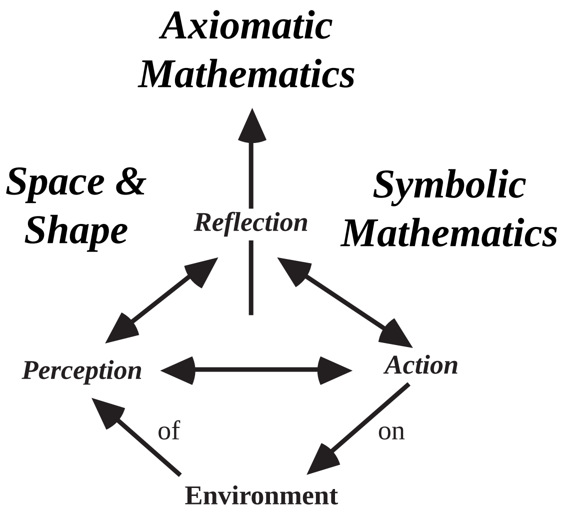


Figure 2: Various types of mathematics

Our interests in this presentation focus on the development of young children, particularly in the symbolic mathematics of the arithmetic of fractions and whole numbers. However, we will also look a little further to see the cognitive way in which arithmetic moves into algebra, and take a brief look at the development of space and shape. Further details of the broader picture can be found in Clements & Battista (1992) (for the van Hiele development of shape and space), Tall et al(2001) (for the development of symbols), Tall, (2001) (advanced mathematical thinking).

Symbolic mathematics

The development of symbolic mathematics begins with arithmetic. This occupies several years of a young child's development. It builds by operating with physical objects (which we term the *base objects* in this particular activity) and continues as follows:

a) preliminary activities such as manipulating and sorting the *base objects*,
b) the coordination of counting words and pointing at objects to give the step-by-step *procedure* of counting,
c) the realisation that counting in different ways always give the same result, so essentially different methods represent the same *process* of counting,
d) the use of the number symbol as a mental concept to be manipulated in arithmetic as an entity in itself giving the *concept* of number.

The use of a symbol as a pivot between a process (carried out by some particular procedure) and the mental concept produced by the process is termed a *procept* (Gray & Tall, 1994). This sequence of building actions on base objects to give a step-by-step procedure, interiorised as a process and conceived as a mental object is called *encapsulation* (Dubinsky, 1991) or *reification* (Sfard, 1991). This gives a sequence of learning involving

a) *base objects* that are operated upon to give
b) step-by-step *procedures* that are interiorised as
c) *processes* conceived as a whole, and encapsulated as
d) *procepts* represented by symbols that pivot between process (counting) and concept (number). Such procepts may then be used as *base objects* to operate upon at a higher level.

The biological basis of cognitive development

The steady process of learning happens by biological developments in the human brain. In general, when links between neurons are stimulated, they are temporarily set on alert and so respond more readily to similar stimulations. If stimulated again whilst on a state of alert, the connections are strengthened, so that in time separate firings are coordinated into a single entity. This biological process—the basis of all human learning—is called *long-term potentiation*. (See, for example, Carter, 1999.)

In the development described above, disparate activities (such as sorting, pointing, saying number words in sequence) become coordinated into an action-schema (counting) that is later conceived as an entity (number). These entities (numbers) then may become base objects at a new level of activity on which to perform the process of addition and develop the concept of sum. At the next level, the process of addition may be repeated to give the process of multiplication and the concept of product.

Cognitive development in arithmetic

In practice, the biological process of mathematical learning does not occur in a neat sequential fashion with each stage being completed before the next is begun. For instance, the formation of mental entities (numbers) is usually still in progress as the child develops a succession of procedures for counting including:

count-all (counting one set, then the other, then putting them together to count all)
count-both (a quick count of the first set, followed by a counting-on of the second)
count-on (considering the first number as an entity and counting-on the second)
count-on from larger (a more efficient form of count-on).

In terms of the procedure-process-procept development, these may be considered as distinct procedures for the process of addition. How far the child has developed when new techniques are introduced radically effects what the child can do. All children learn a variety of *known facts*, such as '3+2 is 5' or '5+5 is 10'. Some children, but not all, see these facts as thinkable entities allowing them to build new *derived facts* from known ones. For instance, knowing '5+5 is 10' may be used to work out '5+4' is one less than '5+5', so 5+4 is 9. Gray & Tall (1994) found that the more successful children use a flexible mixture of efficiently chosen counting procedures and derived facts. However, they found that *none* of the children employing even a single case of 'count-all' was able to generate any derived facts. A divergence in performance seems to be setting in between those who develop proceptual flexibility and those who remain entrenched in primitive counting.

