

Mathematical Intuition, with Special Reference to Limiting Processes

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Mathematical thinking often extrapolates beyond the practical experience of the individual. Limiting processes are a case in point. To understand the nature of thinking processes it is insufficient just to analyse the mathematics, one must try to understand the thought processes themselves. This is of vital importance when we consider mathematical intuition, where thinking does not proceed along logical lines. In this paper we build on suggestions of Hebb and others as to how the brain functions and develop these ideas to give a description of mathematical intuition in cognitive terms. In the particular case of limiting processes we summarize various results which demonstrate the manner in which such processes can be naturally extrapolated to give intuitions of infinity quite different from cardinal infinity.

In the choice of an appropriate terminology to describe the cognitive aspects I have been fortunate to be able to work with Dr S. Vinner and to develop a model of conceptual thinking including some of his ideas. This puts forward possible reasons why an individual can on different occasions have apparently conflicting intuitions and yet sense no cognitive conflict, yet on other occasions cognitive conflict can occur without any explicit reasons being apparent. The various intuitions of infinity are rich in such conflicts.

In the first section of the paper I concentrate on cognitive ideas, introducing the formulation developed with Vinner in §2. Then there follows a section which briefly describes a formal notion of infinity quite different from cardinal infinity. In §4 we will consider various examples of infinite processes and indicate how conflicting intuitions of infinity can arise. All of these intuitions are natural extrapolations of certain parts of finite experience and some of them include a reciprocal idea of the infinitesimally small. The threads are drawn together in the final section when we review the general notion of intuition in the light of the particular examples described in the paper.

§1 Intuition and cognitive structure

As Fischbein observed (1978) at the second conference of IGPME two years ago, there are a variety of descriptions of the meaning of intuition. Bruner (1966) contrasts intuition with analytic thinking, Poincaré (1913) contrasts intuition with logic, as does Hadamard (1945), whilst differing with Poincaré over important details. Skemp (1971) contrasts intuitive and reflective thinking as different modes of mental activity and subsequent work (Skemp, 1979) differs quite considerably from Poincaré's description. Meanwhile, Fischbein (1978) describes intuition as *immediate knowledge* characterised by the properties of *self-evidence*, *coercive effect*, *extrapolative capacity* and *globality*. It is not my purpose here to reconcile the differences between these descriptions but to underline common features through a consideration of the thought processes involved.

A cognitive approach to intuition can be given using the physiological ideas described in D. O. Hebb's *Textbook of Psychology* (1972) chapter 4 or, more recently, in J. Z. Young's *Programs of the Brain* (1978). Hebb postulates that the mechanism of excitation and inhibition of neurons causes activities of cell assemblies in the brain. Sensory information from the eyes, for instance, is conveyed both to the area of the brain which gives rise to the sensation of sight and also, by a network of diverging paths, to other parts of the brain, setting up complex neuronal excitations. The stimulation of certain neurons passes messages to others, which excite or inhibit them, making them more or less sensitive to subsequent stimulation. In this way global mental activity is made up of an amalgam of local processes which have been selectively excited. Faced with a novel situation the brain can only make sense of it by reacting with such an amalgam of mental activity from its currently available cognitive structure. This mental activity may be anything but logical and it is such activity, performed relatively quickly without analysis, which we term intuition. We may wish to qualify this definition by excluding logical or reflective processes, or we may wish to add more precise refinements such as those of Fischbein. For the moment, however, I will concentrate on what I consider the central quality of intuition: the global amalgam of local processes from the current cognitive structure selectively stimulated by a novel situation.

§2 Concept image and conflicts

Vinner (1975) defines an individual's *mental picture* of a concept to be the set of all visual representations (including symbols) that are associated with that concept. He defines the individual's *concept image* (Vinner, 1980) to consist of the mental picture together with all the

properties associated with the concept in the mind of the individual. These ideas have been extended in Tall & Vinner (1980) where the concept image is regarded as the cognitive structure consisting of the mental picture and the properties and processes associated with the concept. Thus the concept image consists of all the mental structure, conscious or unconscious, that shapes the individual's notion of that concept. This concept image need not be totally coherent. Depending on the context, different parts of the concept image may be activated. At any given time the portion of the concept image that is activated is called the *evoked concept image*. At different times different concept images may be evoked and if these are communicated to an observer then the latter may note that there are conflicts in what the individual says or does. However, the individual may not be at all disturbed by the conflicts. Such potential conflicts only become *cognitive conflicts* when they are evoked simultaneously. Fischbein (1978, p. 155) notes:

Two opposite interpretations may coexist for a long time without the subject being aware of the contradiction.

