Mathematical Misconceptions and Music of the Spheres

Inaugural Lecture of Professor David Tall,
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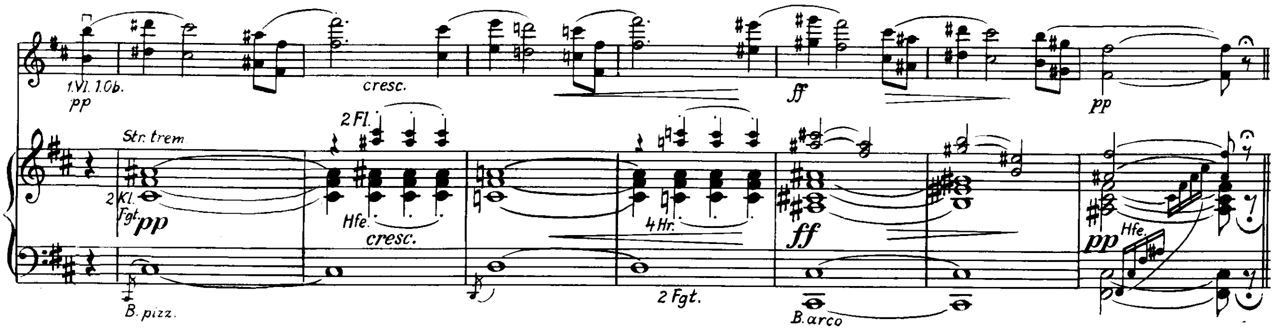
University of Warwick
March 22nd 1995

In his later years as Professor of Mathematics at Cambridge, G. H. Hardy wrote:

> The function of a mathematician is to *do* something, and not to talk about what he or other mathematicians have done. Statesmen despise publicists, painters despise art-critics, and physiologists, physicists, or mathematicians have usually similar feelings; there is no scorn more profound, on the whole more justifiable, than that of the 'men who make' for the 'men who explain'. Exposition, criticism, appreciation, is work for second-rate minds.
>
> (G. H. Hardy, *A Mathematician's Apology*, 1940, 2nd edn 1967, p. 61)

As a mathematics educator, I sense no need for an apology in attempting 'to explain' the thinking of mathematicians, for it is also our duty 'to make' theories about the nature of mathematical thinking. I see the vital role of understanding *how* experts and novices think mathematically and how this knowledge can be of value in teaching mathematics to subsequent generations. In particular I see the expert mathematical mind not as one always full of clarity and cool deduction, but one which is itself susceptible to human misconceptions.



Strauss, Sphärenklänge Walzer, opus 235, bars 14–20

The *Music of the Spheres* is a beautiful Strauss waltz which looks back on a forgotten world of elegance and refinement, in turn referring back to the Music of the Spheres with which the ancient Greeks used to describe the harmonies of their universe. The ethereal sound of the music conjures up the sophisticated rhythms and exquisite harmonies of the Waltz-king, Johann Strauss. But that is our first misconception—it is not the music of Johann Strauss at all, but of his less extrovert, more poetic and melancholic brother, Josef, whom Johann

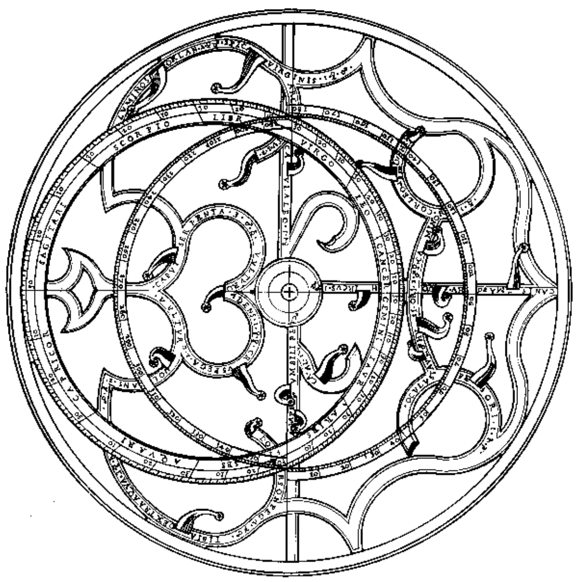
Inaugural Lecture, published in E. M. Gray (Ed.), *Thinking about Mathematics and Music of the Spheres*, Mathematics Education Research Centre, Warwick, 42–52

claimed was “the most gifted of us all.” Familiarity brings with it a lack of critical sensitivity, a willingness to look on events through rose-tinted spectacles. Just as some may look back at those elegant days in the ballrooms of Vienna, so we may look back at the days of our youth when mathematics was taught properly—in those good old days when it was always warm in summer and snow fell on Christmas day. It is a nostalgia which clouds our reason and limits us from seeing the possibilities that lie before us.

In this lecture I will look back at the classical mathematics of the *Music of the Spheres* to see how, from the expert viewpoint of the time, the explanations of the natural world were perfectly reasonable, but that mismatches between theory and reality led to the need for theoretical change. I will take this as a metaphor for some of the expert views of teaching and learning mathematics today, to see how the coloured vision of the expert may involve misconceptions in mathematics education.

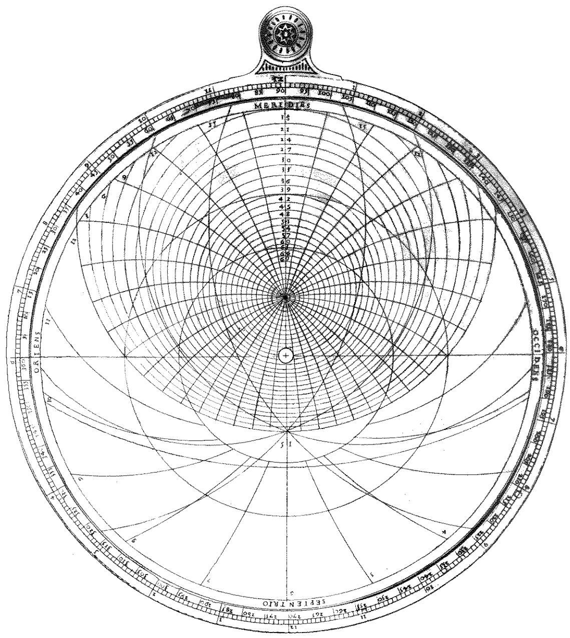
Music of the Spheres

It would be natural for primitive man to see the perfection in circles and spheres. Each day the sun rises as a glowing circle and pursues a curved path through the sky and, in a slightly more intriguing manner, the moon waxes and wanes through its cycle, reaching full circular glory every month. At night the stars reveal a fixed pattern that moves around as points of light on a huge black background. Early civilisations began to observe, record and predict for astronomical and astrological purposes. As technology improved, it became possible to make a circular plan of the stars in the heavens and the constellations of the Zodiac, as on this 16th century astrolabe:



