Operable Definitions in Advanced Mathematics: The Case of the Least Upper Bound

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This paper studies the cognitive demands made on students encountering the systematic development of a formal theory for the first time. We focus on the meaning and usage of definitions and whether they are "operable" for the individual in the sense that the student can focus on the properties required to make appropriate logical deductions in proofs. By interviewing students at intervals as they attend a 20 week university lecture course in Analysis, we build a picture of the development of the notion of least upper bound in different individuals, from its first introduction to its use in more sophisticated notions such as the existence of the Riemann integral of a continuous function. We find that the struggle to make definitions operable can mean that some students meet concepts at a stage when the cognitive demands are too great for them to succeed, others never have operable definitions, relying only on earlier experiences and inoperable concept images, whilst occasionally a concept without an operable definition can be applied in a proof by using imagery that happens to give the necessary information required in the proof.

Mathematicians have long "known" that students "need time" to come to terms with subtle defined concepts such as limit, completeness, and the role of proof. Many studies have highlighted cognitive difficulties in these areas (e.g. Tall & Vinner, 1981, Davis & Vinner, 1986, Williams, 1991, Tall, 1992). Some authors (e.g. Dubinsky et al, 1988) have focused on the role of quantifiers. Barnard (1995) revealed the subtle variety in students' interpretations of statements involving quantifiers and negation. Nardi (1996) followed the development of university students' mathematical thinking by observing and audio-taping their first year tutorials. This highlighted the tension between verbal/explanatory expression and formal proof, and tensions caused in proofs by quoting results of other theorems without proof. Vinner (1991) drew attention to two modes of use of definition – the everyday, and the technical mode required in formal reasoning. In this paper we report a longitudinal study of the individual developments of students in their first encounter with a formal mathematical theory, to see the growth of their use of definitions in building concepts and proving theorems.

We initially formulated the following working definition:

> A (mathematical) definition or theorem is said to be *formally operable* for a given individual if that individual is able to use it in creating or (meaningfully) reproducing a formal argument.

The intention of the study is to track the construction of the concept of least upper bound, and related concepts such as continuity, to see how (or if) operability develops over time and how this relates to the use of the concepts in later theorems such as the existence of the Riemann integral.

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Methodology

Five students were selected to be interviewed (and video-taped) on five occasions during a twenty week Analysis course consisting of sixty one-hour lectures. This followed a long-established syllabus in which the axioms for a complete ordered field were given in the third lecture (using the least upper bound form of the completeness axiom) followed by the convergence of sequences and series, the continuity and differentiability of real functions, the Riemann integral and the Fundamental Theorem of Calculus.

The interviews were semi-structured, in that the same lists of questions were used with each student but were then followed up in response to the students' answers. Students were invited to speak about their experiences of the course and beliefs and attitudes, as well as answering more directly content-related questions. Each interview tried to capture the state of development of the student at that time, focusing initially on recently covered work, then checking on the longer-term development of selected fundamental conceptions including the notion of least upper bound.

Of the five students, Lucy, Matthew and Martin were mathematics majors whilst Alex and Sean were physics majors taking several mathematics courses, including Analysis. Lucy proved to be the most successful of the five; she was invariably able to talk coherently about concepts and theorems but did not memorise definitions, preferring to draft what she knew on paper and then refine her ideas. Matthew worked very hard, attempting to commit material to memory by repeated readings; when explaining things he would sometimes break down and then need to "refresh his memory" by looking at his notes. Alex missed more lectures than the others and was not always conscientious in copying up the notes; he later changed courses without taking the end of year exam, nevertheless he had certain ways of operating which will be central to the discussion which follows. Both Martin and Sean found the requirement for formality bewildering and were unsuccessful in their examinations. We therefore choose to focus on the work of Alex and contrast this with the more successful Lucy and the less successful Sean.

First encounters with the definition of a least upper bound

In the third lecture, the following definitions were given:

> An *upper bound* for a subset $A \subset \mathbb{R}$ is a number $K \in \mathbb{R}$ such that $a \leq K\ \forall a \in A$.
>
> A number $L \in \mathbb{R}$ is a *least upper bound* if L is an upper bound and each upper bound K satisfies $L \leq K$.

