Misguided Discovery

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It seems that the world is changing more now than it ever did before. So we must educate our young to be flexible in their thinking and to be able to face new problems with confidence. Part of this trend is seen in the introduction of problem-solving and investigations into the mathematics curriculum. The problem with this change is that it needs a new kind of thinking by teachers as well as pupils, and it is making demands way beyond those of traditional mathematics. Are we ready to face the new challenge? Is it fair to teachers and to children to introduce new techniques almost overnight without adequate research and adequate teacher preparation?

I very much want to see our educational system succeed, and passionately believe that we must equip our children to face problems that we have never seen. But such a drastic change requires far greater time for preparation, experience and reflection than our system is prepared to give and, as we learn from our mistakes, it is our children who must accept the consequences.

Jamie – as we will call him to protect his identity – is fifteen years old. He is a gifted child who is expected to get a grade A in his GCSE Mathematics. His teacher is faced with the new curriculum and presents the class with an extended project. Instead of teaching children the full intricacies of euclidean geometry, the idea is to encourage them to discover some of the ideas for themselves and to write them up as work to contribute to their GCSE assessment. They have been introduced to the idea of making constructions using straight edge and compass and have experienced a number of techniques: bisecting angles, subdividing lengths into a number of equal intervals, and so on. The investigation is to use straight edge and compass to construct a number of given figures and then to go on to see what angles, other than 30˚, 45˚, 60˚, 90˚ can be constructed.

Jamie attacks the problem with relish. He demolishes the given figures and begins on what he sees as the main event. He attacks the problem systematically. If only he could *trisect* an angle, he could divide 30˚ into three and get 10˚. He could then bisect it to get 5˚, and with a little more effort he could divide this into five equal parts to get 1˚. He could then build up an angle of any size (which is a whole number of degrees). So far, so good. His logic is OK. The only trouble is: *can* he trisect an angle?

Of course Jamie hasn't heard that mathematicians vainly sought a solution of this problem for two thousand years, nor that a young French mathematician who showed

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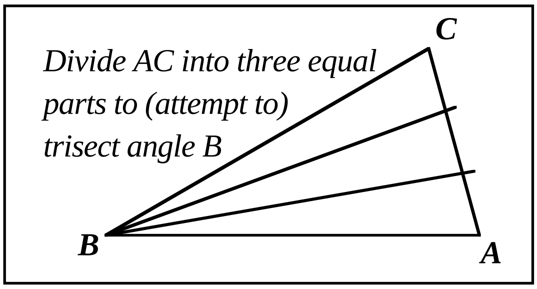
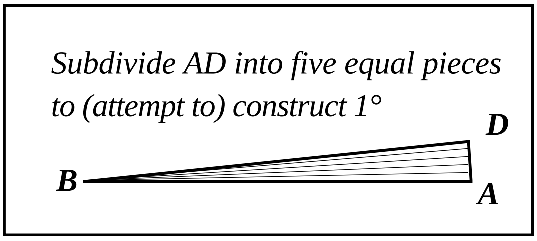
that it was impossible was ridiculed by professional mathematicians of his day before being tragically killed in a duel. Jamie is young, vigorous and untainted by serious failure. He believes it *can* be done.

He has an idea. He knows how to trisect a line, why not use the same method to trisect an angle? So he takes an isosceles triangle with a 30˚ angle between the equal sides and trisects the opposite side.



He does it as accurately as he can and then measures the resulting trisected angles. They all equal 10˚.

He bisects one of these angles, then divides the 5˚ angle into five parts using the same technique.



With very careful drawing, as best as he can measure, each of the tiny angles is one degree. So he has *conjectured* that he can do it, he has *tested* his conjecture empirically, and it works.

He goes on to show how he can construct a number of regular polygons by working out the angles required and constructing them using his theory. Again, he has *conjectured* that he can do it, he has *tested* his conjecture empirically, *and it works*.

Triumphantly he uses a computer word-processor to write his material up neatly, with his solutions all set out in his best possible draughtsmanship. Typical of many boys he writes in a terse, pointed style, stating the results with only the briefest of explanations. The previous year he had the experience of getting every answer correct in a test yet obtained only 87% because he did not explain answers whose truth he considered to be self-evident. But this has not daunted him, for now he believes that his hour has come;

this time he has made a real discovery and he does his best to write it up. It takes many hours, but it is worth it. At last it is finished and it looks good. He submits it for assessment and waits eagerly for the marks...

He gets a grade D. He has failed.

Jamie’s teacher knows that it is theoretically impossible to trisect an angle. He knows that almost all of Jamie’s work is based on a false premise. So it is wrong. But he seems not know that it is possible for the *process* in the proof to be correct, even though the premise is false. *If* Jamie could have trisected an angle then divided a 5˚ angle into five equal parts *then* he could have done all the things he claims. His argument is faultless. His premise is faulty.

At least, his *theoretical* premise is wrong. In a practical sense he has done well. In *Thinking Mathematically* John Mason and his co-authors tell us that the process of conjecture and proof consists of three stages:

> convince yourself
> convince a friend
> convince an enemy.