On the other hand a cognitive conflict may be partly subconscious, in which case the individual may feel a sense of unease and confusion, yet not know the reason for it.

Quite distinct from the complex structure of the concept image is the *concept definition* which is the form of words used to describe the concept. Such a concept definition may be *formal* and given to the individual as part of a mathematical theory, or it may be *personal*, invented by the individual to (partially) describe his concept image. A personal concept definition may be considered as part of the concept image, the formal concept definition may or may not be.

The formal concept definition is usually regarded as part of a formal mathematical theory. In this way it has implications which may be deduced within the theory. Following Vinner & Tall (1980), we shall use the term *potential conflict factor* to denote any part of the concept image or any implication of the concept definition which may conflict with another part or another implication. Within a consistent theory there should be no potential conflicts but it may happen that factors in different formal theories can cause conflict. For instance, in the theory of arithmetic of natural numbers (positive integers), subtraction when possible leads to a smaller answer. In the arithmetic of all integers this is no longer true, leading to a potential conflict between factors from different theories. Another example is given by the definition of a complex number $x+iy$ as the ordered pair (x, y) of real numbers. If we identify the real number x with the complex number $x+i0$ then we introduce a possible conflict, for this identifies x and $(x, 0)$ whilst set

theory distinguishes between the element x and the ordered pair $(x, 0)$. Students may see no conflict here, especially when the identifications occur in different contexts. They simply use whichever convention is appropriate in the given circumstances.

To create actual *cognitive* conflict it is necessary to evoke two mutually conflicting factors simultaneously in the mind of the individual. There is therefore the distinct possibility of a potential conflict between part of the concept image and an implication of the concept definition which never gives rise to cognitive conflict simply because the implications of the formal definition never become part of the individual's concept image.

It may well happen that a formal definition is given early on in a mathematics course but the examples and work which follows creates quite a different concept image in the individual students. For instance a formal definition of a function may be given in set-theoretic terms yet subsequent examples where functions are given by formulae and rules may create a concept image with these restrictions implicitly built in Vinner, (1980). In this paper we will shortly concentrate on concept images of infinite processes.

§3 Measuring infinity

At this juncture I pause briefly to mention the notion of measuring infinity, introduced in Tall (1980d), as opposed to cardinal infinity.

It is possible to have an ordered field F containing the real numbers \mathbf{R} such that certain elements $\alpha \in F$ satisfy

$$\alpha > x \text{ for all } x \in \mathbf{R}.$$

These elements are called (positive) infinite elements because they exceed all real numbers. Because F is a *field* we may add, subtract, multiply and divide such elements. If α is positive infinite, then 2α is an even larger positive element, $-\alpha$ is negative infinite and $1/\alpha$, $-1/\alpha$ are positive and negative infinitesimals respectively.

Examples of such fields are the hyperreal numbers of non-standard analysis (Robinson, 1966; Keisler, 1976a, 1976b) or the superreal numbers (Tall, 1979, 1980a, 1980c). In Tall (1980d) I showed that such systems extrapolate properties of *measuring* numbers rather than *cardinal* numbers. For instance, if we imagine a "point" to be infinitesimal in size, say d , then the (measuring) number of such points in a line of finite length l is the *infinite* number l/d . The number of such points in a line length $2l$ is $2l/d$, which is an infinite number twice as big as l/d . I refer to Tall (1980d) for details. The purpose of mentioning this fact here is to demonstrate that cardinal infinity is not the only possible kind of infinity; another kind is possible with its own theory of arithmetic quite different

from the cardinal concept. There are others: $+\infty$ and $-\infty$ as idealised points adjoined to the real line, a single point at infinity giving the one point compactification of the line or plane, the line at infinity in the projective plane. All these kinds of infinity extrapolate various properties of finite experience. What usually happens is that the finite properties concerned are all inextricably intertwined and extrapolate to give an intuition of infinity which contains a variety of potentially conflicting factors.