This spectrum of performance begins to take its toll when multiplication is introduces. Those who sense 3+3 is an entity (6) and 3 lots of three is an entity (9) are likely to see that 4 lots of 3 is 'three more than nine', namely, 12. The concepts they think about are entities that they can manipulate in their mind. Those still using mainly counting procedures may, of course, make *some* sense of multiplication. However, it involves a great deal more cognitive activity. A child who counts 4 times 3 by one of the counting procedures (say count-on) might say '3 and 3 is … 4, 5, **6**, and another 3 is … 7, 8, **9**, and another is 10, 11, **12**.' The cognitive strain of such an extended procedure is evident. It leads to a spectrum of possible success and failure in which some become increasingly expert handling richer and richer cognitive units whilst others struggle with a greater number of isolated pieces of information.

Figure 3, expanded from Gray, Pitta, Pinto & Tall (1999), represents the widening spectrum of performance. It shows how a single procedure may be applied to solve a routine problem: – a widening range of procedures enables the child to be more

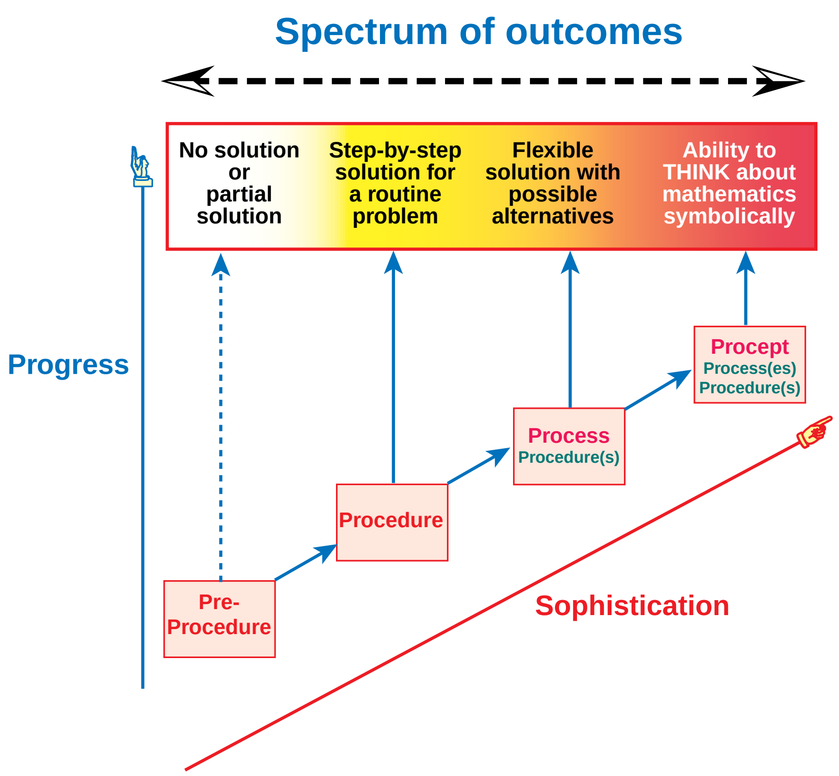


Figure3: A spectrum of performance (based on Gray et al, 1999, p.121).

flexible and efficient, whereas the ability to think *about* symbols and manipulate them in a proceptual manner offers a basis for future development.

This warns us about the subtleties of testing children and drawing inferences from the tests. Routine problems may be solved by a wider range of children without distinguishing between those who use isolated procedures and those with proceptual thinking processes ready for more sophisticated study. It is also consistent with the idea that using unreflected routine procedures may give children procedural tools that lack the flexibility for more subtle developments.

The Case of Fractions

To illustrate the analysis of growth in terms of 'base object-procedure-process-procept) we now turn to the elementary arithmetic of fractions which proves to be easy for some, but impossibly difficult for others.

The notion of a fraction begins with a fraction *of something.* This 'something' is a base object or a collection of base objects that may be a cake, a length of cloth, a group of children or any other of a wider range of possibilities. The process of calculating a fraction of something consists of a two-stage procedure: first subdivide the base object(s) into a certain number of equal pieces, and then take a specified number of them. For instance, $\frac{3}{4}$ of a cake consists of dividing the cake into 4 equal parts and taking three of them; $\frac{3}{4}$ of a class of 24 consists of dividing the class into 4 equal parts, in this case consisting of 6 children in each part, and taking three of them, so that $\frac{3}{4}$ of the class is 18 children. Implicitly, a fraction of a cake is less than the whole cake and a fraction of a class is less than a whole class. However, in the latter case the result is 18 children, which is more than one child. Because of the variety of contexts in which the fraction concept is used, the perceptual idea of a fraction can have a variety of apparent meanings.