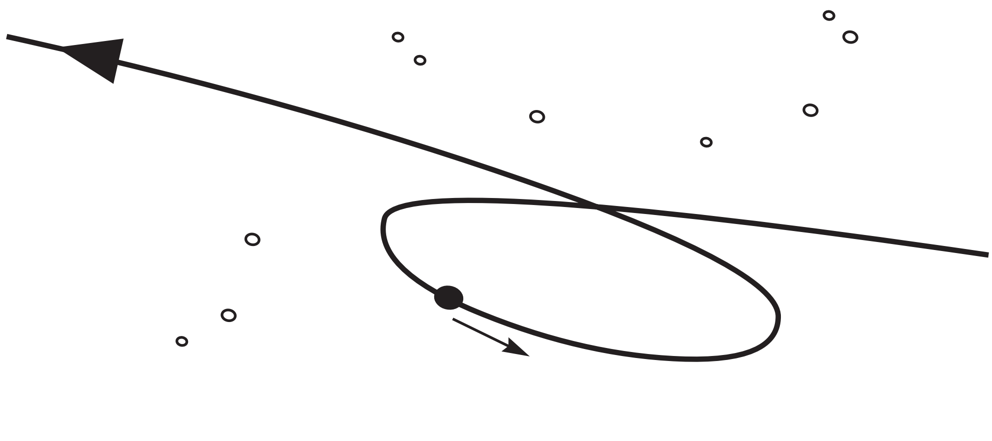
The positions of the stars on the disc or “rete” of an astrolabe.

A second circular plan marked with bearings, degrees of altitude, and circular time lines allowed the first disc to be lined up to accurately specify the positions of the stars at a given time.



The markings to line up the position of the “rete”.

But certain stars are not like the others. Instead of remaining fixed, over a period of time, five stars visibly “wander” around the night sky in intriguing patterns.



The path of a “wandering star”

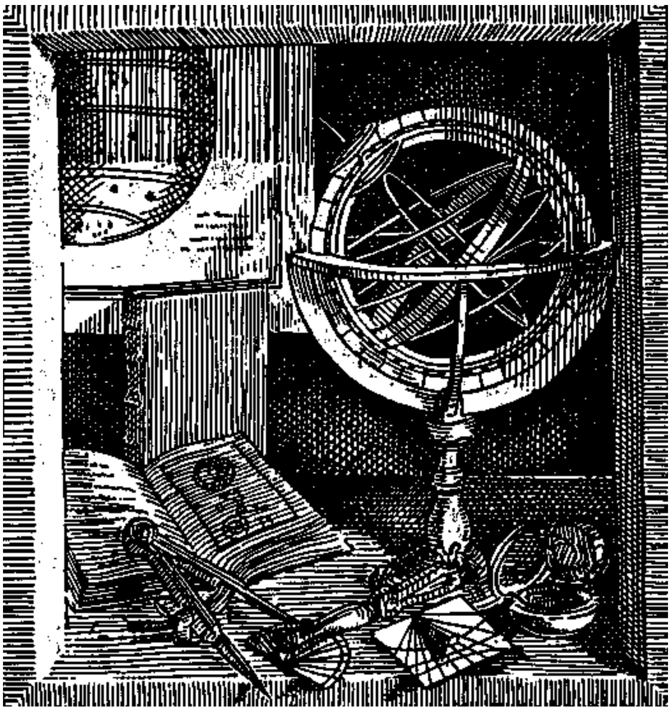
How could these patterns be described? The Pythagoreans in the Fifth Century BC believed that the wandering stars, or planets, each moved on a sphere round the earth, and the description of harmony in terms of ratios of whole numbers suggested a tenuous link between the harmonies of music and the motion of the planets. This is the origin of the “music” or “harmony” of the spheres.

No written records have survived from the Pythagoreans, and it was Plato (c..428 – c. 348 BC) who instructed the members of his Academy to develop a theory of circular motion to model the movement of the planets. First Eudoxus (c. 400 – c. 350 BC), then his pupil Callipus, and then Plato’s pupil Aristotle (384 – 322 BC) proposed successively more complex models based on circular motion on concentric spheres around the earth. The heavenly bodies were considered to be in the order: moon, Mercury,Venus, Sun, Mars, Jupiter, Saturn.

Archimedes (c. 287–212 BC) made a sophisticated planetarium of bronze and glass, driven mechanically by water power, which represented the motions of sun, moon and planets, and the daily revolution of the fixed stars. Writing over a century later, Cicero (106–43 BC) reported seeing the planetarium still in action:

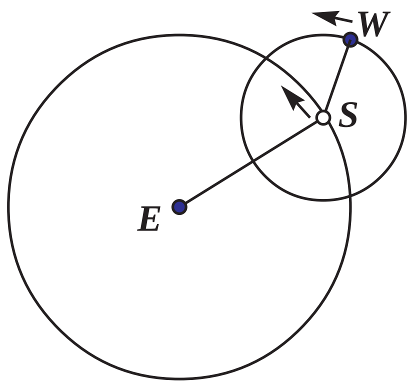
> When Gallus set the sphere in motion, one could actually see the moon rise above the earth;s horizon after the sun, just as occurs in the sky every day; and then one saw how the sun disappeared and how the moon entered the shadow of the earth with the sun on the opposite side. (Cicero, *De Re Publica*, 51 BC.)

