What do we "see" in geometric pictures?
(the case of the blancmange function)

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In geometry, and more generally in pictures representing mathematical ideas, it is important not only to look at the picture, but also to know of the context it is drawn in. Here, for instance is a freehand sketch representing a tangent touching a circle.

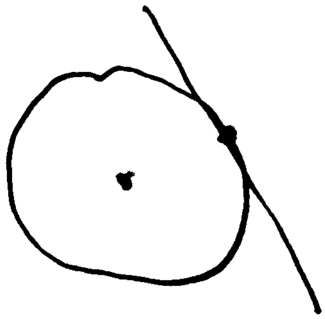

Figure 1: A tangent touching a circle

It does not look like a very accurate representation, indeed, the circle is not circular, the line is not straight and the line doesn't "touch" the circle, yet in some way, a sketch such as this drawn on a piece of paper or a blackboard can provide the basis of a mathematical discussion.

In *A Mathematician's Apology*, the well-known Cambridge Mathematician, G. H. Hardy addressed this question as follows:

> Let us suppose that I am giving a lecture on some system of geometry, such as ordinary Euclidean geometry, and that I draw figures on the blackboard to stimulate the imagination of my audience, rough drawings of straight lines or circles or ellipses. It is plain … that the truth of the theorems which I prove is in no way affected by the quality of my drawings. Their function is merely to bring home my meaning to my hearers, and, if I can do that, there would be gain in having them redrawn by the most skilful draughtsman. They are pedagogical illustrations, not part of the real subject matter of the lecture. (Hardy, 1940/1967, p. 125)

What matters therefore, is not the picture that is drawn in a physical representation, but the picture that we imagine in our mind. Let us use this idea to travel on a flight of fancy. Here are two pictures we have drawn of certain graphs using a computer graph plotter. Without us telling you the formulae we have in mind, can you tell us which graph is "smooth", (in formal terms, which one is everywhere differentiable) and which one is not. In informal terms, you

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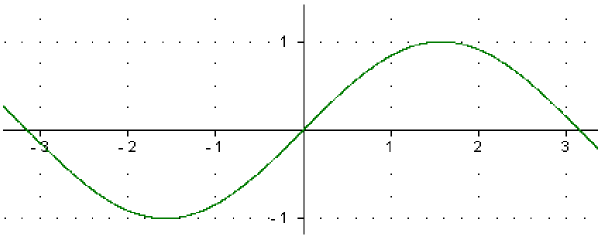
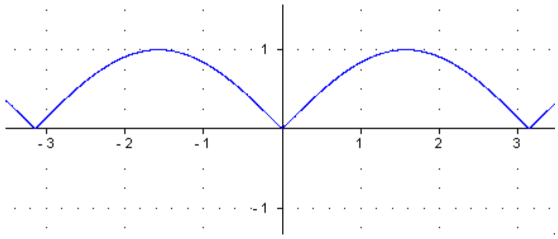
might think about which one might not be smooth, perhaps because it has “corners” where the graph suddenly changes direction.



Graph (a)



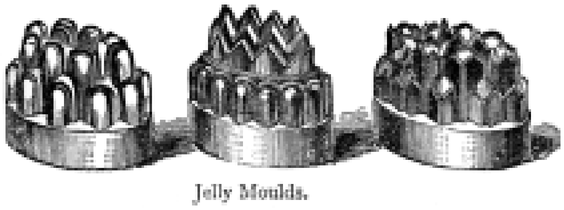
Graph (b)

Figure 2 : Which of these graphs is "smooth"?

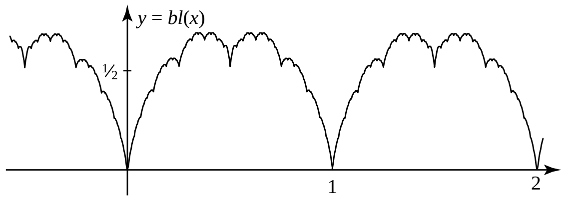
Immediately you are likely to say “the first one looks like sin$x$, so it is clearly smooth”. The second may take a little longer to suggest that perhaps it is the graph of the absolute value of sin$x$, found by reflecting the negative bits below the x-axis up above to give a curve; this is smooth in most places, but clearly has corners at 0, $\pm\pi$, $\pm 2\pi$, and so on. This is a very sensible interpretation, given the information that you have available.