Kerslake (1986) asked a number of 'average' British thirteen-year olds which of the following alternative drawings would help someone who did not know what $\frac{3}{4}$ is:

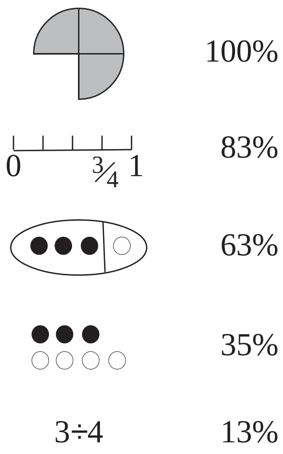


Figure 4: Percentage of average 13 year-olds recommending representations of $\frac{3}{4}$

All the children recommended conceiving $\frac{3}{4}$ as a fraction of a disc; most related it to a point on a number line, fewer were happy with other contexts and a tiny minority recommended it as'3 divided by 4'. Several conceived division as a property of whole numbers, so 3÷4 was simply 'not possible'. When the fraction was changed to $\frac{3}{5}$, the percentages were similar, but *none* of the children suggested relating $\frac{3}{5}$ to 3÷5.

Kerslake then asked fourteen children to share three cakes equally between four. This has a practical solution by cutting the cakes in half to give six halves, four of which can be shared one at a time and the remaining two can be cut into four quarters, which neatly share out so that everyone gets a half and a quarter. All fourteen could see that the half and the quarter gave a total of three quarters of a cake.

However, when an extra person was introduced and the problem became sharing three cakes between five, only ten children out of fourteen could see that each child got three-fifths of a cake and of these, only six could see $3/5$ related to 3÷5. The four who did not see the answer as three-fifths all used their practical method to divide the cakes in half, giving six cakes, which meant one half for each person and an odd half left over. This was cut into roughly five pieces and given one to each child. *None* of these four children could see that the amount given to each person (half a cake plus a fifth of half a cake) was equal to $3/5$. Nor could they relate the solution of the problem "divide 3 cakes between 5 people" with the fraction $3/5$ or the division 3÷5.

Analysing this data in terms of base objects and sharing procedures, we see that all the children could operate in a real world situation to give a practical and imaginative method of sharing the base objects reasonably fairly. However, some did not see the procedure as related to the fraction concept and even fewer related it actually to division of one number by another.

The development to the full fraction concept has a procedure-process structure. The fractions $3/4$ and $6/8$ are very different as step-by-step procedures. To get $3/4$ of a bar of chocolate involves dividing it into 4 equal pieces and taking 3 of them; $6/8$ involves dividing it into 8 equal pieces and taking 6 of them. The second activity has more pieces of chocolate and is a different procedure, both in terms of the sequence of activities and also in terms of the final result. This can act as an obstacle for procedurally focused children who sense the *difference* between the procedures rather than the *sameness* of the final quantity.

In essence, the fractions $3/4$ and $6/8$ invoke different procedures but are effectively the same process. Children who grasp the 'sameness' of equivalent fractions have a great advantage. For instance, $1/3$ is the same as $2/6$ and $1/2$ is the same as $3/6$ so 'a third plus a half' is 'two sixths and three sixths'. This operates like 'two apples and three apples', which is 'five apples'; so 'two sixths and three sixths' is 'five sixths'. Paradoxically, a familiarity with the *mental* properties of whole numbers provides a link back to operating with fractions in much the same *practical* way as with whole numbers. It gives those with a proceptual feeling for whole number a huge built-in advantage dealing with fractions.

Many theorists focus attention to performing mathematics in realistic situations. Our analysis suggests that this can give greater success initially but it may also unwittingly imprison some children in the physical world where activities become increasingly complex. If children remain at the level of procedure, seeing a fraction as a double process of dividing and sharing then $1/3 + 1/2$ may be interpreted as 'divide into three and take one of them' and 'divide into two and take one of them'. Children at this level will find the arithmetic of fractions exceptionally difficult and fail to go beyond pragmatic real-world sharing.