Over the centuries, sophisticated “armillary spheres” were constructed to represent the movement of the heavens and the paths of wandering stars around the earth:

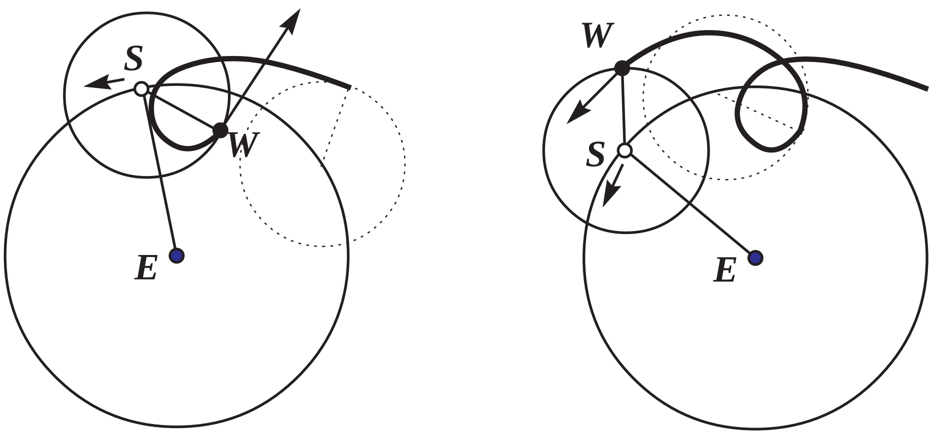


A woodcut of a fourteenth century armillary sphere charting the positions of the heavenly bodies.

The manner in which the wandering stars were able to reverse their direction as they moved against the fixed background was described by Hipparchus (fl. 150 BC) using a new model with the planets moving in circles upon circles:



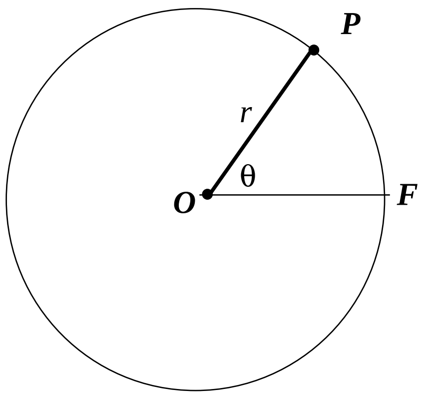
W moves round a circle centre *S*, as *S* moves in a circle round *E*.



The path of *W* is similar to that of the "wandering stars".

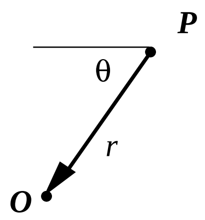
Ptolemy of Alexandria incorporated and developed these ideas in his *Mathematical Collection* (*He mathematike syntaxis*) rechristened *Algamest* ("the greatest") by the ninth century Arabs; it became the foremost reference of astronomy for well over a thousand years.

A simple mathematical thought experiment will show why the theory of "circles on circles" is a good match for the motions of the heavenly bodies. If a point *P* is moving in a circle radius *r* around the point *O* at an angle θ measured from a fixed radius *OF* to the turning radius *OP* then, as the angle θ increases, the point *P* moves round the circle clockwise:



A point P moving in a circle centre O.

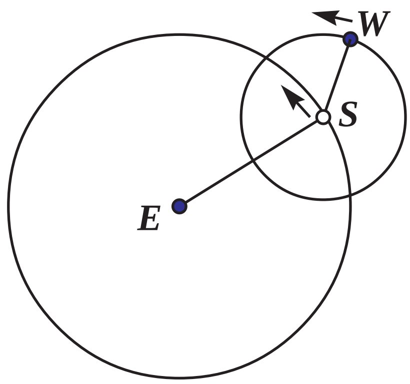
A small creature at P sees the point O is always a fixed distance r away, but in the *reverse* direction:



The view from the point P.

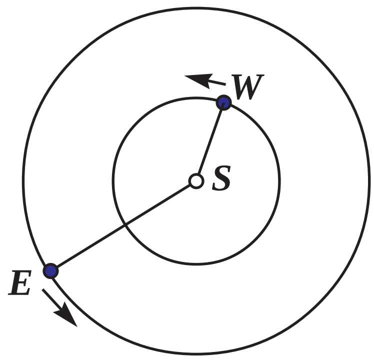
If the tiny creature is unable to sense it is moving and thinks it is staying in the same place, then the point O will appear to be moving around it in a circle in the opposite sense.

Of course, this is what is happening to us. When we look up at the sun moving round the earth, we consider ourselves fixed and the sun as moving. As a wandering star W moves in a circle round the sun S, as we saw earlier, the star appears to be moving on a “circle upon a circle”:



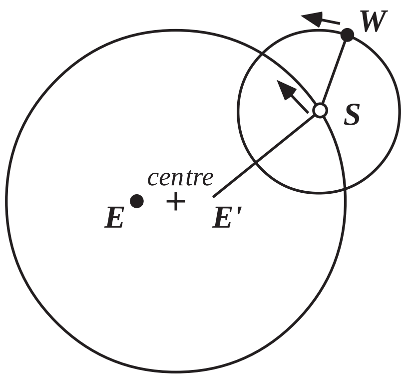
Motion relative to the earth, E.