In the early interviews, all the students showed that they could give the concept some kind of meaning, varying in the relationship to the formal definition. Lucy and Alex focused on the second part of the definition which does not explicitly define the notion of upper bound whilst Sean used his own imagery:

> *Lucy*: Well, say *k* is an upper bound for the set, then we'll say that *m* [the least upper bound] is less than or equal to *k*.
>
> *Alex*: A least upper bound is the lowest number … that is an upper bound. Any number greater than it, no matter how little amount by, it's not going to be, you know it's not going to be, in the set.
>
> *Sean*: The supremum of a set is the highest number in the set.

Lucy and her struggle giving meaning to the least upper bound

By the second interview Lucy was able to verbalise the definition of least upper bound in a manner close to the symbolic form:

> *Interviewer*: If I asked you what was a least upper bound what would you say now? What properties would you say that that's got?
>
> *Lucy*: Well for a start it has to be an upper bound.
>
> *Interviewer*: Right so what does that mean?
>
> *Lucy*: An upper bound for a set *S*, if you take any element of S to be *a*, say, and for all *a* you can find, say the upper bound was *k*, for all *a*, *k* will be greater than or equal to *a* for any number in that set.
>
> *Interviewer*: So that's the definition for *k* being an upper bound.
>
> *Lucy*: … and the least upper bound is also an upper bound but it's the least of all the upper bounds so *l* has to be less than or equal to *k* for all *k* greater than or equal to *a*.

In the fourth interview she is very confident expressing the definition verbally:

> Well it's got to be an upper bound itself and it's got to be the least of all the upper bounds.

But even in the fifth interview, when asked to write down the definition of the least upper bound of a non-empty set $S \subset \mathbb{R}$, she wrote:

> $\forall\, s \in S,\ s \leq \mu$ [saying "μ is an upper bound"]
>
> $\forall\, k \in \mathbb{R}$ s. t. $s \leq k$ and $\mu \leq k$,

After a discussion she modified the last part to "$\forall s \in S,\ s \leq k \Rightarrow \mu \leq k$." Although she had most of the component parts of the definition, she still needed to negotiate the details.

Sean's concentration on his earlier experiences

Sean continued to have difficulty with the concept of least upper bound, as well as the definition, throughout the course.

> *Sean (Interview 2)*: [you get the supremum by] looking at all the elements of the set to find out which is the greatest and choosing that number. I always have trouble remembering whether the supremum has to be in the set.
>
> *Interview 4*: It's the greatest number of … it's a number that's bigger than all the numbers in the set.
>
> *Interview 5*: The set {1, 2, 3} has upper bound 3. [Is 7 an upper bound?] No, it's not in the set.

In the second interview when asked for the least upper bound of the set *S* of real numbers *x* where $x<1$, he suggests "a very small number subtracted from one" or "nought point nine nine nine recurring", thereby maintaining his (erroneous) belief that the least upper bound is in the set. In the fifth interview he is quite articulate about his struggle:

> … when we have theorems in analysis lectures, stuff like supremums are just the basic workings; since I can only just understand these individually, one of these basic foundations, I can't look at all of them together and understand the theorem.

It is a classic case of cognitive overload. However, not only does he seem to have too many things to think about, the individual items not only lack the precision of operable definitions, they seem diffuse and difficult for him to grasp as manipulable mental entities.

Alex eventually learns the definition with apparent meaning

Alex is somewhat erratic in attending lectures so he does not get all the information from the course that he should. In interview 2 he explains an upper bound L for a set S, saying:

> *Alex*: there exists L such that L is greater than or equal to max S – what do you call it – the greatest number in S.
>
> *Interviewer*: Has S got a maximum number?
>
> *Alex*: Yes.

In the third interview he seems to become entangled with the completeness of the reals:

> [the least upper bound is] not necessarily in the set – well it depends – if your subset is a subset of the reals then it's going to be a real number, in which case it's going to be in the set but if you're talking about a subset of the rational numbers, it's not necessarily in it, but it's the lowest number that is an upper bound.