Moving on to Algebra

The difficulties of introducing algebra also benefit from an analysis in terms of 'base objects - procedures - processes - procepts'. In this case, the base objects are actually numbers, but they are no longer explicit. The procedure 'double the number and add six' can be represented as the expression 2*n*+6, but now the symbol *n* represents a base object that is either 'unknown', or 'variable. This is quite unlike arithmetic where there can be actual physical objects to count and share. It can bewilder many children who need a concrete referent to operate on—a difficulty that has been recognised by using apparatus for algebra, such as a physical balance to represent a linear equation. However, such apparatus, which can give an initial boost by linking to the real world, can act as an insight for some but a severe limitation for others.

The notation used violates certain standard conventions, such as reading from left to right. The formula 2+3×*n* does not mean "add 2 and 3, then multiply by *n*', but 'first multiply 3 by *n*, then add 2 to the result.' Then there are formulae such as 2n+6 and 2(*n*+3) that look very different but always give the same output for any given value of *n*.

The notion of equivalence again has a process-procedure interpretation. The *procedure* 'double the number and add 6' involves a different sequence of steps from 'add 3 to the number and double the result'. When written down, they are expressed in different ways, such as 2×*n*+6 and (*n*+3)×2, or the more standard notations 2*n*+6 and 2(*n*+3). However, for any value of *n*, they give the same result. They are different procedures that give the same process.

Once again, many children struggle to interpret algebraic symbols in a manner that has genuine personal meaning. On the other hand, those with a friendly (proceptual) feel for numbers, who can see that the two expressions are essentially the same process, are well on their way to manipulating them as entities in their own right. Thus, equivalent procedures become processes whose symbolism has the dual role of process (of evaluation) and manipulable concept (as algebraic expression). This allows the symbols to operated as meaningful base objects for algebraic manipulation, leading on to more subtle forms of mathematics.

Discontinuities in Cognitive Growth

The mathematical simplicity of successively wider number systems—whole numbers, integers (positive and negative), (positive) fractions, rationals, irrationals, reals, and complex numbers—is evident to mathematicians. The situation for growing children is very different. The (whole) number after 2 is 3, so there are no other 'numbers' between 2 and 3; when the move is made to fractions; there are many numbers in this gap. In whole number arithmetic 'addition makes bigger', when negative numbers are introduced, adding a negative number will 'make the result smaller'. In whole number terms, 8÷4 has the solution 2 but the problem 3÷4 does not have a (whole number) answer. For many young children 3÷4 'cannot be done.' In every case, moving to a more general context produces cognitive conflict.

How do human beings react to danger? For those with self-confidence born from success that has produced a range of strategies of action, the brain produces neurotransmitters to heighten the senses of attention. For those with less confidence, the primitive reaction on sensing danger is to produce neurotransmitters to enhance automatic reactions for flight. The consequence is the suppression of reflective thought. The discontinuities that occur in cognitive growth therefore exaggerate the differential effects on success and failure.

Many teachers themselves will have experienced such failure. It is therefore essential for them to reflect on the *reasons* underlying their difficulties. These are genuine *cognitive* difficulties arising from moving into a new context where old implicit beliefs no longer hold. In the development of whole number arithmetic, fractions, and algebra, not only are there necessary processes compressing step-by-step procedures into symbolic concepts (procepts), but also each development has potential conflicts with previous knowledge.

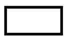
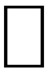
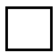
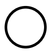
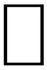
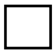
Teaching children therefore requires more than being able to do the procedures of mathematics. Being competent in mathematics may increase a teacher's confidence in their own ability but a fuller understanding of how children learn and the difficulties they face is an essential part of increasing their competence to teach.

What about Geometry?

There is insufficient space here to give the same attention to the development of geometry in this presentation. In outline, the understanding of shape and space occurs through initial perception of and interaction with objects in the real world along the lines expressed long ago by van Hiele (1986) However, in essence, what happens in geometry is the same as what happens in other broad areas of cognitive development. It depends not only on visual perception and physical action but also on reflection using the most powerful tool that distinguishes Homo Sapiens from all other species — language. The young child's perception of the world is quite different from their representation. Figure 5 for example is a picture of me (on the left) and my wife, as drawn by my four-year old grandson Lawrence. If you look carefully, you can see every letter of his name in his own style. He can also describe verbally every part of his representation. He explains he started with my (big) body and tummy button, then he added a head (with beard and glasses), two arms and two legs.

Figure5: Grandpa and Nana

As he gets older, his drawing technique will improve, although it may never be perfect, but the *meaning* of what he sees in pictures will get more sophisticated.