However, if we consider the *sun* fixed instead, and see the earth moving in a circular orbit around it, we see a simpler picture:



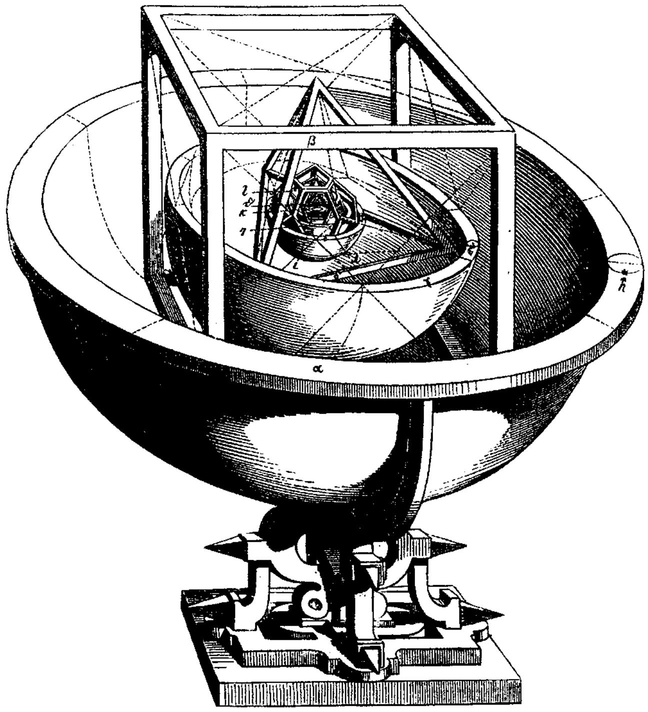
Motion relative to the sun, *S*.

However, even these more complex ideas failed to give an exact match for the observed motion. Ptolemy knew this and had developed a more complex model in which the Earth *E* was placed to one side of the centre of the large circle and the motion of the centre of the smaller circle had uniform angular motion about another point *E′* an equal distance on the other side of the centre.



An “eccentric” version of the theory.

Over a millennium later, Copernicus (1473–1543) instated the sun as the centre of a planetary system but returned to circular planetary orbits in *De Revolutionibus*, published just before his death. He was followed by Johannes Kepler (1571–1630) who believed devoutly that the universe was an ordered mathematical harmony and sought the underlying pattern that held it together. He mused as to why there were only *six* (known) planets and, in a moment of inspiration, linked this to the *five* regular solids. If the six planets were moving on unseen spheres about the sun, could not the five regular solids be inscribed *between* the spheres? (*Mysterium cosmographicum*, 1596).

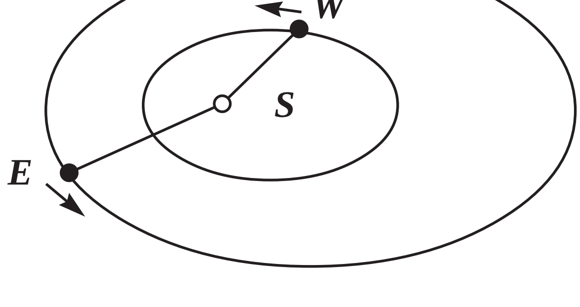


Kepler's relation between the regular solids and the planetary orbits

He later tried to fit a large quantity of data about the planet Mars collected by Tycho Brahe to the ancient theory of circles on circles and failed. Then he had a further inspiration, this time one that would change our view of the heavens. In 1609 he published *Astronomia Nova*, explaining his new theory of planetary motion. On page 285 of 337 pages, after many calculations relating to various possible models which failed to fit the data, he reported how the scales fell from his eyes:

> *O me ridiculum! ... demonstrabitur, nullam Planetae relinqui figuram Orbitae, praeterquam perfecte ellipticam!*
>
> Kepler, 1619, p. 285.

The motion of the planets could be explained so simply, not as off-centre circles, but as *ellipses* with the sun at a focus!



Planets moving in ellipses round the sun as a focus.

In *Harmonices Mundi* (1619) he gave the planets a new form of music which moved up and down as they moved alternately closer and further from the sun. Saturn, the furthest away, sang a sonorous basso profundo, Venus, who moved in a near circle remained on a single note, and eccentric Mercury dazzled up and down as a coloratura soprano:

Kepler’s version of the “harmony of the heavenly bodies” *Harmonices Mundi*, p. 207

The two years 1665 and 1666 were a catastrophe for Britain, with first the Black Death and then the great fire of London. But it was a good time for mathematics. Young Isaac Newton had just completed his degree studies at Cambridge in a course which still referred to the Geometry of Euclid and the Astronomy of Ptolemy. However, he knew the heretical ideas of Kepler’s empirical theory of planetary motion and the theory of Descartes describing geometry in algebraic terms. He returned to Lincolnshire, working alone and undistracted, using the symbolism of algebra to develop the theory of the calculus, applying it with stunning power to explain the mechanics of the heavens.

His use of symbols as a manipulable calculus to support sophisticated computations proved to be a huge innovation given greater impetus by the evocative symbols for differentiation and integration devised by Leibniz. The powerful use of symbols as mental objects for manipulation—and how we might see them from an appropriate viewpoint—will shortly become the focus of our attention.

Misconceptions

The succession of new mathematical ways of looking at the heavens shows how a change in viewpoint can give a profound new insight which often *simplifies* our mathematical models rather than making them more complicated. However, this can have subtle effects in the way in which we as adults, and also those with an even greater expertise in mathematics view the learning of children. In essence, our perceptions are changed by our experiences and we have great difficulty in appreciating the stages that learners pass through in developing mathematical sophistication.

When I was invited to give a presentation to the *International Congress of Mathematicians* in 1994, somewhat tongue-in-cheek, I suggested reasons why mathematicians are good at mathematics but less good at thinking about how others do mathematics:

> **Axiom I**: All mathematicians are born at age 0.
>
> **Axiom II**: to reach the age *M* of mathematical maturity, the mathematician must pass through ages 0, 1, 2, …, *M*–1.
>
> **Theorem**: A cognitive development is necessary to become a mathematician.
>
> **Proof**: Since no child aged 0 has produced any important mathematical theorem, something happens between ages 0 and *M* that makes mathematical thinking possible. (Tall, 1994)

Of course this is a caricature of a mathematical proof, but it was done with a purpose—to show that even mathematicians were once children at a time when they had little of the sophistication that is developed by experts. The argument went on to suggest:

> **Cognitive Principle I:** For survival in a Darwinian sense, the individual [focuses] on concepts and methods that *work*, discarding earlier intermediate stages that no longer have value.
>
> **Corollary**: The individual is likely to *forget* much of the learning passed through in years 0, 1, ..., *M*–1 and the mathematician is likely to attempt to teach current methods that work *for him/her*, not methods that will work for the student. (ibid.)