§4 Infinite processes

In mathematics there are many different kinds of limiting process. There are continuous limit processes such as $\lim_{x \rightarrow a} f(x)$, the notion of continuity itself, the geometric limit of a chord as it approaches a tangent. Then there are discrete processes, including limits of sequences and series, decimal expansions, iterative processes for computing numbers or approximating geometrical shapes. Students meet many of these processes. In England they are usually introduced in an informal manner first, with a formal definition coming much later, if at all. (In the School Mathematics Project the formal definitions of limits of functions and sequences come at the very end of the course.) This means that a concept image is built up long before any formal concept definition is given. With such subtle concepts this is a sensible approach, it may be the *only* practical approach, but it does mean that certain implicit properties which are not part of the concept definition become part of the concept image.

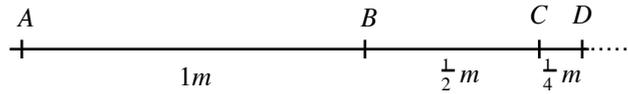
In an investigation Schwarzenberger and Tall (1978) noted that a large number of good mathematicians entering university assumed that if $s_n \rightarrow s$, then s is never equal to s . As one put it

“ $s_n \rightarrow s$ means s_n gets close to s as n gets large, but does not actually reach s until infinity ... Infinity is an imaginary concept invented by mathematicians, useful in describing limits etc.”

Here the concept image includes a fact ($s_n \neq s$) which is not part of the formal concept definition. This fact often survives after long exposure to formal mathematics. In a seminar which I held with final year B. Ed. honours students (shortly to become mathematics teachers), all four students concerned insisted that s_n is never equal to s .

Fischbein (1978, p. 68) reported an investigation with younger pupils (grades 8 and 9) in which two of the questions were:

- (i) Given a segment $AB = 1\text{m}$. Let us add to AB a segment $BC = \frac{1}{2}\text{m}$. Let us continue in the same way adding segments of $\frac{1}{4}\text{m}$, $\frac{1}{8}\text{m}$, etc. Will this process of adding segments come to an end?



(ii) Let us consider question (i). What will be the sum of the segments $AB+BC+CD+\dots$ etc. ?

Out of 107 students, 84% thought the process in (i) would never end, 14% thought it would. In (ii) only 6% thought the sum of the segments would be 2, 17% thought it would be less than 2 and 51% thought it would be infinity. Even before the limiting process had been discussed the concept image intuitively alights on the infinite nature of the *process* rather than the finite numerical limit.

In a paper presented to ICME IV (Tall, 1980b) I reported a questionnaire on limiting processes for mathematics students arriving at university. These were good students (with A or B grades in mathematics A-level). They were asked:

Did you use the notation $\frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x}$ at school? If your reply is ‘yes’, explain (if you can) what the following are:
 δx , δy , $\frac{dy}{dx}$, dx , dy , δ , d .

These are not regular mathematics questions, but they give insight into the individual’s concept image. I was particularly interested in their interpretation of the meaning of dy which I classified to include 22 out of a total of 70 giving infinitesimal-type responses. Sixteen of these specifically mentioned “infinitesimal”, “infinitely small” or “smallest possible” quantities (the term “smallest” almost certainly used in a superlative, rather than mathematical, sense). The other six responded that I was the “limit of δy ” which such comments as

“ dy is the limit as δy tends to zero”,

or

“The only possible meaning I can attach to this is $dx = \lim_{\delta x \rightarrow 0} \delta x = 0$.”

The latter responses once more underline the fact that many students tend to visualise a limit as a dynamic *process* rather than a numerical quantity.

Other topics in the questionnaire show that the *majority* of students regard the limit as a dynamic process (Tall, 1980b).

What effect do such processes have in developing a concept image of infinity? Fischbein (1978, p. 159) has asserted:

“the natural concept of infinity is the concept of *potential infinity*, the only one which Aristotle admitted as meaningful and which was predominant until Cantor.”

