Verbal Evidence for Versatile Understanding of Variables in a Computer Environment

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We have previously reported (Thomas and Tall, 1986, 1988) on experiments demonstrating the value of a computer-based pre-algebra module of work in aiding 11 and 12 year-old pupils to reach a higher level of understanding of the use of letters in algebra than that found in a more traditional approach. We have also put forward the hypothesis that one reason for this success is the way "cognitive integration" (Thomas 1988) of the child's global/holistic and serialist/analytic cognitive abilities leads to versatile thinking. Further, this may be actively promoted using the "enhanced Socratic mode" of teaching (Tall 1986) using the computer as a resource for teacher demonstration, pupil exploration and discussion to develop appropriate concept imagery. This paper considers evidence in support of the theory from interviews with the students involved, taken six months after the computer treatment.

Some Theoretical Considerations

When algebra is perceived, and hence taught, as an essentially logical, serialist activity with little or no recourse to either its inherent structure or its underlying concepts - such as the use of letters as generalized numbers or variables - one would expect this view of algebra to prevail among pupils. A substantial body of research points to just such a lack of understanding as contributing to poor performance in algebra throughout secondary school and beyond (e.g.Rosnick and Clement 1980, Matz 1980, Küchemann 1981, Wagner, Rachlin and Jensen 1984). The results of our work have suggested differential effects between the computer-based approach to algebra, with its emphasis on letters as generalized numbers and the traditional skill-based type of module with its emphasis on acquiring manipulative skills. It seems that the computer work promoted a deep conceptual understanding better, while the other work, as expected, initially facilitated better surface skills. However, when the computer module was combined with the skill-based one then it led to a superior overall performance without detrimental effect on skills. It is our view that the computer is providing an environment in which pupils acquire a global/holistic view of algebraic concepts - relating the symbols on paper to meaningful ideas such as the mental picture of a letter representing a variable number - in contrast to the more serialist/analytic view nurtured by emphasizing the operation on symbols. An illustration of this is the following type of question, with which many will be familiar:

Published in *Proceedings of P.M.E.*, Paris, volume 3, 213–220 (1989)

Factorize $(2x + 1)^2 - 3x(2x + 1)$.

Many pupils faced with this type of question seem locked into a sequential/operational mode of working where they “multiply out the brackets”,“collect together like terms” and factorize the resulting quadratic function. Few are able to apply the versatility of thought to switch from an analytical approach to a global/holistic one which “chunks” together the symbols 2x+1 as a single conceptual entity, allowing them to move more directly to the answer. We believe that the activities carried out in the computer context encourages flexible mental constructs more likely to lead to this global/holistic view.

Evidence For Versatility and Conceptual Understanding

Conceptual understanding in algebra is not evidenced by test performance alone. Correct answers to routine problems may be produced by incorrect understanding and incorrect responses to non-routine problems may have a sensible foundation. In order to examine pupil’s understanding of algebra beyond the test performances indicated in Tall and Thomas (1988), we conducted a number of interviews with selected students and administered a broadly based questionnaire to see if certain phenomena which occurred in the interviews were replicated on a wider scale.

The Interviews

The teaching experiment (Tall & Thomas 1988) had comprised two groups of 13 year-old secondary-school children taken from six mixed ability forms, arranged into 57 matched pairs. The experimental group used the computers for three weeks, following a module of investigational activities, while the controls followed their traditional algebra course. Six months later, all the pupils were given the same traditional module for a two week period, the controls as revision, the others for the first time.

After the post-test in this experiment a cross-section of 11 experimental and 7 control pupils (of comparable performance on the post-test), were given a semi-structured interview lasting about twenty minutes. During the interviews, which were recorded, the pupils were required to attempt certain key questions and to explain their thinking and strategies. The following examples taken from the transcripts of the interviews show a marked difference between the experimental pupils, who often attempted to give a relational explanation for their reasoning, and control pupils, who were more likely to be concerned simply with carrying out routine algebraic processes.

Question : Solve $2p - 1 = 5$.

The following response from a control pupil illustrates the confusion that may arise from mechanically carrying out routine processes:

> Pupil 11 : 2p minus 1 equals 5. If you add the 1 to the 5 that's 6 so, because there's no other minus p, I forget the p and do the 2p minus 1 equals. If you add the 1 to the 5 which is 6 and then you take 1 from the 6... No, I don't get that. I know I've done it but...
>
> Interviewer : What would the value of p be did you say?
>
> pupil 11 : Six.