Such forgetting of earlier struggles is characteristic of many levels of activity. When children were taught initial activities as a precursor to more formal mathematics:

> After the formalization had been taught, or three months later, the practical or pre-formalization work which led up to it was often forgotten or not seen as significant. (Johnson (ed.), 1989, p. 219)

Likewise, when I developed computer software to show students how to zoom in to magnify part of a curve on a computer to “see” how steep it is, I was confident that they would see this as a powerful way to approach differentiation. Two months later, when they had spent a great deal of time on tangents, I asked them how they would teach differentiation to a student new to the concept, and hardly anyone spoke of magnification; the most common response was “as the gradient of the tangent”. They had not lost the notion of magnification, for they used it to say that certain graphs with “corners” did not have a gradient because they “did not magnify to look straight”, but they operated in the classroom with "their current best approach" and only brought the more fundamental approach to mind when their current approach failed them, (Tall, 1986).

To remember how one first learned concepts after many years of expertise at a more sophisticated level is very difficult:

> After mastering mathematical concepts, even after great effort, it becomes very hard to put oneself back into the frame of mind of someone to whom they are mysterious. (Thurston, 1994, p. 947)

Just as the ideas of Ptolemy lasted for a thousand years, *because they worked*, so do experts develop a point of view *which works for them*, but may not be suitable for the growing mind of the learner.

A majestic failure for the experts was the introduction of "the New Math" in the nineteen sixties, where it was assumed that it was only necessary to give truthful definitions in set theoretic terms and the children of the world would be free to build up their own knowledge of mathematics. Even though the axiomatic method based on set-theory has been a success in generating coherent mathematical theories at research level, it proved "not to fit" the thinking processes of growing children.

Axioms and formal proof are a powerful tool for organising mathematics into a logical and systematic theory, but they are the *final* part of the mathematical process, not the beginning of it. It is here that the vision of the expert falters in attempting to formulate a way of teaching the learner. The expert seeks clarity and accuracy of expression. Concepts are very carefully defined, so that they can be used unambiguously in a formal theory. But clear definitions of concepts and lack of ambiguity seem to play little part in the child's earlier learning of mathematics. So what does?

Insights from work at the Warwick Mathematics Education Research Centre

Over the years, a number of different research studies at the University of Warwick have pointed to a phenomenon that suggests a radical rethink of how we view mathematical learning.

The first Director of the Mathematics Education Research Centre, Richard Skemp, is justly remembered as the formulator of a number of highly original theories of mathematical thinking, not least for his work on "instrumental understanding and relational understanding", (Skemp, 1976). However, I wish to draw attention to an idea in his paper which has received less attention than the items in the title, namely, the concept of "faux amis". This refers to words which mean different things in different languages, such as "médecin" for a "doctor" in French, rather than the apparently corresponding English word "medicine". This underlines the idea that certain words, which apparently have a clear and specific meaning in one community, say a community of *experts*, may have a very different meaning to another community, say a community of *learners*.

The second Director, Rolph Schwarzenberger, took a great interest in what were, apparently, student misconceptions about mathematical concepts. In particular, he was responsible for focusing on students' ways of conceptualising infinite decimals, such as √2=1.414…, π=3.14159…, and the most interesting example of 0.999… . When a questionnaire revealed that the vast majority of students thought that "nought point nine recurring" was *less* than one, he asked a sequence of questions requesting students to translate decimals into fractions, including **0.25, 0.05**, **0.3** (re-inforced as 0.3, not 0.3 recurring), **0.333…** (0.3 recurring), and **0.999…** .

Many students were happy to write

0.333… = 1/3

and this led two thirds of them to write

0.999… = 1

(by multiplying the previous answer by three), including over a third of the total who had previously asserted that 0.999… was *less* than one. A number of responses bore evidence of mental conflict, with verbal comments and crossings out. At the time we, as mathematicians, had our ideas why the students were confused, and we responded by writing a paper to tell teachers how students might be taught the ideas of real numbers and limits so that they would "understand" (Schwarzenberger & Tall, 1978). However, the limit concept proved to have difficulties that did not necessarily respond to "good teaching" and this became a significant area of research in the ensuing years.

For instance, Monaghan (1986) found that sixth-formers views of infinity were very little changed by their mathematical teaching and that they viewed repeating decimals as "infinite numbers", in *extent*, if not in size. Their view of the "real number line" had various friendly numbers, such as whole numbers and fractions of which they felt secure, and numbers such as π, e, √2 with which they developed a working security, but infinite decimals were often seen as "improper". Other research studies continue to highlight such problems with decimals and limits (eg Williams, 1991, Cornu, 1991, Monaghan et al, 1994).

In another apparently unlinked area, Thomas (1988) performed research on the learning of algebra with younger children using computers. The learning environment involved children programming variables in BASIC (where the child specified the computations to be made, such as A=2:PRINT 3+2*A, and the computer carried out the computational processes. The children also enacted the internal processes by a game in which they would record the value of variables in boxes drawn on a large sheet of cardboard and carry out the operations themselves. This research showed that the participating children were better at conceptualising expressions such as 3+2*x* than corresponding control children who were more likely to find the expression meaningless—after all, how could they calculate 3+2*x* unless they knew the value of x? This is a well-known phenomenon (e.g. Küchemann, 1981; Booth, 1984) called, at the time, 'lack of closure'—but what did this mean, and why did it cause such difficulty?

An innocent conceptual question in arithmetic (Thomas, 1988) gave further interesting data:

> Is $\frac{6}{7}$ the same as 6÷7?

76% of the children following the experimental computer course said "yes" as compared with only 44% of those following a standard algebra course. At the time we called this the "process-product" obstacle, because:

> … many of the controls did not consider the two notations equivalent because
>
> "$\frac{6}{7}$ is a fraction, 6÷7 is a sum."
>
> This reveals the perception of 6÷7 as a *process* involving value-operation-value rather than as a global entity – the single number – produced by this process. (Tall & Thomas, 1991).

