Success and Failure in Mathematics: Procept and Procedure

1. A Primary Perspective

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"By giving a name to something, you acquire power over it"
(one of "Scott's Laws" proposed by the late Professor Bernard Scott, Sussex University)

Introduction

Why is it that so many fail in a subject that a small minority regard as being trivially simple? In this article we introduce an idea that offers an explanation for the great divergence in performance between those who succeed and those who fail in mathematics. Our initial focus is on the development of numerical concepts by young children, in a second article we will broaden the mathematical perspective to consider some of the mathematics learned in the secondary school and at degree level. By looking at the way in which mathematical ideas are developed by learners we come to the conclusion that the reason why some succeed and a great many fail lies in the fact that *the more able are doing qualitatively different mathematics from the less able.*

The more successful perform in a way which often makes the mathematics seem so effortless for them. If they seem to use so little effort, there must be an internal engine creating the motive force. What is the nature of this engine? We shall see that the more able have a kind of knowledge that *self-generates* new knowledge.

If others fail at mathematics, why is it that they often fail so catastrophically? We shall see that the catastrophe occurs because their mathematical thinking processes are qualitatively different. The less able fail because the mathematics they are doing is *more difficult* than the mathematics of the more able (Gray, 1991).

A major source of the generative power of mathematics is in the use of symbols. It is only in the last two millenia that the power of written symbolism has allowed mathematics to grow and to be passed from generation to generation, culminating in the great explosion of mathematics in recent centuries.

If we subject this symbolism to close scrutiny, we find that, although every attempt is made to refine it to make it explicit and unambiguous, its power lies in a very specific kind of *ambiguity*. It often refers both

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to *something to do*, and to *the result of the doing*. So 3/4 refers to the intention to divide 3 by 4, it indicates a process to be applied, and also to the result of the action of applying the process, the fraction $\frac{3}{4}$. The symbol "–2" refers both to the intention to subtract two, again indicating a process to be applied, and also to the result of the action of subtracting 2, the negative number –2.

First encounters with such symbolism can lead to bewilderment and consternation for the learner in establishing precisely what the symbolism means. It may be easy to "take away 2" but how can one have a negative number which is "less than nothing"? The most able see it for what it is: an amalgam of both process and concept, a process which in most cases is manifest through a procedure to get an answer, and a thing produced through the process or the manifestation of the process which can itself be manipulated as a mental object. This amalgam of process and concept we call a *procept*. By giving this construct a name we will begin to gain power over it. We shall see how we can focus attention on a problem that has baffled mathematics educators and the general public alike for centuries.

The development of mathematical processes into concepts

Number provides our first example of a mathematical process which develops into a concept. It is one of the first mathematical ideas that a child meets. Yet when our colleague Jack Forster asked a group of two hundred student teachers to say what "three" means, he met with total silence. What is it that we expect the youngest children in our school to appreciate yet proves so difficult for us to explain? Perhaps the difficulty lies in the fact that the notion of "threeness" cannot easily be defined in a sentence, but rather depends on an accumulation of experience. This is an insufficient and inadequate explanation; the problem needs further analysis.

The meaning of "three" is only established when the counting process is linked to the cardinal value of the set. Counting is a complex activity, which individual children perform in many different ways, perhaps using counters, fingers, marks on paper etc. It may be vocal or subvocal. It can be forward or backward, starting at one or starting at any given number. At a more abstract level it may involve both counting and keeping a check on the amount counted at one and the same time. Counting involves a focus of attention on each object that is counted (just once) and the co-ordination of this focus with the sequence of number words in an order.

Because of this complexity, it is idealistic to conceive of a single counting process shared by everyone. Each individual develops personal methods of carrying out the actions which are grouped together under the general umbrella of "counting". It is useful to distinguish between the notion of a *process*, which is the general intention to be carried out, and the particular method used by an individual at a given time which we shall call a *procedure*. There are many general processes in mathematics, including the counting process, the addition process, the subtraction process, the process of finding a fraction or the process of solving an algebraic equation. Each of these can be carried out by individual procedures, which may be the result of mechanical action, algorithmic routine or idiosyncratic behaviour.

The different possible procedures which arise are of particular interest when we turn to the process of addition and its related concept of sum. Consider the question:

"What is 2+2?"