Yet, in the fifth interview he suddenly offers the full definition, writing:

> $l \leq s \ \forall \ s \in S$ [saying "l is an upper bound"]
>
> If u is an upper bound [pointing to the previous line, saying "satisfies this as well"] then $s \leq u$.

He explained that, since it arose several times in the interviews, he decided to *learn* it.

The Definition of Continuity

When the definition of continuity is given, the students have already met a succession of ideas including convergence of sequences and series. Lucy works at making sense of each new idea without always having the space to absorb earlier detail. For instance, when studying convergence of series she has no recollection of the definition of convergence of a sequence but, when asked for the definition of convergence of a series, she responds with a precise formal definition of Cauchy convergence of its partial sums. Although she tries to memorise the definition of continuity, it "would not stick", and she builds it up from its parts (sometimes by visualising the page of notes in her imagination). By the fifth interview she is fairly fluent with the definition of continuity and when asked to explain the definition of f being continuous on $[a, b]$ she constructs her own version in steps, writing the following down in order (1), (2), (3):

$$\underbrace{\forall x, x_0 \in [a,b]}_{(2)} \quad \underbrace{\forall \varepsilon > 0 \ \exists \delta > 0 \ \text{s.t.}}_{(1)} \underbrace{|x - x_0| < \delta \Rightarrow | f(x) - f(x_0)| < \varepsilon}_{(3)}.$$

Further conversation leads to this being modified as she explains:

> I write down everything and say, "no that's wrong", and then I work backwards.

Neither Sean nor Alex ever come to terms with the formal definition of continuity.

> *Sean (interview 4)*: It means the function at that point minus the function at some other point very near it is less than or equal to the epsilon.
>
> *Sean (interview 5)*: I can remember just about the definition of continuous but I tend not to use it and still think of continuity as drawing the graph without taking your pencil off the paper.
>
> *Alex (interview 5)*: It's a line you can draw without taking your finger off the board.

The Riemann Integral

The definition of Riemann integral $\int_a^b f$ of a (bounded) function f is introduced later in the course. The interval $[a,b]$ is partitioned into what (in the course) is called a *dissection*, D, consisting of sub-intervals $a = x_0 < x_1 < \ldots < x_n = b$. From the definition of least upper bound it is deduced that on any interval a bounded function has a supremum (least upper bound) and an infimum (greatest lower bound). Using the supremum and infimum of the function in each sub-interval to give an upper and lower rectangle allows the upper sum $U(f,D)$ and lower sum $L(f, D)$ to be computed, as the total area of upper and lower rectangles, respectively. Proving that $L(f, D_1) \leq U(f, D_2)$ for any dissections D_1, D_2 shows that the set of lower sums has a least upper bound L and the set of upper sums has a greatest lower bound U. If $L=U$, then the Riemann integral is defined to be $\int_a^U f = L = U$. If f is continuous, then it is proved that $L=U$ so the Riemann integral exists.

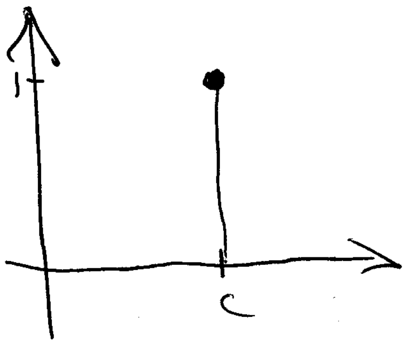
When the three students meet this sequence of theory, Lucy is already able to speak about both the definitions of least upper bound and continuous function and write them down (though sometimes with errors), Alex can write down one but not the other and Sean can write neither. So how do they cope with the definition of Riemann integral?

Lucy is able to discuss it intelligently, though needing occasional assistance. For instance, she says that a dissection of an interval consists of “lots of little bits” and, after a suitable notation is suggested, is able to describe both upper and lower sums, and upper and lower integrals as “the inf and sup of the upper and lower sums.” She has a broad grasp of the overall framework including most definitions and statements of theorems, such as “a continuous function on a closed interval is bounded and attains its bounds”, although she is still “trying to understand it at the moment but I haven’t got it quite.” However, she confidently explains why the function $f(x) = \begin{cases} 1, & x = c \\ 0, & x \neq c \end{cases}$ has a Riemann integral in terms of a zero lower sum and an arbitrarily small positive upper sum.