In geometry, children begin by recognizing shapes perceptually, and sense their similarities and differences. They are told about circles, squares, triangles, rectangles and so on. Imagery is initially based on perceptual clues in the form of specific visual prototypes. For instance,  and  could be prototypes for a rectangle, with  a prototype square and  a prototype circle. For a child at an early stage, this is a rectangle:  but this:  is not (because it is a square). This:  may not be a square (because it is a “diamond”, or because it is not “square” with the paper).

As the child’s cognitive apparatus matures and language develops, it becomes possible to talk about such prototypes; for instance, a rectangle has four sides, all its angles are right angles and the opposite sides are equal. A square has *all* four sides equal, so it is different from a square. Names fit visual prototypes and verbal descriptions, and they are not initially hierarchical.

Language becomes more sophisticated and conceptualisations begin to allow hierarchies to be developed. For instance, a square has all angles right angles and four equal sides, so in particular its opposite sides are equal and it is also a rectangle.

As the descriptions become more firmly meaningful, deductions of various kinds become possible. Given a paper triangle with two sides equal, if it is folded along the line through the top vertex and the middle of the base, then, physically, the sides match, so the base angles match as well. In this way, for some children, definition and deduction comes into play: *if* I know this is true (that the triangle has equal sides) *then* I know that something else is true (the triangle has equal base angles). In the later parts of secondary school, this can lay the basis for more sophisticated Euclidean proof.

Geometric shapes develop more sophisticated meanings. Initially a straight line is something drawn with a pencil and has thickness. Later it represents something that is *perfectly* straight, with no width, but with length that can be extended as desired. The child begins with a physical, perceptual view of geometric figures and may, over a period of years construct a perfect platonic figure in the imagination. (Even Plato was unable to speak when he was born and he had his own development to Platonism). Proof plays its role in the Euclidean geometry of perfect platonic figures (though it can also operate at the physical level too).

A further development is possible in which axioms are written down for a geometrical context and formal deductions are made in a logical context (figure 6). This level is only met by experts, but it is an advantage for even a teacher of young children to have some insight into the development from perception through the increasing sequence of growing sophistication to formal proof.

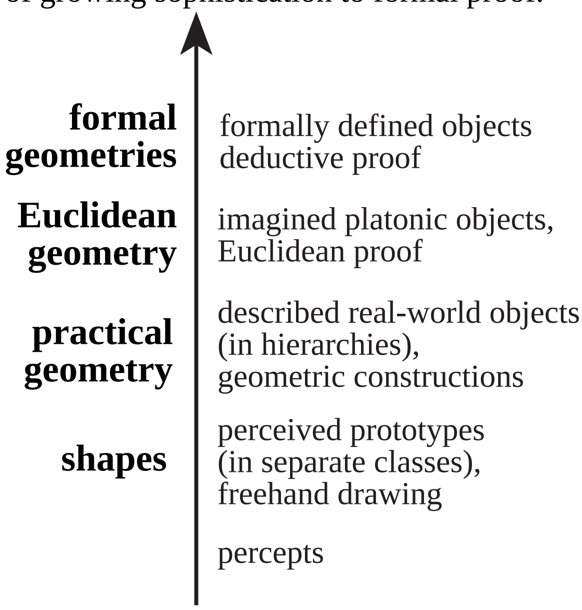


Figure 4: cognitive development of geometrical concepts (Tall et al, 2001)

Competence and Confidence in Teachers of Primary Mathematics

In preparing students to be teachers of primary mathematics, I have advocated that they need to have a real insight into how mathematics develops *cognitively*. This means more than how to *do* mathematics, it requires more than a sophisticated reasoning *why* it works. It means an awareness of how children do mathematics in different ways and why they encounter conflicts and difficulties. Such a study is more than refreshing one's knowledge of school mathematics in the same way that it was formerly learnt by the individual. It is more than simply covering more modules in more advanced mathematics. It means starting to reflect on one's own experiences to see *why* certain things were difficult, or even impossible, at the time.

Such a strategy can be assisted by re-thinking how arithmetic works at the simple and profound manner of Li Ping Ma. This requires time and sharing of ideas, not only during a training course but on a continuing basis throughout one's career, so that the new teacher can be inducted into the culture of helping children learn and the experienced teacher can grow reflectively through a life-time of improvement.

In summary, competence in being able to do school mathematics and confidence in doing it well are essential for the teacher of mathematics. However, confidence and competence in *teaching* young children requires more. We have identified a bifurcation in mathematical growth in which more sophisticated thinkers have built-in engines to support even greater sophistication whilst slower children use isolated procedures that work for a time but fail to generalise. Coping with this conundrum and developing methods to help all children in suitable ways continues to be a focus of study for us all.

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