It was with the publication of Gray (1991) that the key to unlocking these disparate experiences came to light. Working with younger children, (ages 6 to 12), Gray investigated how they solved addition and subtraction problems with small numbers when they did not immediately know the answer. He found—as had many before him—a wide range of solution processes. But from the broad spectrum, when he did something that was "not politically correct" by comparing responses from pupils the teachers considered to be the "more able" with the "less able" throughout the primary school, he found the solution processes tended to diverge. When the "less able" could not do a problem, they invariably *counted* the solution (often inefficiently) or used their fingers or other physical supports to develop idiosyncratic routines to solve individual problems in ways which often did not generalise to larger numbers. Meanwhile, the "more able" not only knew more facts (as would be expected), but when they did not know, they often used other known facts to build a solution or, when they counted, they invariably selected an efficient method to do so.

Two examples on video

Put another way, the less able often used inefficient and inflexible counting processes which placed a great strain on their cognitive resources whilst the more successful reduced the strain by using the number symbols flexibly—either as *processes* to calculate a result, or as *concepts* to think about and manipulate mentally.

This idea of processes (such as counting) being reconceptualised as concepts (such as number) had been around in the literature for a long time, at least since Piaget asserted that

> mathematical entities move from one level to another; an operation on such 'entities' becomes in its turn an object of the theory, and this process is repeated until we reach structures that are alternately structuring or being structured by 'stronger' structures. (Piaget 1972, p. 70).

In recent years, conversations with Sfard and Dubinsky—who respectively speak of "reifying" or "encapsulating" a process as an object—has refocused our thoughts on this phenomenon (Dubinsky, 1991; Sfard 1991; Sfard & Linchevski, 1994). It seems foolish now to see the struggle in our Ptolemaic conception as we moved in ever-decreasing circles, produced various theories to "explain" what was going on. Then in a conversation, Eddie Gray and I realised what was missing – *no-one had encapsulated the process of encapsulation!* No-one had given *a name* to the "thing" that could be conceived *either* as a process *or* as a concept.

On the spur of the moment the name "*procept*" was born – as a symbol that could be thought of as a process or a concept. It is with affection that I remember Rolph Schwarzenberger arriving at the critical moment to greet our new term with some scepticism because he "doubted its etymology". I laughed and said something to the effect that when you are growing new plants you don't start by pruning them, you begin by putting manure all over them (although perhaps I used a different word from 'manure').

Bernard Scott, who was head of Sussex Mathematics Department when I started my first job as an assistant lecturer, often said that "by giving a name to something, you acquire power over it." So it happened.

In the context of symbolism, the *faux amis* of Skemp could now refer to the different meanings of *procept*. A problem like "8+6" for one child would evoke a counting *process*, for another a number *concept*, so the same symbol can mean different things to different children. Even more interesting, it can also mean different things to the same person at different times. The successful thinker uses symbolism not in a precise unique way, but in a *flexible* and *ambiguous* way to evoke either *process* or *concept* and learning often proceeds with a symbol used to signify a process becoming also used for the product of that process. The *process* of counting becomes encapsulated as the *concept* of number, the *process* of addition becomes the concept of sum, the *process* of repeated addition becomes the *concept* of product, the *process* of division of whole numbers becomes the *concept* of fraction, and so on and so on.

In arithmetic we saw the success of those who used the symbolism flexibly as process and concept and the lesser success of those who thought mainly in terms of counting procedures and seemed to become locked in a procedural strategy of learning a growing collection of mechanical rules.

This in turn sets up even more serious problems in algebra. Children who consider arithmetic symbols as *processes* will see 5+6=11 not as a concept, "5+6 *equals* 11", but as a *process*, "5+6 *makes* 11". Seeing an algebraic symbol $3+4x$ as a process which cannot be performed when x is unknown will lead to the "lack of closure" reported in earlier research.

This gives insight into why many children cannot make sense of algebra, and why some may be able to cope with an equation like

$4x+3=11$

yet fail with a more complex equation like

$4x+3 = 17–3x$.

It is not simply that the second is more complicated because "x" appears on both sides, but a problem of interpreting the meaning. If the symbolism is intepreted only as *process*, the first equation might be read as

"four times a number plus three *makes* eleven,"

and this can be mentally manipulated *as numbers* to give

"four times the number is eight,"

and so

"The number is two!"

The second equation, $4x+3 = 17–3x$, is of a different order of difficulty because it cannot be conceived as a single arithmetic process giving a known result. Children who see symbolism mainly as process may read it as two different processes deemed to be "equal". However, it is not the *processes* which are equal, but the number *concepts* produced by the two evaluations. Understanding requires a flexible view of symbolism as process and concept, without which the likely fall-back position is one which attempts to rote-learn a collection of solution procedures without having any sound conceptual structure. Hence the likely failure in algebra of children who do not develop a flexible view of symbolism.

The third area of research mentioned earlier—the meaning of infinite decimals and limiting processes—also involves symbols doubling as process and concept. Students who see infinite decimals as *processes going on forever and never finishing* are bound to conceive of them as "improper" quantities which never end. This relates to difficulties that students have with limits, which seem to be seen as a "variable quantity" that "gets close" rather than a fixed value that can be approximated as near as required. The focus seems to be more on the *process* of getting close than the *concept* of limit.

Implications for Mathematics Education

The notion of procept proves to have implications throughout primary school arithmetic, secondary school algebra, senior school calculus and university analysis, indeed, everywhere that processes are given a representation as a symbol which can itself be manipulated as a mental concept. It offers a new viewpoint from which to observe the learning process when mathematical symbols are manipulated.