In the final interview, Sean claims no knowledge of the Riemann integral, saying:

> I can only say integrate using A level knowledge – I don’t know how you’d do it using the theorem.

After the interviewer has talked through the definition of the Riemann integral, Sean is asked about the integral of $f(x) = \begin{cases} 1, & x = c \\ 0, & x \neq c \end{cases}$. He draws a graph with a vertical line from $(c, 0)$ to $(c, 1)$ and turns to his notion of integration as the area under the graph explaining:



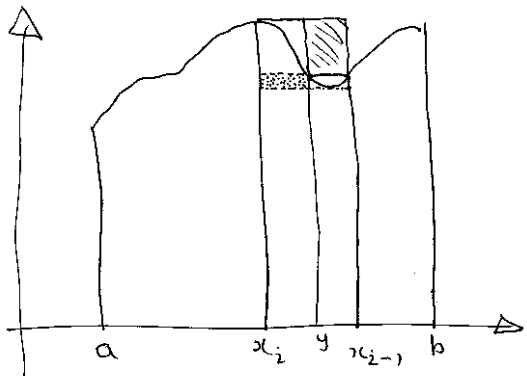
> there’s no way you can determine the thickness of that line and when you let things tend to zero you can’t account for that, so there’s no way you can work out the area in that bit.

When the interviewer offers a demonstration that this function is integrable in the Riemann sense by considering upper and lower sums, Sean responds:

> Ha – I can’t see any flaws in your logic, but I don’t like it. … Because if you try to define an area you say it’s something contained in this thing. … If you had a hole where there was nothing you could have great trouble finding the area at that point.

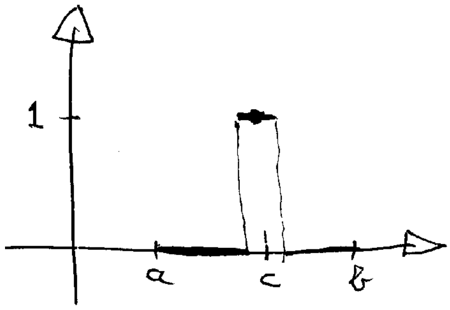
He has no operable definitions and no meaningful concept of proof from definitions. Instead he attempts to translate everything into his own intuitive terms and ends up with a diverse system of ideas that is just too unwieldy to make any sense.In contrast, Alex produces a most fluent account of the definition of Riemann integral. In particular he remembers and uses accurately the notations U(*f*, *D*) and L(*f*, *D*) introduced for the upper and lower sums for a function *f* and dissection *D*, and *U*, *L* for the supremum and infimum of the upper and lower sums. He is asked how to define the integral, and replies

> *Alex*: Well the integral is – oh dear – the integral is when that equals that [pointing to the symbols *L* and *U*] it's that becomes – oh – it's the supremum of the lower bounds and the infimum of the lower [sic] bounds become – no wait – he did write the integral sign with a lower thing for the lower integral – or something – which was the supremum of this ([pointing to *L*].
>
> *Interviewer* (aside): The supremum of all the *L*s – yes, fine.
>
> *Alex* (pressing on): … and you get your upper integral which is the infimum of your upper thing – so what happens is – like – the integral is when these two equal each other so you've got to like take more divisions, cos if you've got another division in there then – it's that thing where you've got *L* – if you've got D_1 which is a number of divisions here and you've got D_2 which just contains another one, then you've got $L(D_1) \le L(D_2) \le U(D_2) \le U(D_1)$.
>
> That's because you've now got another little partition up there (points to the diagram) so that plus – it's like, this isn't – this part and that part are now your upper, so you've lost some – whereas your lower has gone to there so you've actually gained some and so it goes closer and closer together, as you get more things until those are equal.