Introducing a very wrinkled function

The “sensible” interpretation, however, depends on the fact that those drawing the graph have used the regular sine curve that we all know. The only information we have given you is a single picture of the graph of each curve; we never told you the formula for either function, so you are just guessing from what you can see. We have a much more devious idea in mind and we will now begin to unfold it. Let us begin by introducing you to a function you may never have seen before, called the *blancmange* function. This name was suggested by John Mills, one of our colleagues, who saw a similarity in shape to an English pudding made of flavoured milk jelly. Figure 3 (taken from a late nineteenth century Victorian cook-book entitled *Warne's Every-day Cookery*) shows three different moulds for making jellies. Beside it is the blancmange function, showing the family resemblance. We should warn you however, that while blancmange puddings seem moderately smooth (and sticky) to the touch, the blancmange function is extremely wrinkled in a very subtle way.



Jelly moulds for making blancmanges



The blancmange function

Figure 3: The origin of the name of the "blancmange"

The joke here is that the English word “blancmange” is made up of two French words “blanc” (white) and “mange” (eat). When the blancmange function was first introduced to French academics and teachers in a Seminar in Paris, the audience were very puzzled. “What is this ‘white-eat’?” they said. The origin of the name was explained and the French were satisfied. It is what the French call a “pudding”. As always the French and the English have some difficulty understanding each other, in this case because the English use a French term “blanc-mange” and the French use an English term “pudding”.

So what is this fabled “blancmange” function? It is a function whose construction was specified by the Japanese mathematician Takagi in 1904. Perhaps we should call it the “Takagi” function after him. Perhaps … But the English-French word “Blancmange” which causes confusion in both countries is so much more fun! To continue with the pudding metaphor, we should think of the ingredients that make it up.

The recipe for making a blancmange

The blancmange function is built up in stages. The first stage is a saw-tooth, which rises linearly to ½ and falls to 0 in every unit interval (figure 3). This function could be called “the distance to the nearest integer”. It may also be calculated for any x as follows:

- calculate the integer part of x, which is the largest integer n such that $n \le x$,
- calculate the decimal part $d = x - n$,
- if $d \le \frac{1}{2}$ then $s(x) = d$, otherwise (if $d > \frac{1}{2}$) then $s(x) = 1 - d$.

For instance, if x=3.23, then n=3 and d=0.23 and, since $d \le \frac{1}{2}$, this gives $s(x)$=0.23. On the other hand, if x=2.75, then n=2 and d=0.75 and now $d > \frac{1}{2}$, so $s(x)=1-d$ = 0.25.

This procedural definition may look strange to those only used to having functions given by formulae involving polynomials, trigonometric functions, and such like, but it is much easier than any of these to define in a computer program. In this sense, in these days of computers, this “saw-tooth” is a very natural function (Figure 4.)

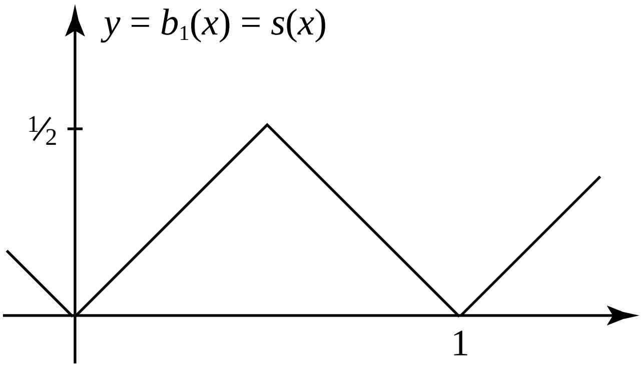


Figure 4: The saw-tooth, y=s(x)

This is the first approximation to the blancmange function.

$b_1(x) = s(x)$.

Next we take a half-size saw-tooth $s_2(x)$, which rises to 1/4 and falls to 0 twice in each unit interval (figure 5). This can be calculated by the formula:

$s_2(x) = \frac{1}{2}s(2x)$.