The human brain works in a massive simultaneous-processing manner and only copes with the complexity by filtering out most of the activity and only focussing on a small quantity of mental data (Crick, 1994, p. 61). It therefore benefits from representations which compress information in a way that can be handled easily by the short-term focus of attention. The encapsulation of long counting processes as immediately available number concepts compresses knowledge into a form that is easily usable by the human brain. A *procept* is available as *process* to perform sequential computations and also as a *concept* respresented by a compact symbol that can be easily manipulated in conscious attention. The child who is less successful at compressing processes to concepts has a harder task to achieve – a harder task for an already stressed cognitive structure.

How unfair mathematics is! It becomes *easier* for the gifted and *harder* for the less successful, but, perhaps, ’twas ever thus. And, almost certainly, this is in direct contradiction to the principles underlying the National Curriculum where ladders of “levels” are placed in each subject for the growing child to climb. The principle appears to be based on the democratic idea that “all children are equal” and “go through the same stages, but possibly at a different pace”.

The theory of procepts suggests that this is a totally inadequate view. It predicts that, in subjects such as arithmetic and algebra where processes are successively encapsulated as concepts using symbols, children who get stuck at the “process” level have an enormous task before them to make any sense of successive levels. At this stage they are *not* doing the same thing as their more successful peers who are mentally handling symbols as both processes to *do* and concepts to *think.* The children who do not encapsulate do not have any meaningful concepts to think with (Linchevski & Sfard, 1991).

This suggests that the idea of a single ladder for all is misguided. Like the Ptolemaic view of circles upon circles, it is a theory which *appears* to explain approximately what happens and seems a good enough approximation to allow broad measurements to be made to measure children’s progress up the ladder. As a statistical guide to progress, discrepancies in data between different children are not regarded as important and it proves to offer a simplistic indicator of relative success of children taught by different teachers in a different schools. But there are *very significant* qualitative differences developing in thinking processes which are less easily measured and these are having very serious effects on the learning of our children.

For instance, children who do not see arithmetic symbols as flexible processes and concepts are hardly likely to see algebraic symbols in a flexible light. Therefore algebra will be meaningless to them other than a task to rote-learn procedures to get answers. The remainder of the “algebra ladder” is therefore a *different ladder* for them from that seen by the flexible thinkers and is likely to be a highly inappropriate ladder to climb.

Conversely, the single ladder also militates against the learning of more successful children who think in a different way and are being denied access to topics which are too difficult for others. For instance, the concept of fraction arises through compressing the process of sharing and is therefore a *procept*. It is typical of an idea which is impossible for those who fail to encapsulate whole number processes as concepts yet straightforward to those who have a flexible "proceptual" view of number as process and concept.

Fractions are vanishing from the school curriculum because they are not democratically possible for all on a single ladder. The consequences are disastrous. In a recent letter to the *Guardian*, Sykes and Whittaker (1994) noted a sharp drop in performance in fractions on students entering a university course on Business Management. Only 50% of the students, with average A-level grades of 22 points (grades BBC) can multiply one half times two thirds correctly. Of these students, only 16% gave the correct answer to "the square of 0.3" in 1994 compared with 66% in 1987 (before the change to the new GCSE). If this is typical of studenta with these A-level results, what chance will university students have of doing probability which requires simple arithmetic of fractions, or of handling rational functions in algebra, whose arithmetic is a generalisation of fractional arithmetic?

Algebra is also "too difficult" for the majority and so insufficient exposure is being offered to those who have flexible thinking processes that are more suited to handling the symbolism. Indeed, when I participatied in the design of the SMP 16–19 A-level, it was explicitly designed for students with an expected low attainment in algebra.

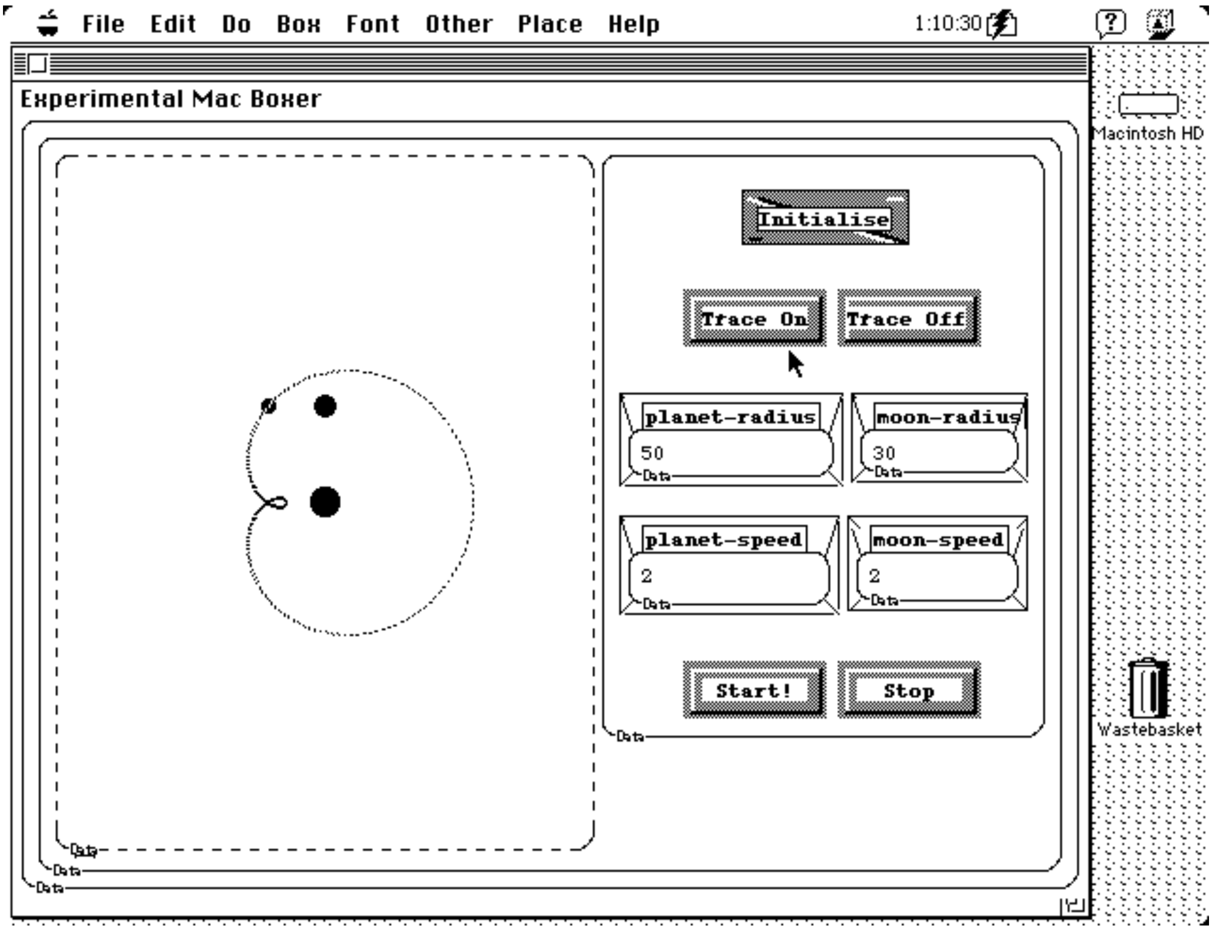
Mathematics education is bringing in a welcome change to focus on "using and applying mathematics" which includes aspects of mathematical communication and mathematical proof. But, strangely, the effects on proof may not be what is hoped for. Instead of concentrating on logical deduction, so many of the investigations in school lead from arithmetic exploration to seeking a generalisation in algebraic terms. The by-product is focused less on logical deduction and more on the expression of algebraic generality, which it has been suggested above is not well understood.

