Construction of Objects through Definition and Proof

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David Tall

Mathematics Education Research Centre
University of Warwick
COVENTRY CV4 7AL

Introduction

The major focus of the discussion of the PME Working Group will be on theories of cognitive development appropriate for undergraduate learning of advanced mathematical concepts. Two of the papers (Dubinsky, Sfard) concentrate on the encapsulation/reification of process as object to give an operational approach to mathematics. A third (Artigue) considers the need to introduce a concept as a tool to solve a problem before making it the object of reflection at a higher level. The purpose of this paper is to remind ourselves that the formal presentation of advanced mathematics is currently through definition and proof. After all, if one cannot define what a "group" or "vector space" is, how can one begin to make general statements about it? It is by no means my contention that a formal approach is the best way to learn advanced mathematics, but we, as educators, must reconcile any cognitive approach with that development pursued by the wider community of mathematicians of which we are part. This must be done either by meeting the community beliefs part way, or offering a viable alternative.

The fundamental question I pose is this:

- How can students requiring insight into formal mathematics be helped to make the difficult transition to a definition-proof construction of knowledge? (This in itself requires a theory of how such knowledge construction works).

This is not to say that *all* students require such a transition, indeed there may be a whole spectrum of approaches appropriate for the needs of individual students and only a small minority may need to address this problem.

Operational Mathematics prior to Formalism

In preparing students for formal mathematics, I agree with Sfard that "operational mathematics should precede structural mathematics" but I would add the qualification *in the early stages*. When a formal approach has been used several times, such an approach to mathematics can itself become operational and so – for advanced mathematical thinkers – the compression and power of the formal approach can become a natural mode of operation.

It should also be emphasised that the growth of human thought in the individual builds on what that individual already knows. In early encounters with the process of encapsulation – counting becoming number, addition through counting-on becoming sum, repeated addition becoming multiplication – it is quite clear that children need to go through the process first to be able to conceive of the object at a later stage. Children who can carry out the procedure of counting but have not encapsulated it as the concept of sum find multiplication very difficult, even impossible.

But the process of encapsulation *itself* can be encapsulated. The more advanced thinker can realise that a new construction will lead to a kind of object whose properties can be investigated *before* or *at the same time as* the individual is becoming familiar with the process required to construct it. By using alternative representations (often visual), meaning can be given to objects produced by processes before those processes are studied in detail. For instance, the notion of a solution to a differential equation can be given geometrical meaning as an object – a solution path in the plane having prescribed gradient – before the methods of solution themselves are studied. I would therefore contend that much more flexible modes of learning are available to advanced mathematical thinkers than just the need to first routinise a repeatable action as a process and then to encapsulate it.

More generally, mathematical objects become easier to understand when they can be fitted into a meaningful cognitive context which may not be through "encapsulation of a process as an object". The case of complex numbers – where the process of taking the square root of a negative number was carried out without giving a meaning to $\sqrt{-1}$ for a century and a half – was given meaning through representation as points in the plane. In the 1830s, when a complex integral could be seen in terms of properties of visible paths in the plane, all discussions of the "meaning" of complex numbers evaporated.

The roles of symbols and other representations are also vital in understanding the process of encapsulation of a process as an object. Working with Eddie Gray with young children, we now see a wide chasm between those who flexibly manipulate number symbols, deriving new facts from old, and those who always need to carry out a counting procedure. It is the flexible use of a symbol to evoke either a process of computation or the product of the process that proves to be central. In defining a *procept* to be a combination of three things – a process, an object produced by that process, and a symbolism to denote either of these – we are only giving a name to an established concept in the minds of children. Dubinsky is right to say that it is no more than the encapsulation of a process as an object. However, it has a more profound function. Giving the name “procept” to this concept essentially *encapsulates the process of encapsulation itself*, allowing it to be manipulated on a higher level. An old colleague of mine, Bernard Scott, said “By giving a name to something, you acquire power over it”. In essence by using the name “procept” we can begin to formulate the concept of encapsulation as part of a wider theory.