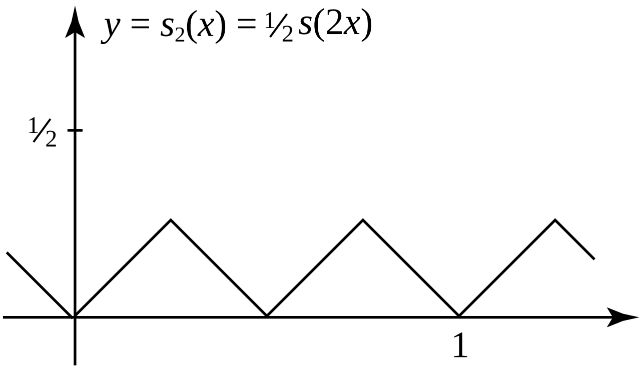


Figure 5: a half-size saw-tooth

Adding them together gives the second approximation $b_2(x)$ where

$$b_2(x) = s(x) + s_1(x)$$
$$= s(x) + \frac{1}{2}s(2x).$$

(figure 6).

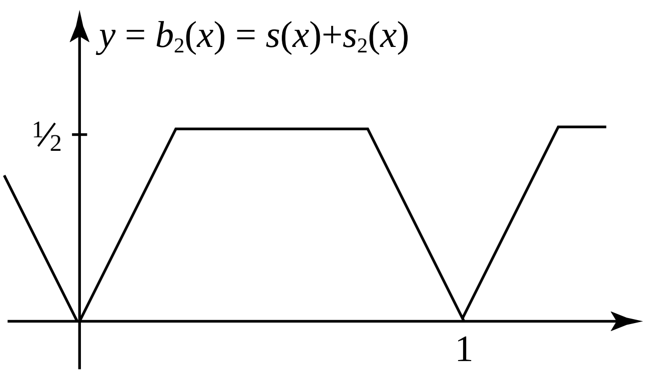


Figure 6: the sum of a sawtooth and a half-size sawtooth

We next take a quarter-size sawtooth (figure 7) and add it to $b_2(x)$ to get the third approximation (figure 8).

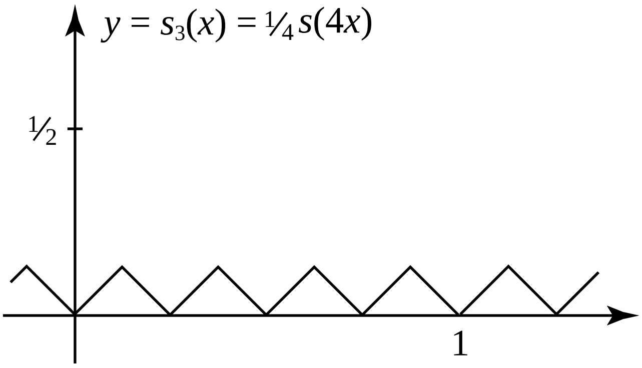


figure 7: a quarter-size sawtooth

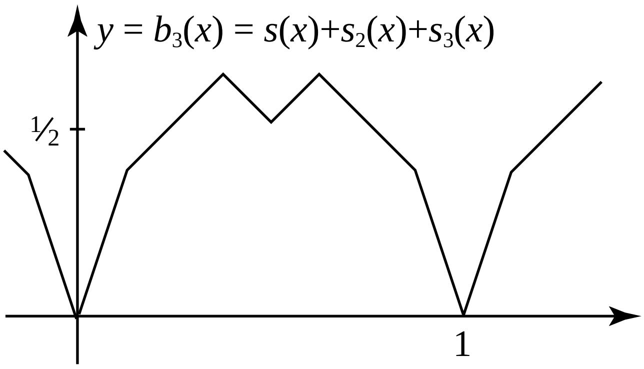


figure 8: adding a quarter-size sawtooth to $b_2(x)$ to give $b_3(x)$

We proceed in this manner, each time taking a sawtooth half the previous size and adding it to the approximation to get a better approximation to the blancmange. We may imagine this going on for ever, but in a practical situation, such as a drawing on a TV monitor, the added saw-teeth soon get so small that the picture stabilizes (after eight or so stages on most modern VDUs). (Figure 9.)