The obvious answer is "4". But how is it calculated? It requires the *process* of addition. When the child attempts this for the first time, the addition process is usually manifested through a procedure which involves counting two sets each with two objects. At its simplest, it requires either taking two objects and two more, then counting them all saying "one, two, three, four", or carrying the first two in the head and counting on from two, saying "three, four". Thus "2+2" signals both the counting *process* and the *product* of that process: the number 4. The symbol 2+2 evokes both the *process* of addition and the *concept* of sum.

The use of a symbolism to mean both process and product occurs throughout mathematics. A *number* such as "three" involves both the process of counting and the concept of number. The symbolism 2+2 stands for both the process of addition and the concept of sum. This phenomenon occurs again and again:

- The symbol 4×3 stands for the process of multiplication "four multiplied by three" which in procedural terms may involve repeated addition to produce the product of four and three which is the number 12.

- The symbol $\frac{3}{4}$ stands for both the process of division and the concept of fraction,

- The symbol +4 stands for both the process of "add four" or shift four units along the number line, and the concept of the positive number +4,

- The symbol –7 stands for both the process of "subtract seven", or shift seven units in the opposite direction along the number line, and the concept of the negative number –7,

- In the second article on more advanced mathematics we will consider further examples in algebra, trigonometry, calculus and mathematical analysis, showing the all-pervading nature of symbolism which evokes both a process and a concept produced by that process.

The notion of procept

The idea of a process giving a product, or output, represented by the same symbol is seen to occur at all levels in mathematics. It is therefore worth giving this idea a name:

> We define a *procept* to be a combined mental object consisting of both process and concept in which the same symbolization is used to denote both the process and the object which is produced by the process.

Usually the object is produced by a specific procedure – counting gives the sum, repeated addition gives product – but we shall see in the second article that this does not always hold in sixth-form mathematics and here we shall find the lack of an adequate procedure leads to its own peculiar problems.

As with Scott's law, we will find that the giving of a name to this idea begins to help us gain power over it which enables us to explain the vast difference in success between those who succeed and those who fail.

A procept is, of course, a special kind of concept. It is usually first met as a procedure, then a symbolism is introduced for the output of that procedure, and this symbolism takes on the dual meaning which evokes both procedure and its output. As a child learns mathematics, the introduced symbol takes on a life of its own. It can be *written* (say, 3+2), it can be *read*, it can be *spoken* ("three plus two"), it can be *heard*. It is an external object that different people can share, so it has, or seems to have, its own external reality. It is the *construction of meaning* for such symbols, the processes required to compute them, and the higher mental procedures required to manipulate them, that constitute the abstraction of mathematics. Indeed the *ambiguity* of notation to describe either procedure or output, whichever is more convenient at the time, proves to be a valuable thinking device for the professional mathematician.

A procept is organic. It soon grows richer than the single process which generates it. Different symbolism may represent the same processes but different procedures may give the same product, as in the case of 4+1 and 3+2. The first, as a procedure, might involve counting on one from four whilst the second counts on two from three. However, the result of both procedures is the same number, "five". The procept of “five” therefore grows richer in its interior structure, giving a wider flexibility in which it may be decomposed in different ways, to be reorganized and transformed into an equivalent symbolism which represents a different procedure but the same product. We therefore envisage a procept as being *plastic* – something flexible that can be re-moulded and reconstructed at will.

For instance, the idea the 3+2 is 5, leads to several equivalent ideas, that 2+3 is 5, or that if “3 plus something is 5” then the “something” must be 2. This flexible use of composition and decomposition of numbers means that subtraction can be seen as just another way of looking at addition, so that the undoing of addition as subtraction is just a manipulation of flexible knowledge rather than the counting process manifest through the use of a procedure which may be the inverse of that used for addition.

It is in the use of procepts that we consider lies the major difference between the performance of the more able and the less able in mathematics. In the development of skills, it is the failure of some to see the processes encapsulated as procepts which leads to a catastrophic division between those that can do mathematics and those that cannot. Those who see processes interpreted as procedures and want only short term success – how to *do* something – are more likely to condemn themselves to a merry-go-round of procedure after procedure with success in being able to perform the current task, but inability to coordinate the processes into any larger coherent theory. Coordinating processes in time is a difficult activity. Those who develop flexible procepts have mental objects that both enable them to *do* the process and *manipulate* the symbols conceived as objects. The plasticity of procepts allow them to be flexibly decomposed and reorganised at will. Because of the richness of conceptual linkages, less needs to be remembered because more can be reconstructed. Thus proceptual thinkers are doing a qualitatively different kind of mathematics that is – for them – conceptually far easier.