Here the explanation is solely in terms of the operations with no reasons for their use being cited. This may be compared with the following reasoning from one of the experimental group pupils :

> Pupil 2 : Well find out what minus 1 so you would add 1 to that so you get rid of the 1, so that would be 6 and then its obvious that 2 times 3 equals 6, so p would be 3.

The pupils in the interviews were also asked to compare the above equation with

$$2s - 1 = 5.$$

This was in order to see if they were able to conserve equation (Wagner 1977) under a change of variable. A distinct difference in the type of comment between the two groups shows the superior understanding in this area of those pupils who had used the computer.

Control group :

Those unsure of the relationship :

> Pupil 10 : s could be 3 as well.
>
> Pupil 12 : So s could be 3 as well.
>
> Pupil 13 : They could both equal 4.

Those who needed to solve both equations:

> Pupil 11 : Well what I have put is 2p equals 6 and 2s equals 6.
>
> Pupil 14 : 2s...add the 1 and 5, 6 er 2 and 2, 6, 3 times, so s is 3 as well.

Experimental (computer) group :

> Pupil 1 : I can say that p and s have the same value...it's the same sum.
>
> Pupil 2 : Well they are both the same...Yes, because they are both the same but different letters.

Pupil 3 : They are both...p and s both equal 3.

Pupil 4 : It's just a different letter but it would have to be 2 times 3 minus 1 equal to 5.

Pupil 5 : The same. Just using a different letter.

Pupil 8 : It is 3 the p and s...because they are basically the same sum, but are different letters.

Pupil 9 : They are both the same. It's the same apart from the letters,exactly the same except the letters.

These pupils offer verbal evidence of a global/holistic view of the equations enabling them to develop the understanding of conservation of equation by seeing the common structure of the equations. This concept of conservation of equation under a change of variable was further tested with several of the children by the use of an extension to the first question above to :

Solve 2(p + 1) - 1 = 5.

The insight of the computer group pupils is shown by their comments:

Pupil 1 : Yes, p equals 2.

Interviewer : How did you work that out then?

Pupil 1 : Well its the same, but its plus 1, so minus 1 add 3.

Pupil 2 : Oh it would be 2.

Interviewer : Can you tell me why?

Pupil 2 : Because p plus 1 if that's 3 its the same as the last one only the p is less because you've got to add 1 to the sum.

Deep and powerful insights such as these, which are facilitated by a global/holistic view leading to the structure of the equations was not matched by the controls. Instead we have:

Pupil 15 : Say p plus 1, there is already 1 plus p plus another one, I'd say that was 2p, and then outside plus another 2 that is 4 minus 1 is 3 I would say.

Interviewer : So what is the answer?

Pupil 15 : p equals 1 I would say.

Extension of algebraic ideas

Research has indicated that the type of algebraic equation where there are variables on both sides of the equation is considerably more difficult, since it involves algebraic manipulation (of variables) rather than arithmetic (e.g. Herscovics and Kieran 1980). Neither experimental or control pupils in the the experiment had been taught to solve this type of equation. It was hypothesized that the relational understanding of the experimental pupils would lead to their greater ability in handling such equations. Several interviewees were asked to tackle the question:

Solve $3x - 5 = 2x + 1$.

The replies again gave evidence of superior understanding on the part of those who had used the computer.

Controls :

Pupil 15 : I'd say it was minus 2x and here you've got 3x, 2x plus 1x,so I'd put that as 1x

[Writes $3x - 5 = 2x + 1 = 1x$]

Interviewer : And is that the answer?

Pupil 15 : Yes

Hence, although the surface operation of subtracting 2x is carried out it does not seem to be in the context of any understanding of an overall purpose in the question, and no reasons for the operation are given. One of the pupils in this group had lost sight of the objective altogether:

Pupil 12 : I'm trying to work out how you could take 5 from that to leave that.

Interviewer : Can you see any way of doing it?

Pupil 12 : You would have to find the value of x before you could start.

In contrast, the experimental group pupils given this question responded more purposefully :

Pupil 1 : Well the value of x must be the same because it's in the same sum... I'm thinking that maybe take x some number away from both sides. That wouldn't leave anything in there to go on. You'd have nothing there if you take 2x away and 1x minus 5 equals plus 1.

[Writes $x - 5 = +1$]

Interviewer : So how might you do it now?

Pupil 1 : I was thinking maybe get rid of this and forget about that 4 by putting, adding 5 to both sides - that should do it - so it would be 3x equals 2x plus 6...try to take x away.

[Writes $3x = 2x + 6$]

Shortly after this he solved the equation.

Pupil 2 : You would add 5 to that to get rid of the minus 5 and then that plus 6 so it would be 3x equals 2x plus 6....Well that plus 6 has got a bigger x because 2x plus 6 equals 3x, that means another 6 would be equal to x, so make that 3x as well...Well x equals 6.