The status of the various components of Alex's discussion is intriguing. In the absence of an operable definition for continuity, he makes no formal mention of it or of any other property that would cause the upper and lower sums to be arbitrarily close. Using the diagram he sees the infimum and supremum as points on a ("continuous?") graph and imagines them becoming close as the intervals decrease in width. He *does* have a definition of least upper bound which is more formally operable and, by showing every upper sum exceeds every lower sum, he is clearly showing that the lower sums are bounded above and have a least upper bound, with the corresponding greatest lower bound for the lower sums. However, on occasions he seems to imagine variable upper and lower sums getting "closer and closer together." In this sense he is working with the *process* of moving towards a limiting value rather than using the definition of the limit concept.

When faced with the integral of $f(x) = \begin{cases} 1, & x = c \\ 0, & x \neq c \end{cases}$, he draws a picture and focuses on the limiting behaviour of upper and lower sum as the interval width tends to zero.



Interviewer: Now if you took your partition – your lower sum is always going to be zero and your upper sum is only going to differ from zero in the interval or two intervals that contains that. If you take these very small …

Alex: Yeah – if you're timesing that distance there by this height, it will disappear.

Interviewer: So do you think that's integrable or not?

Alex: That's not going to be integrable – no, that's just going to give us zero isn't it?

Interviewer: Is the upper sum going to be zero?

Alex: The upper sum's going to be – that is going to be integrable in that case because the lower sum is always going to be zero, and the upper sum is going to go to zero as that gets smaller.

This extract shows interesting uses of the present and future tense. The interviewer asks "do you think that's integrable or not?", intimating his view of the state of the function. The response "that's not going to be integrable" suggests a sense of process, perhaps relating more to the process of allowing the interval width to tend to zero.

Discussion

In defining the notion of "formally operable definition" we hoped to have a construct which enabled us to see instances of the successful use of definitions in theorems to build a systematic formal theory. Of the three students here, Sean claims to "only just understand the 'basic foundations' individually" and "can't look at them all together." Definitions for him are not operable and the ideas are so diffuse that he cannot comprehend them. He does not understand the notion of a definition being used to prove anything. Alex, a more sophisticated but erratic performer, has no formally operable definition of continuity, but he is eventually able to formulate the notion of least upper bound in an effective manner and use it operably in the definition of Riemann integral even though other ideas (such as continuity) are not formally operable. Lucy is more effective, but the definitions are not committed to memory, rather constructed and re-constructed in a struggle for coherent meaning. She "writes down what she knows" then says "that's not right" and "works backwards". Throughout the interviews, sometimes with the assistance of an ongoing dialogue, she is able to build impressive links between the materials, even when there are a significant number of gaps in them. During the lecture course, she often continues to have difficulty with proofs long after she has been presented with them. In other words, the "operability" of the definitions are for her an ongoing struggle. There are at least two distinct components of operability, the giving of meaning to the definition itself, not only through examples, but through the development of a range of strategies for its use in different theorems.

A telling difference between the students is the manner in which particular verbal expressions may be helpful or unhelpful in moving towards the formal definition from the very outset. Lucy's initial conversations about the least upper bound are well-targeted from the outset whereas Sean states that "it's the biggest element in the set." Alex has some aspects of both. When he interiorises the definition of least upper bound, it gives him the impetus to use the focused idea in the subtle construction of the notion of Riemann integral – a task which Sean is not even able to start.

We hypothesise that there is an important principle underlying this observation which is more than just the use of particular terms. If a student is focusing mainly on the essential properties in the definition then, in meeting new examples, there is the possibility of focusing *only* on these essentials, thus greatly reducing the cognitive strain. A more diffuse view of the possibilities means that successive examples may have a variety of extra detail that can cloud the issue. The former approach has prior focus on the "intersection" of the properties of the examples, the latter must sort out the important essentials from the "union" of the examples with their subtle irrelevancies that can lead to cognitive overload. This research has considered a mathematics lecture course "as it is". New research is required to see if an explicit focus on the use of properties in a definition can lead to a better comprehension of systematic proof. It is not just a matter of how quickly the ideas are encountered in a mathematics course, but of the individual's capacity to focus on the role of the ideas in the overall theory.

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