Indeed, we can begin to talk about different kinds of procept, for instance, *operational* procepts, such as 3+2, which have a built in procedure to compute the result, *template* procepts, like 3+2*x*, which have an implicit process (add 3 to 2 times *x*) but can only be computed when a value for the symbol *x* is substituted into the template, and *structural* procepts, such as $\sum_{n=1}^{\infty}\frac{1}{n^2}$ which do not have an obvious procedure of computation (it can be done by complex contour integration!). There is empirical evidence to support the hypothesis that intuitive beliefs in operational procepts from arithmetic can confuse students when meeting template procepts in algebra, and structural procepts in analysis cause further confusion, since each concept fails to work in ways which seem implicit in earlier instances.

Symbolism plays a powerful role because it allows compressed communication to occur between individuals who can give the symbolism a meaning. A number word such as “five” can be *said*, can be *heard*, can be *written*, can be *read*, so that the senses begin to endow it with a shared meaning as concrete as a physical object. However, symbolism alone, without any sense of meaning can, as Skemp says, lead to “rules without reasons” or, as Sfard observed at Assisi, “processes without objects”. Meaning which links the new ideas to other already functioning cognitive structures is clearly a major advantage.

The cognitive function of “meaning” is straightforward. Short-term memory skills are very limited. The need to be able to cope with great complexity can only be solved by compression of knowledge, in which symbolism often plays a central role. The ability of some symbolism to work dually, evoking either a known process to compute a result, or an object to be manipulated at a higher level, is particularly favourable to reduce cognitive stress. Likewise the ability to link new ideas to old representations in the mind which themselves are in a form that can be readily manipulated can only increase the chances of success.

The transition to formal mathematics

The transition from operational, or representational, or other forms of intuitive mathematical knowledge to formal mathematics still requires serious study. Indeed I commend this study as a central topic for future empirical investigation and theory-building in the Advanced Mathematical Thinking Working Group. Although conflicts between concept images and concept definitions have often been documented, what papers are available in the literature that seriously study the process of definition–proof development? Without understanding this process, how can we prepare students for the transition to this form of thinking?

In a mathematical treatment a definition is given (perhaps in conjuction with some examples) and then theorems are proved about any object which satisfies the definition.

Cognitively this involves serious difficulties. For instance if a "group" is defined as a set G and a binary operation $\bullet$ satisfying certain properties, is there actually a "thing" called "a group", as opposed to many different examples which satisfy the *properties* of the group definition. When I prove properties of "a group", am I thinking of a *specific instance* of a group from those that I know, or *a prototype* of a group which is "typical" of a range of examples, or am I to think of an abstract thing called a group?

For, if there is to be a "thing called a group", then this thing must satisfy *all* the properties that can be deduced from the definition and *only* those properties. In practice therefore, what a mathematician sees as "proving" theorems about groups can cognitively involve the *construction* of the properties of "a thing called a group". For instance, we can call it G, and it has a unique identity which we call e, and every element $a \in G$ has a *unique* inverse a^{-1} satisfying $a \bullet a^{-1} = a$ and powers of elements defined by $a^1 = a$ and $a^{n+1} = a^n \bullet a$ satisfy properties such as $a^m \bullet a^n = a^{m+n}$ etc, etc. All these properties must be proved from the definitions by students who already know that they are trivially true in all the known examples. As the chain of deduction grows, the links may seem weaker and the status of results that are proved and those that are known may become confused. (I can remember being confused for a time over 2+2=4, which I *knew*, but I needed to *prove* in a new thing called a field". And if I could do this, could I *prove* that 2^{10} was bigger in an ordered field than 10^3?)