As a *primary* intuition this may be so, but a process of teaching which regards the natural numbers as a well-known set N blurs the distinction between potential and actual infinity. In a questionnaire I asked a group of 42 final year students if the natural numbers existed as a coherent mathematical idea. All 42 responded in the affirmative. The next day I expressed my amazement that all of them had said ‘yes’ and asked if any of them were concerned with the difference between actual and potential infinity. We had a five minute discussion on the topic and at the end I asked those who felt if necessary to distinguish between potential and actual infinity to write ‘yes’ on a piece of paper, otherwise ‘no’. There were 42 no’s. None of the students felt there was an actual difference between the two. Modern set theory, with that all important set N , has retrained our intuition. Many university students go through a stage where they accept the actual infinity of a *set* but only a potential infinity of a *process*.

At a less sophisticated level many young children still go through a period of distinguishing between potentially infinite and actually infinite sets. They realise at a young age that the process of counting is potentially infinite and its actual infinity cannot be realised. As we shall shortly demonstrate many students come to recognise potential infinity as a reality but actual infinity as a mathematical fiction. This comes about, not through cardinal consideration but through limiting processes, for instance through $\lim_{n \rightarrow \infty} s_n$, $\lim_{x \rightarrow \infty} f(x)$, $\lim_{x \rightarrow a} f(x) = \infty$.

As an example I returned to a questionnaire given to students in 1977 which considered infinity in the context of limiting processes. Mathematics students were asked whether they had met the concept of the limit of a sequence $s_1, s_2, \dots, s_n, \dots$ in school. They were asked to find the following limits (if they exist) :

$$\lim_{n \rightarrow \infty} \left(\frac{1}{2}\right)^n, \lim_{n \rightarrow \infty} \frac{n^2}{2n^2 + 1},$$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{9}{10} + \frac{9}{100} + \dots + \frac{9}{10^n}\right), \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right).$$

Next they were asked to give the definition of the limit of a sequence (if they could) and then to say whether $0.\dot{9}$ (nought point nine recurring) was equal to one or less than one. Finally there came the questions which concern us now:

- (a) Is “ ∞ ” a real number?
- (b) What is it?
- (c) Explain the reason behind your answers for (a) and (b).

The sequence of questions effectively prevents a cardinal response from being evoked. There were 36 replies and 17 gave correct responses to

calculating the four limits (0, $\frac{1}{2}$, 2, ∞ , or no limit, respectively). (Only one of the 17 claimed not to have seen the definition of the limit of a sequence either formally or informally.) Of these 17, who ostensibly can operate the notion of limit with success in the examples, what was their evoked concept image of infinity? Only two thought infinity was a real number. One gave no reason, simply stating

“[It] is a bigger number than the biggest one you can think of.”

The other said

“ $\infty \in \mathbf{R}$ because $\sum_{x=0}^{\infty} x = \infty, x \in \mathbf{R}$.”

(He had written $\sum_{x=0}^{\infty}$, but scrubbed out the ∞ over the Σ sign.) The rest (who said infinity was not a real number) gave a variety of responses,

“it is an idea useful in seeing the trends in various functions and series.”

“a kind of limit.”

“a figment of man’s imagination to put a value to a number which does not exist. It must not be confused with an imaginary number (i.e. complex).”

“it is a concept which illustrates the fact that there is no top limit to the size of a number.”

“it is a symbol used to denote the ‘limit’ of a sequence s_n if

“ $\forall M > 0 \exists n_0(M): s_n > M \forall n > n_0$.”

“I regard ∞ not as a number at all, but as a concept, it is used to indicate that we have found something beyond our understanding because of its sheer size.”

“A concept invented in order to give an end point to the real numbers, beyond which there are no more real numbers.”

“ $\infty = \lim (1, 2, 3, 4, 5, \dots)$.”

In the context of limits, the evoked concept of infinity was a limiting concept or something large, beyond understanding. Although we could read cardinal implications into certain responses, never was one-one correspondence explicitly evoked. Why should it be: *it has never been taught!*

The reasons given for this evoked concept tended to concentrate on why was not a real number:

“ ∞ cannot be either written as a decimal or represented as a length. It therefore does not qualify under the given definition of a real number.”

“If ∞ is a real number it mucks up behaviour of some axioms adds other problems (odd or even?)..”

“It is not real because it has no fixed value you can’t say $\infty = x+1$ where x is a real number.”

