Calculus and Analysis

The *calculus*, invented simultaneously in the 17th century by Newton and Leibniz, consists of two general procedures, one to compute the rate at which quantities change (differentiation), the other how they accumulate (integration). These are unified by the *fundamental theorem of calculus* which essentially says that the differentiation and integration are inverse operations: *doing* the one is equivalent to *undoing* the other. Thus integration can be performed by inverting the process of differentiation.

Analysis is the rigorous study of the technical details of such processes, starting from the simple axioms for numbers and arithmetic. But, whereas arithmetic operations can be carried out directly, problem-solving may require indirect methods involving *undoing* such operations. For instance, the square of a number can be calculated by multiplication but a square root such as $\sqrt{2}=1.414...$ is computed as a succession of improved approximations. Analysis is founded on the *axioms for real numbers*, the theory of *functions* (calculating values by well-defined processes), and the theory of *limits* (of functions and sequences of approximations). Upon these logical foundations the algorithms of differentiation and integration may be formally defined using the limit concept.

In some countries (including the USA) a calculus course with a pragmatic blend of calculus algorithms and practical analysis is studied by a wide range of college and university students. An advanced placement course in calculus is also available in high school. Elsewhere the study of calculus at high school (with students aged 16–19) is more the norm, with formal analysis studied by mathematics specialists at university.

Courses in calculus and analysis are increasingly coming under scrutiny as it becomes apparent that students are not being as successful in them as might be hoped. The problems lie in the subtle complexities of the mathematics.

Traditional teaching of calculus begins with an intuitive introduction to limits of functions $f(x)$, in particular, the limit of the gradient $(f(x+h)–f(x))/h$ as h gets small. This limit is called the *derivative.* By manipulating the symbols for specific functions $f(x)$ a calculus of derivatives is developed. Integration of many functions is performed by attempting to reverse this process. Students can become adept at the manipulations of symbols in a procedural way but this may have few links with the real meaning.

In Dina Tirosh (Ed.), ‘Mathematical Topics of Instruction’, in T. Husen & T. N. Postlethwaite, (Eds.) *The International Encyclopaedia of Education*, Second Edition, Pergamon Press. pp. 3680-3681, 3686.

The formal study of analysis can prove an even greater stumbling block, The notion of limit may be a logical foundation for the formal structure of analysis but it proves to be a difficult starting point for learning. Intuitively, a sequence $a_1, a_{2,}, \ldots, a_n, \ldots$ is said to tend to the *limit* a if the terms get closer and closer to a as n increases. Formally the definition is more complex. It says that for any specified error $e>0$ there must exist a corresponding number N such that when $n>N$, the terms a_n differ from a by less than e. Cognitive research shows that this conflicts in subtle ways with the intuitive mental image, for instance students may believe that the terms literally get closer and closer without ever reaching it, although the formal theory does not require this – only that the definition is satisfied. Handling the definition to make formal deductions proves to be far too intricate for the average student.

The arrival of the computer has enabled the subject to be attacked in new ways. For instance, interactive computer graphics can be used to represent graphs of functions and to investigate how fast they are changing. The rate of change of the function may be found by magnifying the graph and simply *looking* to see how steep it is. The process may then be linked to the numeric processes which drew the picture and to the corresponding symbolic processes of the calculus. Thus visualisation using a computer can give cognitive foundations for concepts which were previously only performed as numerical and symbolic manipulations.

Modern software can also carry out all the numerical, graphic and symbolic manipulations previously taught in traditional calculus. Appropriate programming languages allow students to program logical operations such as the definition of limit by procedural methods and reflect on the processes thus carried out. In these ways the computer can provide a context in which students can atempt to construct analysis concepts in a meaningful interactive context.

The full implications of analysis in a technologically rich culture will take some years to begin to be fully appreciated. It will not only provide new ways of teaching and learning calculus and analysis, it will alter the very conception of the subject itself. Traditional analysis proves many results by contradiction: to show something is true, assume it is false and show that this cannot happen. The computer focuses attention on things that can actually be *constructed*. It offers practical algorithms to give genuine solutions to real problems. Modern analysis will surely evolve to take advantage of such powerful facilities.

David Tall