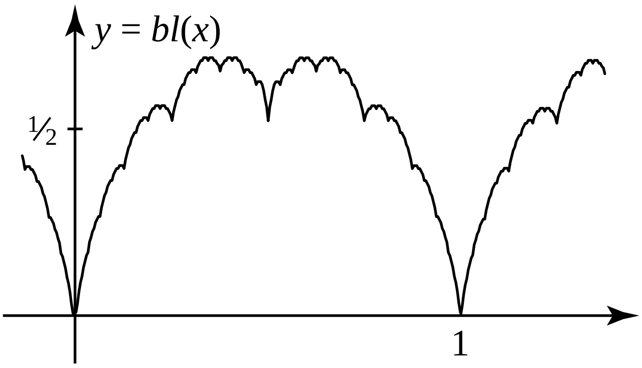


Figure 9: the blancmange function

The blancmange function is continuous but nowhere differentiable

It is easy to show that the blancmange function is continuous. Since $bl(x)$ is the sum of saw-teeth

$$bl(x) = s(x) + s_2(x) + \ldots + s_n(x) + \ldots$$

where

$0 \le s(x) \le \tfrac{1}{2}$ and, more generally, $0 \le s_n(x) \le \tfrac{1}{2^{n+1}}$,

we see that the approximations

$$b_n(x) = s(x) + s_2(x) + \ldots + s_n(x)$$

satisfy

$$0 \le b_n(x) \le \tfrac{1}{2} + \tfrac{1}{4} + \ldots + \tfrac{1}{2^{n-1}} \le 1$$

so that the sequence $b_1(x)$, $b_2(x)$, ... is an increasing sequence bounded above by 1 and therefore tends to a limit less than one.

More than that, the difference between the approximations, $b_m(x) - b_n(x)$ where $m>n$, satisfies

$$\begin{aligned} b_m(x) - b_n(x) &\le \tfrac{1}{2^n} + \ldots + \tfrac{1}{2^{m-1}} \\ &= \tfrac{1}{2^n}\left(1 + \ldots + \tfrac{1}{2^{m-n-1}}\right) \\ &\le \tfrac{1}{2^{n+1}} \end{aligned}$$

so the convergence is *uniform*, and the limit $bl(x)$ — being the uniform limit of continuous functions — is *continuous*.

However, it is a very wrinkled kind of continuous function that doesn't look smooth anywhere. When the blancmange function is magnified, a strange thing happens: tiny blancmanges can be seen all over the place!

To understand why this happens, consider what would occur if we left out the first sawtooth and added up the rest. We would start at a half-size sawtooth and successively add sawteeth each half the previous size. The result would be the same shape as before, but would be only half the size. It would give *a half-size blancmange* (figure 10). (The formula for the half-size blancmange is simply $\tfrac{1}{2}bl(2x)$.)

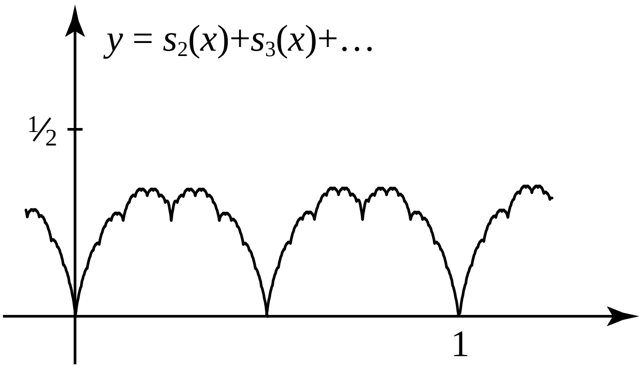


Figure 10: a half-size blancmange

Adding the first saw-tooth to this sum, reveals the full blancmange as the sum of the first sawtooth and a half-size blancmange.