In a wide range of mathematics where symbols are used as process and concept, the theory of procepts allows us to look at the situation in new ways. Instead of seeking the precision of a set-theoretic approach with precise definitions for things as *objects*, (a 'set' of things with certain properties), we see the alternative of the flexibility, ambiguity and duality of an approach to symbolism in a manner that seems more suitable for cognitive growth – through constructing *processes* that have meaning, then encapsulating them as *concepts* that can be meaningfully manipulated. In other words, *thinking as mathematicians think*, rather than *doing as they* ***say*** *they do!*

Developments at Warwick Mathematics Education Research Centre

The age of information technology is upon us, so in addition to understanding the processes by which individuals learn mathematics, the focus is also being turned on the complementary way in which technology can complement human thinking. For instance, the computer can carry out routine algorithms of arithmetical computation and algebraic manipulation, so they allow the learner to take the opportunity of investigating the properties of the concepts whilst the computer carries out the processes. This gives new opportunities in learning previously unavailable, though the availability of computers to do these processes does not remove the need for the human mind to understand them in an appropriate sense.

In the Mathematics Education Research Centre, various researchers are investigating the role of the computer in learning. The role of visualisation using computers has been a central topic for research and development, for instance, in a visual approach to calculus (Tall, 1986), which has been the basis for much development of the SMP 16–19 curriculum. The computer has featured widely in the use of spreadsheets for modelling mathematical and scientific ideas (Beare, 1994), developments in using integrated programming systems for Logo programming (e.g. Pratt, 199?) and relationships in collecting, visualising and analysing data using portable computers (Ainley, 1994; Pratt, 1994). The following picture shows a microworld, designed using the software *Boxer* by Dave Pratt, to investigate the path of a satellite moving in circles round a planet, which is itself moving in circles round the sun:

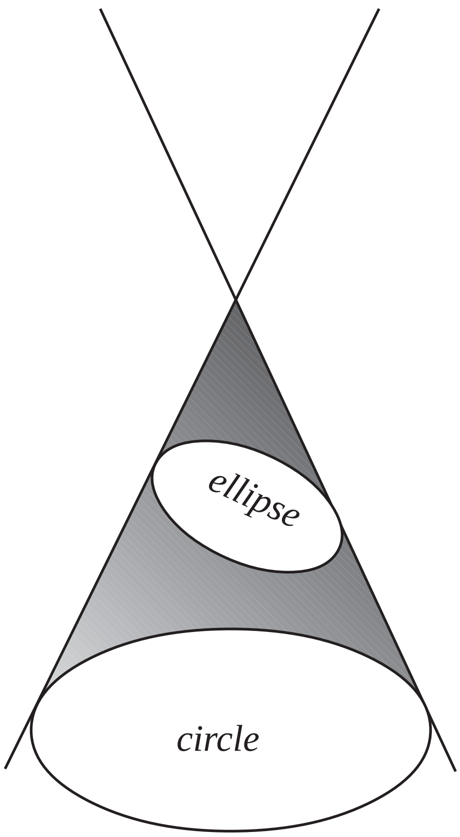


Investigating the path of a satellite moving in a circle on a circle, programmed in the *Boxer* environment

This microworld allows investigation of the “circles on circles” problem of the Greeks either through the act of programming the problem or, in a quite different way, through using the program to perform experiments in a way far more immediate than the calculations that must have been performed by the Greeks.

The Research Centre continues to consider the growth of mathematical concepts from the first days of schooling through to university level both in terms of visual and symbolic cognitive development, and of the reflective thinking strategies involved in problem-solving. Its aim is to seek new and simpler ways of understanding what it is that is happening in mathematical thinking. To do so we should learn from history, that we may have the data already to hand in our culture, if only we know how to look at it from the right point of view. For instance, it is not the only the *precision* of symbolism that gives it power, it is the manipulable way it can be used ambiguously as process to *do* and concept to *think* about mathematics.

The Ancient Greeks pondered the movement of the planets and chose a more complex theory of circles upon circles to explain the movement of the wandering stars, when a change in viewpoint would help to see a simpler phenomenon. After all, *O me ridiculum!*, all that is necessary to see that a circle is an ellipse is to change the viewpoint!



VIewing a circle as an ellipse using a conic section

The Greeks knew this in their theory of conic sections from the work of Appollonius of Perga, *Conic Sections* in the 3rd century BC. However, they did not see it as relevant in the context of describing the dynamics of the universe, preferring instead the mathematics inspired by the *music of the spheres.* Are we to let history repeat itself and fail to see how children grow to be mathematicians?

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