Young Jamie did two out of three. He convinced himself, he convinced not one, but several friends in his class that he was empirically correct, in fact they were very impressed. But he failed to convince the enemy. The enemy, in this case the teacher acting as assessor, did not check the accuracy of Jamie’s empirical evidence, for if he had done, he might have been a little more sympathetic. In Jamie’s putative trisection of 30˚, a little intricate trigonometry shows that the actual angles are 9.9˚, 10.2˚ and 9.9˚ – so close to an exact trisection that it is unlikely to be seen as different in the drawing of a fifteen year old – even in the drawing of a professional draftsman for that matter!

Jamie’s class have done a lot with calculations to make them as accurate as possible. Jamie tried to get the most accurate possible trisection of an angle. And it was close! But not close enough in the eyes of the teacher.

I sympathise with Jamie. When I was sixteen I was told that it was not possible to trisect an angle and I promptly spent the next fortnight trying to do it. I remember distinctly noting that the “angle at the centre was twice the angle at the circle” and somehow tried to get the angle at the circle redrawn abutting the angle at the centre so that the sum of the angles gave a new angle divided into two parts, one twice the other. I never managed the construction; I was unsuccessful but not defeated. Years later at university I was shown that constructions using straight edge and compass essentially involve drawing straight lines and circles, and finding where they meet is equivalent to solving linear and quadratic equations. More complicated constructions lead to combinations of quadratic equations which turn out to be quartic equations, equations of degree eight, then sixteen, and so on. There is no way that one can solve a *cubic* equation by such a construction and it is easy (for a more mature mathematician) to show that $x=\cos 10^\circ$ is the root of such a cubic equation. Hence trisecting the angle 30° is *theoretically* not possible, but Jamie got it close as a gnat's whisker.

In carrying out his investigation, Jamie had done (almost) all the right things: he had a *plan*, he attacked the problem *systematically* and, where he made *conjectures* he *tested* them in the best ways he knew how. He is a *problem solver*. His only failure is that he lacks the knowledge and sophistication to see a very subtle error, which amounts to the distinction between what can be done *theoretically* with a theoretical straight edge and compass, and what he can achieve *practically* with his own instruments.

Perhaps if he had tried his trisection idea on larger angles, he might have seen that the middle of the three angles is actually a bit bigger than the two on either sides. But he didn't. That is a level of sophistication far beyond what might be expected of a novice problem-solver.

Yet it is well-known in mathematics that some of the greatest advances come through making such mistakes. A favourite quote of mine comes from Poincaré, a great and visionary mathematician who was himself not above publishing an incorrect proof of a theorem:

> It is impossible to study the works of the great mathematicians, or even those of the lesser, without noticing and distinguishing two opposite tendencies, or rather two entirely different kinds of minds. The one sort are above all preoccupied with logic; to read their works, one is tempted to believe they have advanced only step by step, after the manner of a Vauban[1] who pushes on his trenches against the place besieged, leaving nothing to chance. The other sort are guided by intuition and at the first stroke make quick but sometimes precarious conquests, like bold cavalrymen of the advanced guard.

Jamie is such a bold cavalryman, cut down in his prime. Yet the assessment of his progress failed to reward him for attempting to make such a visionary leap into the dark. What is needed is the realization that the child should be rewarded for successfully completed a number of phases of the problem-solving process and that he needs careful guidance to encourage him to review his work more carefully. To deny the existence of mistakes in problem-solving is to deny the process itself, for from erroneous conjectures, can come improvements, refinements and true progress.

[1] Sebastien de Vauban (1633-1707) was a French military engineer who revolutionized the art of siege craft and defensive fortifications.

I believe that the role of the teacher is as a *mentor,* who must use the wisdom of Solomon to know when to stand apart and let the child learn through making mistakes and when to intervene to guide the child into a fruitful and powerful path. The child should be encouraged to discover, and also to reflect on those discoveries to learn to become an independent thinker. But when the child lacks specific tools to do a job and moves into an area where serious errors may be committed, there comes a time when the mentor should intervene and help to *guide* the discovery. It is a difficult equilibrium for the teacher to maintain and beyond my own ability to consistently make the right choice.

The achilles heel of our new system is that we have introduced new (and relatively untried) techniques of *teaching* and *assessing* problem-solving with insufficient teacher preparation to carry them out and with little attempt to understand the *learning* processes involved. When the teacher adopts the complementary role of assessor to judge what it is that the child learns, it seems that the role of teacher must fall temporarily in abeyance. When the teacher has little experience of children operating in such an unfamiliar context and even less experience of assessing what it is that the child is doing, then, without appropriate guidance, the child may suffer.

Jamie has learnt from his experience. What he has learnt is that he should not attempt to do anything imaginative which may rock the boat. To succeed in what remains of his GCSE and recover the failure in his first investigation, he told me, with some feeling, that he must "keep his nose clean" and only attempt "fairly standard things that the teacher can understand". I do not think that this what we want children to learn from their investigations. But Jamie has discovered it, and it is a misguided discovery.