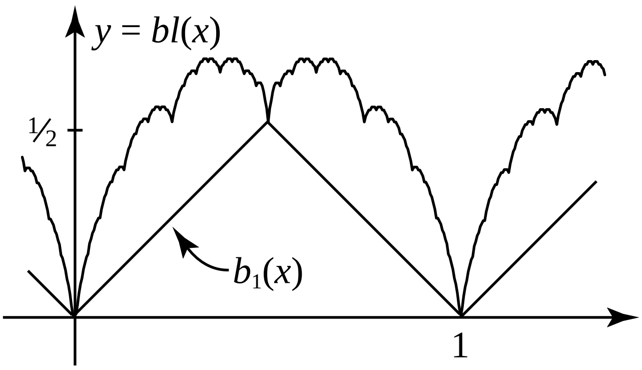


Figure 11: The blancmange function as the first approximation $b_1(x)$ plus a half-sawtooth

In this case the half-size blancmanges are set on each side of a saw-tooth, sheared upwards. It is as if each half-size blancmange is on a flat plate turned through an angle of 45°. Instead of sliding off the plate like blancmanges made out of jelly, it is sheared only in the vertical direction by an amount increasing from zero at the lowest part of the tooth to ½ at the highest part (figure 12).

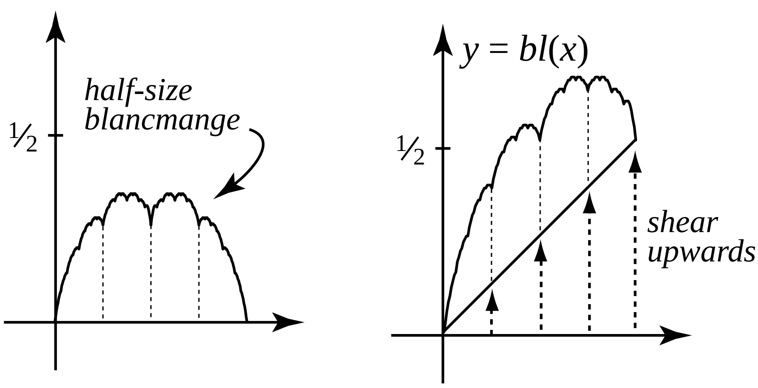


Figure 12: shearing a half-size blancmange

The idea may become clearer when we move to the next tooth. Leaving out the sum of the first two teeth $b_2(x) = s(x) + s_2(x)$ shown in figure 6, we start to add up the rest of the teeth starting at the third (quarter-size) sawtooth and adding all the succeeding saw-teeth. This gives *a quarter-size blancmange*. The full blancmange is then revealed as the sum of the first two teeth $b_2(x)$ plus a quarter-size blancmange (figure 13).

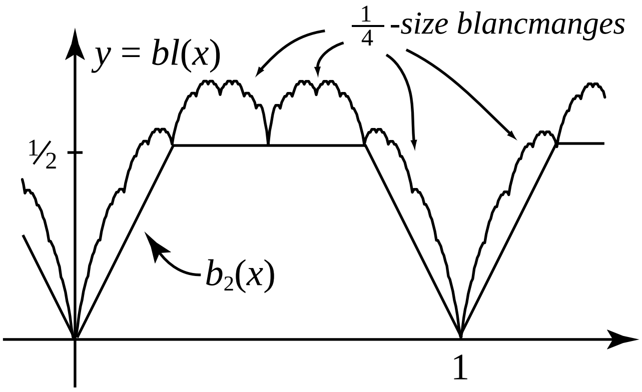


Figure 13: The blancmange showing quarter-size blancmanges

In general

$$bl(x) = b_n(x) + 1/2^n\text{-size blancmange.}$$

Each approximation $b_n(x)$ is made up of straight-line segments over each x-interval of length $1/2^n$. Over each interval the $1/2^n$-size blancmange is added to give the full blancmange function. Figure 14 shows $b_4(x)$ superimposed on the blancmange function. There are tiny blancmanges clearly visible on each horizontal portion of $b_4(x)$ and on the other portions the blancmange is sheared to sit on the line-segment.

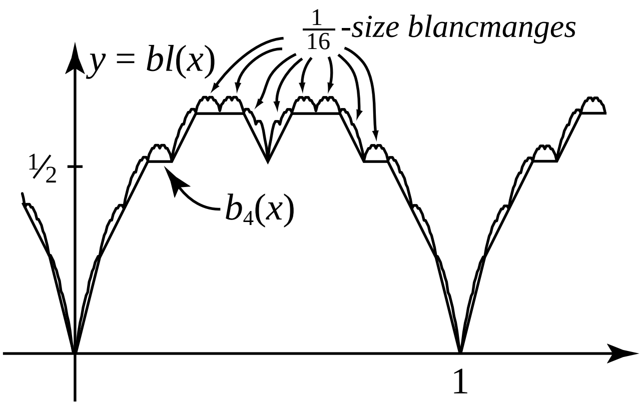


Figure 14: blancmanges growing everywhere!

