LOOKING AT GRAPHS THROUGH INFINITESIMAL MICROSCOPES WINDOWS AND TELESCOPES

An Introduction to Calculus using Infinitesimals

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1. Introduction

The differential triangle of Leibniz for a real function f is found by taking an increment dx in the variable x, finding the increment dy in y=f(x) and drawing the 'triangle', in Fig 1. Here ds is increment in the length of the graph, where

$$ds^2 = dx^2 + dy^2$$

and the derivative of f is

$$f'(x) = \frac{dy}{dx}$$

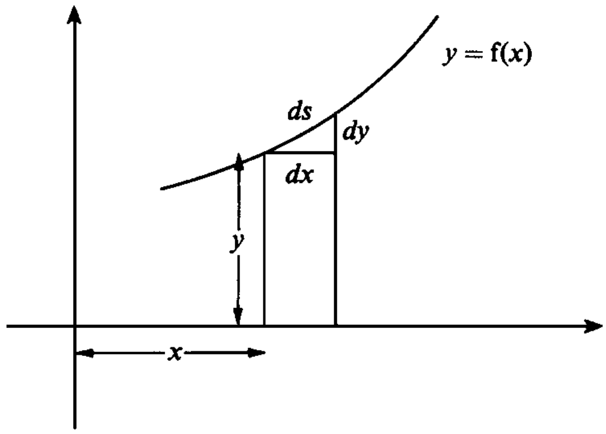

FIGURE 1.

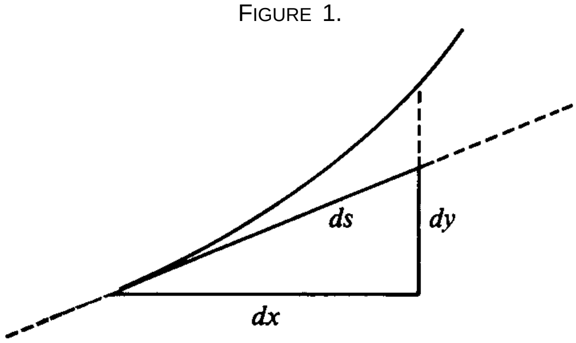

FIGURE 2.

The reader may already feel uneasy about the preceding description. Surely an accurate picture is as in Fig. 2. Here dy is the increment measured up to the tangent, not to the graph, and if $ds^2 = dx^2 + dy^2$, then ds is the increment along the tangent, not along the graph. Of course, if dx is extremely small, then Fig. 2 approximates to Fig. 1. Leibniz imagined dx to be an infinitesimal, and that Fig. 1 was accurate within infinitesimals of higher order. In the nineteenth century the arrival of the analysis of Weierstrass and his school banished infinitesimals from mathematics. Pictures like Fig. 1 and geometric interpretations after the style of Leibniz were also banned.

This article will give a simple algebraic description which accords with Leibniz's theory.

The first such modern approach was through the epoch-making work of Robinson [1]. That requires the use of first-order logic and an axiom almost as strong as the axiom of choice. Here a less powerful version is used which is still strong enough to handle the calculus of analytic functions. Its advantage is that everything is done with straightforward algebra which has a geometric interpretation. The algebra gives a simple set-theoretic description of a function which describes the intuitive notion of looking through a microscope with 'infinite magnification'. Looking through such a microscope, we see that an infinitesimal portion of the graph looks like a straight line and the magnified picture reveals the differential triangle of Leibniz. In a complementary manner there is also an algebraic description of 'looking at a graph at infinity'. This reveals that a graph with an asymptote looks the same as the asymptote itself at infinite points.

For the convenience of the reader, we briefly discuss the notion of an infinitesimal in §2 and describe the superreal numbers in §3, already previously explained in [21 and [31. We then get down to the set-theoretic ideas of microscopes, windows and telescopes in §§4, 5, 6. The differential triangle appears in §7 and applications in integration and length of curves appear in §§8, 9. Asymptotes are viewed through windows in §10.

2. What is an infinitesimal?

A positive infinitesimal may be thought of as a quantity greater than zero yet smaller than any positive real number. (Similarly a negative infinitesimal is smaller than zero but larger than any negative real number.) How can that be? Surely there is something inconsistent about the definition of a positive infinitesimal. If ε were such a quantity, then $\frac{1}{2}\varepsilon$ is also positive but smaller than ε. In the late nineteenth century such an argument was put forward to demonstrate that infinitesimals cannot exist. The repercussions of this viewpoint still reverberate in our modern mathematical culture, leading to the rejection of the infinitesimal idea. But this is a misconception of the possible nature of an infinitesimal.

Returning to the first sentence of this section, we see that a positive infinitesimal ε satisfies

$0 < \varepsilon < a$ for every positive real number a.

If such an ε were a real number, then indeed $\frac{1}{2}\varepsilon$ is positive and putting $a = \frac{1}{2}\varepsilon$ yields a contradiction. There is a way out of the dilemma: it is simply that an infinitesimal is not a real number.

We may conceive of an infinitesimal by supposing that we have an ordered field F such that F contains the real numbers $\boldsymbol{R}$ as a subfield. A positive infinitesimal ε is then an element of F which is *not* in $\boldsymbol{R}$ satisfying

$0 < \varepsilon < a$ for all positive $a \in \boldsymbol{R}$.

In [3] it is shown how the field $F=\boldsymbol{R}(x)$ of rational expressions

$$\alpha(x) = \frac{a_0 + a_1 + \ldots + a_n x^n}{b_0 + b_1 + \ldots + b_m x^m} \quad \text{(where } a_i, b_j \in \boldsymbol{R} \text{ and } b_m \neq 0.)$$

in an indeterminate x may be considered as such an ordered field. Briefly, if $\alpha(x)$, $\beta(x) \in \boldsymbol{R}(x)$, then we define

$$\alpha(x) < \beta(x)$$

to mean:

for some positive real number k,
$\alpha(t) < \beta(t)$ for all $t \in \boldsymbol{R}$ such that $0 < t < k$.