We can see that this pupil starts off with a serialist/analytical approach, but accompanied by clear reasons for the steps taken. However, in the middle of the question the pupil is versatile enough to change viewpoint to a global/holistic one and see the equation in terms of its balancing structure, enabling the equating of an extra x with 6.

The Questionnaire

A questionnaire given to 147 pupils, whilst not giving the opportunity to follow up answers as in an interview, gave evidence of a wider dispersal of the phenomena found in the interviews. It included three types of questions; one where they were required to explain, with reasons, whether two algebraic expressions were equal or not; one where they had to explain to an imaginary visitor from Mars the meaning of some algebraic notation and the third where harder algebraic questions, beyond the level they had studied, were to be attempted.

Question	Experimental Proportion Correct	Control Proportion Correct	z	p
Is $\frac{6}{7}$ the same as 6÷7 ?	0.76	0.44	3.38	<0.0005
Is 2+3c the same as 5c ?	0.41	0.31	1.24	n.s.
Is 2(a+b) the same as 2a+2b ?	0.57	0.31	2.69	<0.0005
Solve 13-y=2y+7	0.43	0.27	1.83	<0.05
Simplify 5h-(3g+2h)	0.24	0.08	2.16	<0.025
Solve 17-3e>2	0.31	0.13	2.37	<0.01

Table 1 - A comparison of some questionnaire facilities

Error	Experimental Proportion Making Error	Control Proportion Making Error	z	p
3 + m = 3m	0.09	0.27	2.54	<0.01
ab = a + b	0.06	0.13	1.77	<0.05
b - 2 × c = (b - 2) c	0.09	0.23	1.77	<0.05
3 + 2m = 5m	0.04	0.13	1.57	n.s.

Table 2 - A comparison of some questionnaire errors

The results in tables 1 and 2 from selected, and the fact that *the controls did not perform significantly better than the experimental group on any question*, support the hypothesis that the experimental students have a better understanding of algebraic notation. Moreover, it also seems that one of the main failings of the controls is that the traditional skill-based module has encouraged a predominantly left-to-right sequential method of processing algebraic notation. In contrast to this, the computer group, seem to have a better, more global, view of the notation which in turn has reduced the occurrence of some of the more common notational errors such as conjoining in addition and the wrong use of brackets. An interesting example of this, although arithmetic rather than algebraic, is the first question in table 1, where many of the controls did not consider the two notations as the equivalent because

"$\frac{6}{7}$ is a fraction, 6÷7 is a sum".

This is a good example of a response which is based on sound conceptual reasoning, but one that is limited because it implies the inability to encapsulate the *process* 6÷7, as a single conceptual entity. The encapsulation occurred far more often amongst the computer group, again underlying what we believe is a more flexible global view.

The difficulties that pupils had with the question

> Is 2(a+b) the same as 2a+2b ?

again revealed the difference between the symbols representing *a process* and the *result* of that process as a conceptual entity. So firmly had it been ingrained in them that “calculations inside brackets must be done first” that the symbol 2(a+b) is read as “first add a and b, then multiply by 2” whilst 2a+2b requires both multiplications to be carried out before the addition, that they saw the *processes* as being different rather than the *results* being the same. Even so, the experimental group were once again more likely to attempt to surmount this conceptual obstacle, one student proposing an interesting way out of his dilemma:

> Pupil 1 : Well its brackets, so you’ve got to add these two numbers before you times it [...]
>
> Interviewer : You can’t see any way round that problem?
>
> Pupil 1 : I know there is one, but I can’t find it. [...] Unless you went along and put a+b equals c and then put 2 times c, but that’s a long way round.

Conclusions and further research

Through interviews it is manifestly clear that the students involved in the enhanced Socratic approach had developed a more versatile understanding of the concept of variable, in which they were able to encapsulate the algebraic processes as objects and to chunk information in expressions in a way which enabled them to take a more versatile approach to solving algebraic problems. However, it should be noted that it has not proved possible to follow up the initial three week algebra module with further algebraic experiences using the computer and, subsequently, the classes have been reorganized in a way which has led to a variety of different experiences for pupils matched in pairs during the experiment. Some eighteen months after the delayed post-test, a similar test has revealed that the difference between the experimental and control groups is no longer statistically significant. We have still to administer interviews to see if there remain differences detectable by these means. This suggests that, although computer experiences may be able provoke different kinds of understanding in the short and medium term, if these experiences are not continued then their effect may wane in the face of the overwhelming influence of more recent experiences.

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