Thus over any segment of length $1/2^n$ there is a tiny $1/2^n$-size blancmange function. It is the supreme non-example. It is a function that *never* looks smooth anywhere. In technical terms, it is a continuous function that is *nowhere* differentiable.

The best time to introduce such a function to students is a matter of personal taste for the teacher. Many may consider it too difficult for weaker pupils at an early stage but my inclination would be to approach it experimentally fairly early on using a prepared piece of software which enables the user to magnify any part of the graph. Though the *technicalities* are difficult, the idea of a wrinkled graph is not. It helps students realize the special nature of those smoother graphs that look straight when small portions are highly magnified. This highly specialized property is the foundation of the differential calculus. (Tall, 1982, 1985.)

Some surprising ideas

The blancmange function has surprising properties. Being nowhere differentiable is strange for a start. But, by the very way it is defined, it is clearly symmetric in each unit interval on either side of the vertical line through the middle of the interval. Since it visibly has a maximum value, it has (at least) two, symmetrically placed on either side of its centre. It is actually more interesting still. Looking back at figure 14 you can see a number of little blancmanges at the top of the graph and each of these has at least two maxima. Zooming in and looking a little closer you will see many, many maxima. In fact the blancmange function has an *infinite* number of maxima in each unit interval. Because it is not differentiable, it is not possible to find these maxima by differentiating and finding where the derivative is zero. Perhaps you might like to think where these maxima are and how high the blancmange function is at these points.

Another trick is to consider a small-scale blancmange that is one thousandth the size. It can be given by the formula:

$$r(x) = \frac{bl(1000x)}{1000}.$$

Its height lies between 0 and $1/1000$. If you draw this function to a normal scale on a piece of paper, it is too small to distinguish from the x-axis since it is less than $1/1000$ of a unit high. (Figure 15).

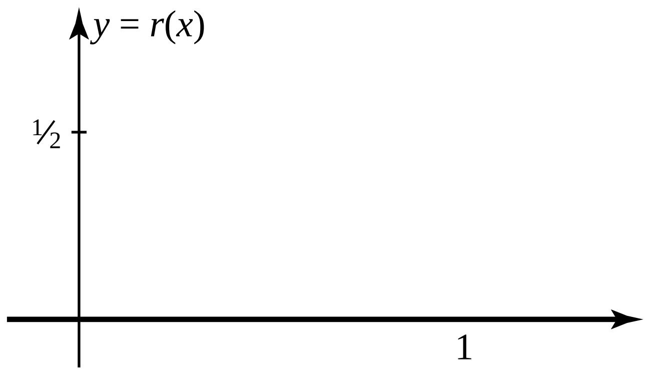


Figure 15: The graph of the tiny "rough" function, $r(x)$

But if you zoom in by a factor of 1000 or so, you will see the tiny blancmanges appear (figure 16).

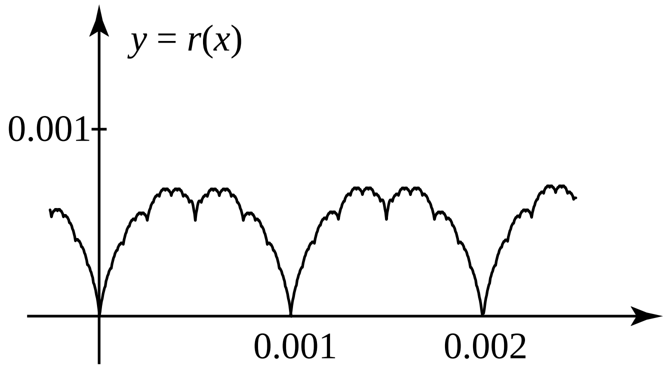


Figure 16: The graph of the "rough" function, $r(x)$, highly magnified

Now let us take *any* function *f(x)* which has a derivative, such as $f(x) = x^2$ or $f(x) = \sin x$, then the graph of $g(x) = f(x) + r(x)$ cannot have a derivative anywhere. (Because, if it did, then $r(x) = g(x) - f(x)$ would have derivative $g'(x) - f'(x)$, and we already know that *r(x)* is so wrinkled that it is nowhere differentiable.) The two graphs of $f(x)$ and $f(x)+r(x)$, however, look the same to a normal scale (figure 17, 18).