“ ∞ has no multiplicative inverse since $1/\infty$ is taken to be 0. $\infty \times 1/\infty$ is thus 0 instead of 1. It’s an odd quantity to work with, multiplying it by anything or adding anything to it is pretty meaningless.”

“If ∞ is a real number you could add 1 (or more for that matter) onto it and ‘create’ something larger than infinity, which is a contradiction. Perhaps it can be looked upon as a kind of limit, for no matter how many real numbers you add together, you never arrive at ∞ , you only approach it.”

In these responses I detect a positive effect of teaching: students are often taught that arithmetic with infinity leads to contradictions. On the other hand there are intimations of the potential nature of the infinite process, of the unapproachable size of infinity. Nowhere is there any mention of cardinal infinity.

In all of the above examples there is the suggestion of a single unattainable positive infinity. However there is a *secondary* intuition which allows different sizes of infinity by considering the *rate* at which infinity is approached.

As n tends to infinity, so do n^5 and 2^n , but 2^n tends to infinity “faster” than n^5 . This is a well known mathematical result (“for $x > 1$, x^n tends to infinity faster than any power of n ”).

With a class of 20 average university students (grades C and D at A-level mathematics) we computed various limits on hand calculators. An interesting example is $s_n = n^5/(1.1)^n$ which increases to begin with, getting very large as n increases, then after $n=80$ or so its behaviour changes and for much larger n it tends to zero. On a written assignment students were asked to compute various limits, including

$$n^2/(n^2 + 1) \text{ and } n^5/(1.1)^n.$$

The students had been taught to write the first as

$$\frac{n^2}{n^2 + 1} = \frac{1}{1 + 1/n^2} \rightarrow \frac{1}{1 + 0} = 1.$$

One student responded

$$\frac{n^2}{n^2 + 1} \rightarrow \frac{\infty}{\infty} = 1.$$

I interviewed her and pointed out that a similar “computation” would give

$$\frac{n^5}{(1 \cdot 1)^n} \rightarrow \frac{\infty}{\infty} = 1.$$

She replied:

“no it wouldn’t, because in this case the denominator is a *bigger* infinity, and the result is zero.”

In this case she was very close to extrapolating properties which lead formally to a kind of non-standard analysis, with different infinities as in Tall (1980d), though in the current mathematical paradigm of cardinal number her response would likely be rejected.

Historically it is interesting to note that Leibniz had an intuition of various orders of infinity, suggested by the kinds of functions for which he computed derivatives. He said that as the earth is infinite with respect to a ball, so the distance of the fixed stars is doubly infinite with respect to the ball (Leibniz, ed Gerhard, 1849–63, pp. 350, 389) quoted in Boyer, (1949, p. 212). Meanwhile Leibniz rejected actual cardinal infinity to avoid the conflicts inherent in the concept as earlier enunciated by Galileo. How times change; now we accept cardinal infinity of Cantor and reject the measuring infinity of Leibniz, despite the non-standard theory of Robinson (1966) and his school.

If we look at the intuitions of children, we still find notions of measuring infinity. For instance in Fischbein *et al.* (1979, p. 75) the sum of $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$ is given as “ $s = 2 - \frac{1}{\infty}$ because there is no end to the sum of segments.”

Likewise measuring aspects appear where infinity is not evoked explicitly. For example in Fischbein *et al.* (1979, p. 73), a pupil justifying a one-one correspondence between a line and a line-segment asserted:

“The line has no magnitude and therefore it is possible to establish a correspondence between a big point of the line and a small point of the line segment.”

How we distort the intuitions of the young by viewing them through the spectacles of a cardinal paradigm! And in saying that I realise how a “measuring infinity paradigm” is another form of distortion, though it has certain intuitive properties. (Non-standard analysis has an axiom which says that any “elementary” property remains true in the extended theory. Therefore extrapolating “elementary” properties leads to true intuitions.) Following Lakatos (1978), who speaks on behalf of incomplete theories in history, one should make a plea that the intuitions of student should be respected whether they fit with an accepted formal theory or not. Who

knows, a later mathematician may canonise such an intuition with a new formal theory!