When writing *Foundations of Mathematics* with Ian Stewart, we faced this dilemma by stopping at various points and establishing that certain "facts" could now be assumed in the given context without quoting a proof. For instance, once we had established field properties, we could now use these without proof, but concentrate on new properties to do with order, then when these were established we moved on to a new level where arithmetic and order were assumed without further proof, but completeness properties needed to be established with painstaking detail. This pragmatism had some measure of practical success but it was never empirically tested in any experiment other than noting that it seemed to work.

Students are quite capable of using definitions in serendipitous ways, for instance, using different versions appropriate for different examples, as in the following quote referring to limits of functions:

> And I thought about all the definitions that we deal with, and I think they're all right – they're all correct in a way and they're all incorrect in a way because they can only apply to a certain number of functions, while others apply to other functions, but it's like talking about infinity or God, you know. Our mind is only so limited that you don't know the real answer, but part of it. (Williams, 1991, p. 232)

In other contexts definitions operate as *descriptions*. For instance, in a dictionary, the definition of a word simply evokes a context in which the word already has a meaning. It evokes an idea which the individual is expected to be able to put in context. In early school geometry the definitions of shapes such as triangles, squares, rectangles often act more as descriptions than definitions. Is a square a rectangle? What are the definitions? Can you *prove* a square is a rectangle? An isosceles triangle is a triangle with two equal sides. Can you *prove* that an isosceles triangle has equal base angles (and vice-versa). Proof here is often seen as coherent relationships between definitions which function as much to *describe* the objects as to *define* them. So it is that the student's concept image of definition and proof, based on previous experience, may be at odds with the notions of definition and proof in advanced mathematics.

The case of vector spaces and groups illustrate some of the problems faced by the student. Examples given of these two mathematical objects are subtly different. Examples of vector spaces tend to be given geometrically, as vectors in two or three dimensions, with vector addition and scalar multiplication having physical meanings. In essence, examples of vector spaces involve familiar *described* geometrical objects.

Examples of groups are different animals. Often the elements of examples of groups are transformations, for instance, permutations of a finite set, or transformations of a geometric figure with given symmetries. These elements are dually *processes* (transformations), which are composed by being performed sequentially, and *objects* (elements of the group). They are *procepts*.

This ambiguous use of symbol as process or object allows the expert to flip between one conception and the other. But it can cause great difficulty with a student in the early stages, which may account for some of the difficulties experienced by students meeting groups in this way for the first time. Certainly the operational aspects of groups, with the strangeness of lack of commutativity, and the dual meaning of elements as process and object, prove to be more difficult than the operational aspects of vector spaces. The difficulties are then compounded when the transition to a formal approach is attempted in either case.

Summary

I am sorry not to give a clear answer to the question I have posed – to explain how students can make the transition to a definition-proof construction of knowledge. However, it is one role of an initiator in a discussion meeting to pose questions for others to discuss. I can see that there is mileage in an initial operational approach to build up a concept image appropriate for later formalities, indeed the notion of a tool-object dialectic highlighted in Michéle Artigue's paper is also consonant with this, because it advocates the implicit use of mathematical ideas as tools to solve problems prior to reflection to turn them into mathematical objects.

I also believe strongly in the need for versatile knowledge in which symbolism can act as a pivot between process for action and concept for manipulation, and other representations can be selected for use when they present information in a more appropriate manner. As the individual is sucessful at more subtle modes of operation I see greater flexibility available in learning strategies. I find it inappropriate to accept "universal rules" that say that one mode of operation *must at all times* precede another.

I see the notion of encapsulation/reification of process into object as more clearly defined by the pioneering work of Sfard and Dubinsky, each giving clearer indications how the mechanisms might work. But I do not see this encapsulation acting in isolation as the *only* mechanism for constructing new knowledge. In particular, we need to attend to the methods of knowledge construction used by successful mathematicians. Such methods are also likely to be appropriate for some of our students who will grow into successful mathematicians. But they may not be appropriate for many other students who will undoubtedly need methods appropriate to their own needs.