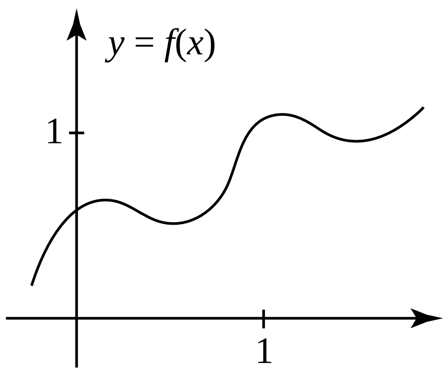


Fig. 17: An everywhere differentiable function

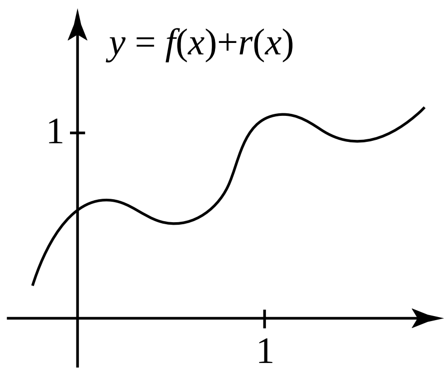


Fig. 18: A nowhere differentiable function

This is an amazing insight. Two graphs *f*(x) and g(*x*) differ by less than a $\frac{1}{1000}$ of a unit so that they cannot be distinguished in a picture drawn to a normal scale. Yet one is differentiable everywhere and one is differentiable nowhere! This means that by looking at a single picture alone, without having any other information, *you cannot see whether the graph represented is differentiable or not!* Of course, if you have the facility to zoom in on a computer drawn graph, the difference may appear at some higher magnification, but you can never be sure. What happens if a much tinier rough graph were added, say the graph of $10^{-10^{100}} bl(10^{100} x)$? How can you ever be sure that at some unfathomably small size a tiny wrinkle may not be added to the curve? The only way to be sure is to know precisely what the function is. A single picture is not enough.

Looking at the Original pictures

Now it is time to look at the pictures at the beginning of the article. The first may look like sin*x*, but what we did not tell you is that its formula is $\sin x + \mathrm{r}(\mathrm{x})$. So when it is magnified a thousand times or so, it reveals the rough contours of the tiny blancmanges growing everywhere. So, contrary to expectations, graph (a) is not smooth at all, it is not smooth *anywhere*.

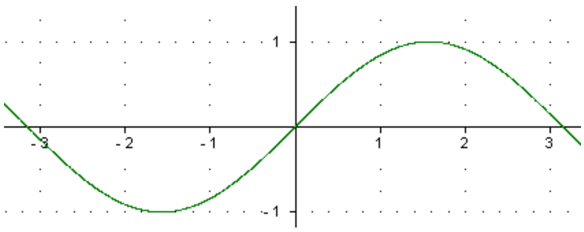


Figure 19: Graph (a), y=sin(x)+r(x)

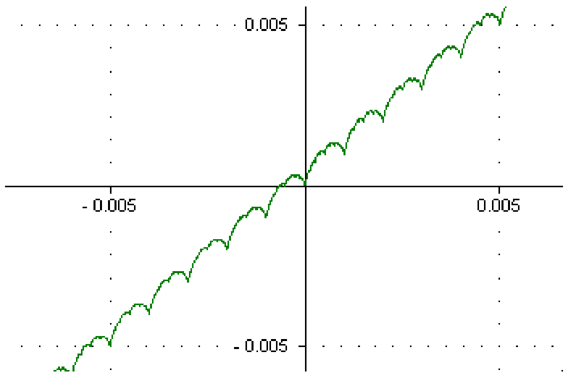


Figure 20: zooming in on graph (a)

Now you may be getting suspicious about graph (b). You are right to do so. We cheated you again. This is *not* the graph of the absolute value of sin*x*, which may be written as

$$\begin{aligned} y &= |\sin(x)| \\ &= \sqrt{\sin^2(x)}. \end{aligned}$$

If you were to zoom in on that graph, then it *would* have a corner at the origin (figure 21).