This divergence which occurs at all levels of mathematics, between those who use interpret processes only as procedures and therefore make mathematics harder for themselves, and those that see them as flexible procepts we call the ***proceptual divide*** (Gray & Tall, 1991). The difference between success and failure lies in the difference between procept and procedure. Proceptual thinking includes the use of procedures where appropriate and symbols as manipulable objects where appropriate. The flexibility provided by using the ambiguity of notation as process or product gives great mathematical power.

The examples which follow consider the encapsulation of procedures into procepts at different stages of mathematical development. At any stage, if the cognitive demands on the individual grow too great, it may be that someone, previously successful, founders and asks “tell me how to do it”, anxiously seeking the security of a procedure rather than the flexibility of procept. From this point on failure is almost inevitable. It is for this reason that mathematics is known chiefly as a subject in which people fail, and fail badly, and fail often.

Example 1: The number concept

The number concept is a procept. It embodies both a process (counting) and the output of the procedure which is the implementation of the process (number) within the same symbolism. We have seen how a child learns the sequence of counting numbers in order, and begins the ritual of counting sets of objects by pointing at each in turn and reciting the number words. “One, two, three four”. There are “four” things in this collection. The same underlying procedure can be done in different ways: the order in which the numbers are counted can vary. Each of these procedures gives the same product “four”. Having the number concept means not only being able to count, it means being aware that different ways of carrying out the same counting process will give the same result.

Children may view number in two different ways, one as a PROCESS of counting – manifested as an uncrystallized procedure which is not regarded as a flexible concept – and one as the PROCEPT of number which involves counting, the knowledge that different ways of counting give the same result, and the crystallized number concept all in one flexible package. Thus the attaining of the procept of number is what Piaget would call “conservation of number”. The one additional item in our formulation is the dualism of the notation: the number “five” being built out of process yet being considered simultaneously of concept.

This is one of the first places in learning where a qualitative difference may occur between individuals. “Slower learners” do not form the procept of number early on. If such children are to move on to addition, they will be at a serious disadvantage. The mathematics that they will have to do will prove to be significantly more difficult for them. Whilst those with the procept of number will be able to manipulate the symbols flexibly in the mind to do arithmetic, the less able will now have the much more difficult task of coordinating processes – the counting process and the addition process, both manifest through a series of procedures which become one super-procedure – and then performing them in sequence. It is a far more complex task to carry out two processes in time, one after the other and to attempt to conceive of them in the mind as a single entity. It is far easier to manipulate symbols on paper which can be seen simultaneously and handled much more fluently. It is here that the proceptual divide occurs between those who are handling processes, coordinated sequentially in time, and those manipulating symbols for procepts which may be seen simultaneously on paper. Those who are fortunate to think using flexible procepts have fundamentally easier mathematics to do than those who operate by carrying out processes.

Such a divide is also embodied in Ausubel's (1968) differentiation between *meaningful* and *rote* learning, or Skemp's distinction between *relational* and *instrumental* understanding (1976). However, the theory we give here has an extra ingredient. It is not just the *relating* of one idea to another, or the giving of a *meaning* to a process or concept. It is the ability to give meaning to the process *in a flexible way that allows process and concept to be interchanged at will*, often without any distinction being made between the two.

Example 2: Addition

Addition can only be carried out meaningfully when the number procept has been established as embodying both process and concept. However, in the early stages, more elementary forms of addition occur which are either processes or combinations of procept and process.

For example, *counting all* is a strategy used by young children in which, given the sum "3+2", is translated into a procedure which involves first counting a set of three objects, then a set of two objects, then collecting the two sets together and finally counting the whole set "one, two, three, four, five". Thus the count-all strategy consists of *three* distinct counting procedures, one after another. Each separate procedure is a manifestation of the counting process. Thus COUNT-ALL consists of PROCESS plus PROCESS leading to a third PROCESS. Given the fact that these occur successively in time, it requires considerable cognitive effort to link the input numbers 3 and 2 to the output 5. So the child using count-all is less likely to see the triple counting procedure as leading to the number fact.