§ 5 Conclusions

In the previous section various aspects of infinity have been considered. Although a primary intuition of potential infinity exists in many children I postulate that modern mathematics at university promotes secondary intuitions of actual infinity of sets contrasting with potential infinity of limiting processes. In the first place limiting processes support the notion of a single unattainable (positive) infinity, but further consideration of the rate of approach to infinity can lead to the intuition of different infinities.

The latter involves a complete arithmetic of infinities and infinitesimals as in Tall, (1980d). This has different properties from cardinal infinity and so conflicting intuitions may develop. There may in addition be other intuitions of infinity not fitting into either of these theories.

Thus the concept of infinity which varies from one individual to another need not be globally coherent and may contain potential conflict factors. As Fischbein has observed, the individual need not be aware of these factors unless they are evoked simultaneously in a context seeking overall consistency.

I contend that these factors occurring in the concept of infinity are a typical case of intuition in particularly extenuating circumstances. They reveal the amalgam of local mental processes giving an intuition by using current cognitive structure in a manner which extrapolates previous experience without a complete intervening logical process.

References

- Boyer, C. B.: 1949. *The History of Calculus and its Conceptual Development*, reprinted Dover 1959.
- Bruner, J. S.: 1966. Towards a Disciplined Intuition. In *Relevance of Education*, 98–113, Penguin, 1974.
- Fischbein, E.: 1978. Intuition and Mathematical Education. *Proc. 2nd Int. Conf. for the Psych. of Math. Educ.* 197–202.
- Fischbein, E., Tirosh, D. & Malamed, U.: 1979. Is it possible to measure the intuitive acceptance of a mathematical statement? *Proc. 3rd Int. Conf. for the Psych. of Maths. Educ.*, 67–79.
- Hadamard, J.: 1945. *The Psychology of Invention in the Mathematical Field*, reprinted Dover 1954.
- Hebb, D. O.: 1972. *Textbook of Psychology*, 3rd edition, Saunders.

- Keisler, H. J.: 1976a. *Elementary Calculus*, Prindle, Weber & Schmidt.
- Keisler, H. J.: 1976.b *Foundations of Infinitesimal Calculus*, Prindle, Weber & Schmidt.
- Lakatos, I.: 1978. Cauchy and the Continuum, *Mathematical Intelligencer* 1, 151–161.
- Leibniz, G. W.: 1849–63. *Mathematische Schriften* V. ed. Gerhard.
- Poincaré, H.: 1913. Mathematical Creation. In *Foundations of Science* New York: The Science Press.
- Robinson, A.: 1966. *Non-Standard Analysis*, North Holland.
- Schwarzenberger, R. L. E. & Tall, D. O.: 1978. Conflicts in the Learning of real numbers and limits, *Mathematics Teaching*, 82.
- Skemp, R. R.: 1971. *The Psychology of Learning Mathematics*. Penguin.
- Skemp, R. R.: 1979. Goals of Learning and Qualities of Understanding, *Proc. 3rd Int. Conf. for the Psych. of Maths. Educ.*, 197–202.
- Tall, D. O.: 1979. The Calculus of Leibniz – an Alternative Modern Approach, *Mathematical Intelligencer*, 2, 54–55.
- Tall, D. O.: 1980a. Looking at Graphs through infinitesimal microscopes, windows and telescopes. *Mathematical Gazette*, (to appear).
- Tall, D. O.: 1980c. Infinitesimals constructed algebraically and interpreted geometrically. *Mathematical Education for Teachers*, (to appear).
- Tall, D. O.: 1980d. The notion of infinite measuring number and its relevance in the intuition of infinity. *Ed. St. in Math.*, (to appear).
- Tall, D. O.: 1980b. Intuitive infinitesimals in the calculus, (Poster presented to ICME IV).
- Tall, D. O. & Vinner, S.: 1980. Concept image and concept definition in mathematics with particular reference to limits and continuity (paper presented by first author to conference of Swedish Mathematicians, UHA, St. Andrew's, Scotland. March 1980).
- Vinner, S.: 1975. The Naive Platonic approach as a teaching strategy in arithmetic. *Ed. St. in Math.*, 339–350.
- Vinner, S.: 1980. Concept image, concept definition and the notion of function (in preparation).
- Young, J. Z.: 1978. *Programs of the Brain*, OUP.