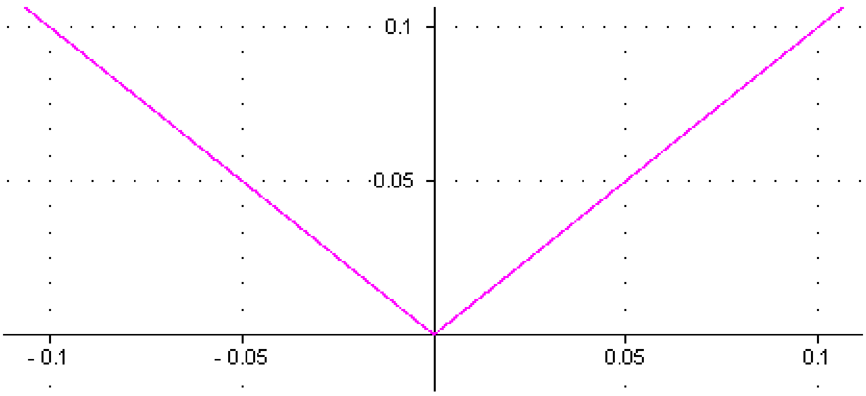


Figure 21: Zooming in on the graph of y=|sinx| at the origin

But we did not draw this graph. What we actually drew is nearly the same, but not quite. Our graph (b) has the formula

$$y=\sqrt{\sin^2(x)+0\cdot 000001}.$$

Zooming in on $y=\sqrt{\sin^2(x)+0\cdot 000001}$ where x=0 gives

$$y=\sqrt{0\cdot 000001}=0\cdot 001$$

and the magnified graph is given in figure 23.

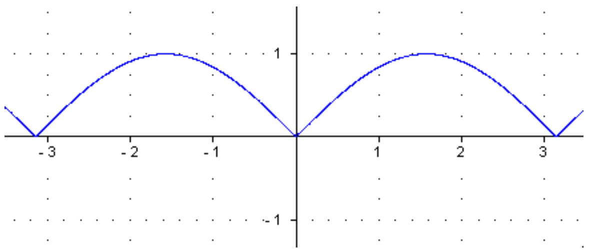


Fig. 22: Graph (b) $y=\sqrt{\sin^2(x)+0\cdot 000001}$

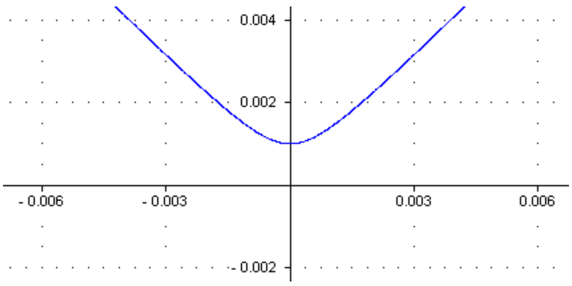


Fig. 23: Zooming in on graph (b) at the origin

So graph (b) *has no corner at all*. It is quite smooth. In fact, it is a formula made up of standard functions and can be differentiated in the normal way.

This discussion has shown that we cannot expect to interpret a graph from a single picture alone unless we are sure of precisely what it represents. Our intuition may tell us that the original graph (a) looked as if it had corners and graph (b) looked smooth, but now we know precisely what the pictures represent, we see that it is almost the reverse. Graph (a) is indeed smooth everywhere, and graph (b) is so rough that it is not smooth anywhere. Don't be fooled in future. If you are told to look at a picture and not told clearly what it represents then your intuition may mislead you. Experience is a great trainer, but in mathematics you need the kind of experience which clearly tells you the full meaning of the situation.

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Note: The papers of David Tall are available as downloads from: http://www.warwick.ac.uk/staff/David.Tall.