Counting on is a more subtle procedure. It occurs possibly through the realization that to count the first set is simply a repetition of a counting process which involves recitation of the initial number names and presenting the number name of the last tagged element as the cardinal value of the whole set, so a short cut is possible in which one simply counts on from the next number name after the number in the first set. Counting on 3+2, simply involves saying two numbers (starting after the "three" of the first part) to get "four, five", so that result is "five". Here the first number is seen as a *procept* and the second as a process. COUNTING ON consists of PROCEPT & PROCESS.

But the counting process here is a subtle double-counting procedure. It is necessary both to count-on "four, five" whilst keeping track of the number of elements counted (two). Often one of these counting procedures is done by using concrete elements for support. For instance, a child might move two fingers to keep track whilst counting "four, five". Alternatively it is possible to use a number line or a ruler to start at the number "three" then to point at successive numbers whilst counting "one, two" to end up at "five".

The complexities involved in counting-on, especially for the less able child, can lead to two possible outcomes. For the less able child, counting-on may become the favoured procedure to do addition. It may take an input, say 8+4, and produce a counting-on "nine, ten, eleven, *twelve*", giving a result 12. As this occurs *in time*, the child may be successful in getting the result, yet not relate the earlier input (8+4) to the output 12. For this child counting-on is but a *procedure* for getting a result. It is less likely to lead to the remembering of the result as a known fact.

For another child, who is able to relate input and output, this may crystalise into a procept, in the form of a known fact, "8+4 *is* 12". In this more fortunate case, counting on takes the form of PROCEPT & PROCESS giving PROCEPT.

Over a period of time the latter is more likely to lead to the pupil developing a collection of flexible "known facts". These could, of course, be learnt by rote, just as the chorus of the song "Inchworm" sung by Danny Kaye in the film "Hans Christian Andersen" goes:

> One and one are two,
> Two and two are four,
> Four and four are eight,
> Eight and eight are sixteen,
> Sixteen and sixteen are thirty-two...

But a rote learnt phrase has no generative meaning. One may be able to repeat the words "four and four are eight", but this gives no clue as to the meaning of four and *five*.

At the highest level a known fact is seen as a procept. The known fact "3+2 is 5" can be immediately recalled as a number bond "3+2=5", but if necessary it can be decomposed either as procept plus process (counting on) or even process plus process (counting all) but, more likely, it will simply be visualized as an iconic array of five objects broken down into a three and a two.

The less able child may make some steps in this direction. But known facts are harder to learn and with fewer facts it is difficult to use them flexibly. Some known facts are learnt before others (e.g. "adding one" (3+1=4), "adding two to an even number" (4+2=6), or "doubling"

(4+4=8) or “number bonds making ten” (8+2=10)). Even if a less able child collects together some of these, the burden of using them may be great. For instance Stuart (aged 10) was asked to calculate “8+6”. He said:

> “I know 8 and 2 is ten, but I have trouble taking 2 from 6. But 4 and 4 makes 8, and 6 and 4 makes 10, and the other 4 makes 14.”

He is here successful, but leans on the number bonds 8+2=10, 4+4=8, 6+4=10 based around “sums making ten” and a “double”. He cannot cope with 6–2 and his methods may collapse with other number pairs that are unavailable to him. Such children may find that using the few known facts at their disposal causes considerable strain and seek the solace and security of counting procedures.

*This underlines the need for children to **know** their number bonds*. The national press is full of demands for children to know their tables. Our experience shows that the problem is further down. Children need regular practice to establish all the number combinations to twenty in a context which enables them to use them proceptually. There are some favoured number bonds that are learnt easier than others (add on one, doubles such as 4+4) and there are others that are accentuated (such as number bonds that add to ten) but there remain others, such as 3+5 or 6+8 which tend to get missed out and would benefit from regular practice. Otherwise we may have children like Stuart, who are beginning to manipulate numbers relationally but may eventually fail because of the lack of appropriate number bonds.

Once more we are faced by the *proceptual divide*, now at a higher conceptual level. The child who sees addition only as a procedure is faced with the difficult task of coordinating different mental procedures whilst the more able child who develops a collection of flexible known facts can use them to derive other facts: for instance, “8 and 9” is seen as the double “8 and 8” plus one, giving “seventeen”. The child on the borderline between the two may be in a delicate state of balance where successful use of derived facts may lead on to further success, but failure through inadequate development and use of number bonds may degenerate to the security of counting.

In this way the more able child moves from a body of flexible *known facts* to building up a technique of re-assemblage of procepts which give new *derived facts*. At this stage mathematics starts to get *easier*. Because so much can be derived, less needs to be learned. The use of procepts builds up a feed-back loop in which known facts are re-combined in new ways to produce new known facts. The more able child has an organic knowledge structure which grows under its own internal energy, almost without any seeming effort. Arithmetic becomes more sophisticated and progressively simpler to do.

Meanwhile the less able child has a different goal in mind, the goal of mastering the procedure of counting and applying it to more complicated tasks. The procedure of counting-on – which may be proving very successful – makes the child a victim of its own limited success. Although individual sums may be correctly carried out by counting on there is less likely to be a feed-back loop giving the explosion of knowledge characterising the more able child. The procedure of counting-on does not give a coordination between input and product which leads to new known facts. Only with extensive practice may some facts be learned incidentally but many of these are retained in isolation. The child therefore has much *harder* mathematics to do: each time a sum uses the same (counting on) procedure but as the sums get more difficult there is little flexible structure on which to build.

If the child obtains success through counting-on, then this leads to a security in the use of this system which makes it far harder to progress. Sums involving several digits must now be performed with counting-on subroutines that drastically extend the length of the procedures involved, placing a greater strain on the weaker child, leading to likely failure.

This is a savage indictment of the belief that children should be allowed to develop their own personal modes of performing arithmetic. For if the less able child develops a highly personalized method of coping with a limited range of sums (often by counting using various parts of the body or various finger configurations to represent different numbers), then that child may develop methods which *do not generalize*. There may be short term success with small numbers, but catastrophic failure with more general problems. In the day-to-day running of a classroom short-term success may be more immediate and instantly rewarding, but if it is at the cost of eventual failure, it is a devastatingly bad strategy.

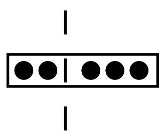
Allowing children only to do number work at their own pace from work-cards designed at their own level can actually disguise the symptoms of eventual failure. The child may succeed at addition sums more slowly through counting procedures, yet may be developing the very strategies which lead down a cul-de-sac. Only through discussion and *listening* to a child talking through the processes being used can one hope to diagnose the possible development of inappropriate strategies.

Example 3: Subtraction

Subtraction, as a reversal of addition proves to be far easier as part of a proceptual structure rather than as a reverse of a procedure. In the former case, if a child has a meaningful concept of 4+2 being 6, so that 6 can be decomposed into 4 and 2, then subtraction is already built in to the structure. If 2 is taken away from 6, then 4 remains. If it is not, we may end up with the phenomenon of Stuart who has "awful trouble taking 2 from 6".

There is worse to come. If addition is seen only as a procedure, such as "counting on" using a number line, the individual may attempt to "take away" by counting back. Thus 9 take away 3 is performed by pointing at the number 9 and counting back three: 8, 7, 6. Here subtraction is seen as a true reversal of the procedure of addition, but it is a view fraught with danger. Counting back may be easy on a concrete number line, but the mental operation of counting back is one of the most difficult ways of performing subtraction. It requires a reverse counting of the number sequence (9), 8, 7, 6, with a corresponding count of the number of terms counted (1, 2, 3), giving a double counting procedure of great complexity compounded further by the difficulty of starting at the right place (counting down from 9 requires starting the double counting at 8). Once again, the child locked in procedure is faced with the far greater task.

For the more successful child, the flexible procept of addition and the existence of appropriate additive number bonds can lead easily into derived subtraction facts, once again with a feedback loop which makes the procepts even more flexible and powerful. The following now all mean the same thing: "3+2=5", "2+3=5", "3+something=5 means something=2", "2+something=5 means something =3", and so on.



2 + 3 is 5
or 3 + 2 is 5
or 5 – 3 is 2
or 5 – 2 is 3
....

Contrast this with the intriguing example of a particular slow learner responding to the problem "5 take away 3" using a limited number of known facts:

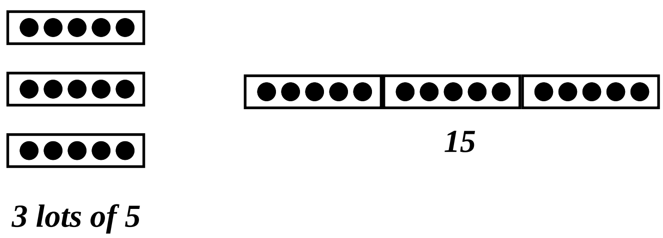
> "That's one I always have difficulty with. But I know two and two is four, so two and three is five, so five take away three is two..."

This amazing piece of mathematical deduction shows the mental powers of the child considered to be slow. He does not lack mathematical flair. But he has a limited array of known facts and uses his knowledge to derive facts by such a tortuous route that even a simple sum is a major voyage of discovery. It is no wonder that our intrepid voyager falls down when the journey gets just that little bit longer. He has the knowledge of how to travel but his map is in tiny parts that do not fit so easily together. Although he is beginning to use his knowledge imaginatively, the product of his procedure is not readily added to his knowledge base, and there is no general feed-back loop to link subtraction flexibly to addition. If he fails as the problems increase in difficulty, then he is likely to revert back to an earlier strategy – counting – which increases the burden of difficulty that he faces.

Example 4: Multiplication

Multiplication may be seen initially as repeated addition: 5×3 is five repeated three times. Again the notation is a flexible procept. If the five and the three are seen as processes, not procepts, then repeated addition through adding five to five to five is unbearably difficult. (“six, seven, eight, nine, ten ... that’s two fives, eleven, twelve, thirteen, fourteen, fifteen ... that’s three fives”.) However, if it is seen as a procept, two lots of 5 might be seen as 10, then 10 and 5 as 15, by combining and recombining the constituent parts.

Flexibly linking the product to a visual array can help see that different processes give (essentially) the same product:



Thus it is that the procept takes on different representations, which are still considered to be the same product.

How is a child who still sees addition as counting to be expected to cope with multi-digit multiplication? One of our student teachers was confident that she could teach anything to any child, provided that she had the time. She had reached a point in the National Curriculum where a child must multiply a three digit number by a single digit number. A slower learner was faltering, so she went through a simple example : “234 times 2”, explaining that first the 4 must be multiplied by 2, so the child counted on “five, six, seven, *eight*”, then the 3 must be multiplied by two, so the child counted on “four, five, six”, then the 2 multiplied by two (“three, four”). She found that the child seemed to be able to do it when she was there to help, even when “carrying” was involved, but there were so many steps to carry out, that when the child was left to his own devices, the procedure collapsed under the weight of all the counting.

Example 5: Place value

Place value uses notation in a powerful way. The two 3s in the symbol 353 are used with entirely different meanings, the first being 3 hundreds, the second 3 units. The place value notation not only represents a number as a combination of units, tens, hundreds, ..., it does so in a canonical way in which the number of bundles of each is between 0 and 9. Initially a child may think of a number in the same way as a name, perhaps living at number 47, where the symbols 4 and 7 have no more separate meaning than the two ms in “mum”. True meaning can only be given when the individual digits are conceived as procepts: both the process of counting and the concept of a group of objects that can be regrouped in various ways. The symbol 5 might be seen as 2+3 or 4+1, and this extends to 12 being seen as 10+2 or 32 as 30+2. It requires the *procept* of number (rather than just the process of counting or the knowledge of the sequence of number words) to be able to view place value as both a grouping procedure in which 452 is the procedure of grouping “4 hundreds, 5 tens, 2 units”, and the result of the procedure: the number 452.

Conclusion

What we have attempted to do in this article is to consider the ambiguity of symbolism in the initial stages of mathematics. What we have seen is the inherent ambiguity of mathematics symbolism and the development of procepts out of the applied actions of mathematical process. The *proceptual divide* occurs between those who complete this transition and those who fail.

Our analysis points to the weakness of the less able child turning to the security of procedures rather than the successful use of procepts. Therefore the additional practice at such procedures may only make the differences greater, not close the gap.

In general classroom activity it is essential for the teacher to *talk* to individual children and to *listen* to how those children are performing their arithmetic calculations. Simply allowing them to carry out idiosyncratic procedures may actually be leading them up a cul-de-sac of eventual failure at more advanced arithmetic.

This burden imposed on the less able, who are constrained to perform harder (procedural) mathematics rather than the more powerful and easier proceptual mathematics, provides a challenge which seems have little hope of resolution by traditional means.

As a first stage to resolving the problem our "*cri de coeur*" must be that we look closely at the ways through which children achieve success: the methods that bring about short term success may lead to long term failure.

Within the next article, we extend the notion of procept to higher mathematics and consider points where the proceptual divide may occur. As a further attempt at resolution of the problem we conclude on a positive note and consider ways in which we may concentrate on the concept rather than responding to immediate need through reliance on a procedure.

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