Geometrically this is equivalent to drawing the graphs of $\alpha(x)$ and $\beta(x)$, and noting that the graph of α is below that of β over some open interval $(0, k)$. For instance, with such a definition we find

$$x^2 < x$$

because, taking $k = 1$, when $t \in \boldsymbol{R}$ and $0 < t < 1$ then $t^2 < t$ (Fig. 3).

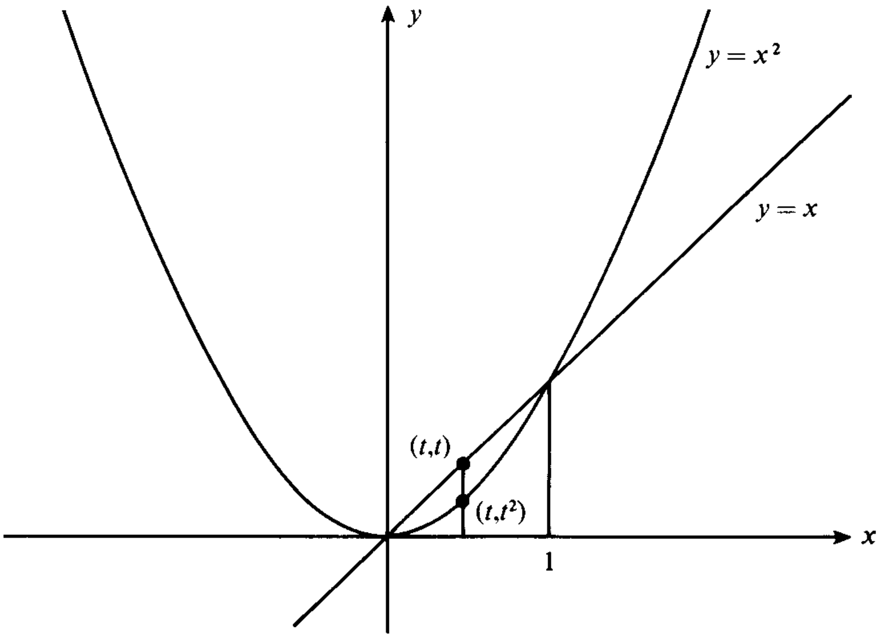

FIGURE 3.

It is essential to check that this definition of order on $\boldsymbol{R}(x)$ satisfies the axioms of an ordered field. In particular the order axioms are:

(i) $\alpha < \beta$, $\beta < \gamma \Rightarrow \alpha < \gamma$.

(ii) Given $\alpha, \beta \in \boldsymbol{R}(x)$, then precisely one of the following holds: $\alpha < \beta$, $\beta < \alpha$ or $\alpha = \beta$.

(iii) If $\alpha < \beta$, $\gamma < \delta$ then $\alpha+\gamma < \beta+\delta$.

(iv) If $\alpha < \beta$, $0 < \gamma$ then $\alpha\gamma < \beta\gamma$.

It is a routine matter to show that these axioms are satisfied and that $\boldsymbol{R}$ is an ordered subfield of $\boldsymbol{R}(x)$.

If a is a positive real number, then the graph of $y = a$ is a horizontal line above the horizontal axis and, for $0 < t < a$, the graph of $y = x$ is below this line (Fig. 4).

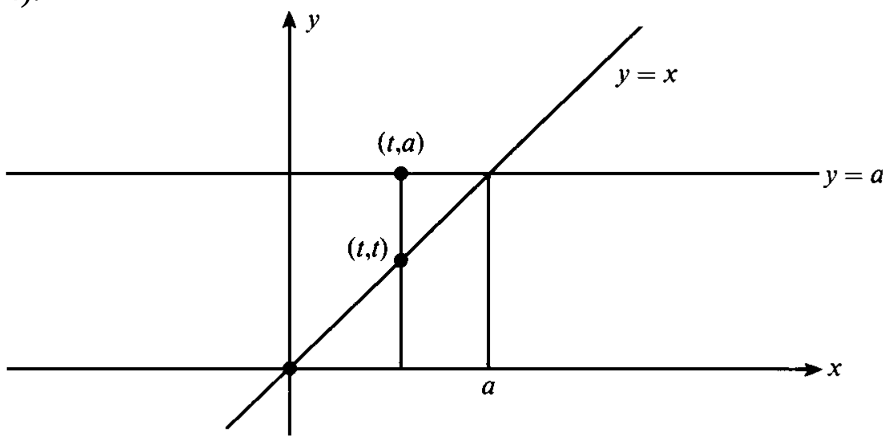

FIGURE 4.

Using the given definition of order on $\boldsymbol{R}(x)$, this means that $x < a$ for any positive $a \in \boldsymbol{R}$. Clearly $0 < x$ using the same definition; thus

$$0<x<a \text{ for all } a \in \boldsymbol{R} \text{ such that } a > 0,$$

and this means that x is a positive infinitesimal. *But note* that *x is not* a *real number!* The field $F= \boldsymbol{R}(x)$ contains the real numbers $\boldsymbol{R}$ and an infinitesimal $x \in F$ where $x \notin \boldsymbol{R}$.

3. The superreal numbers $\mathfrak{R}$

Now that we see that infinitesimals *can* exist, in the shape of rational functions, we move on to a different system which will prove to be precisely what we need in the calculus.

To be able to use infinitesimals geometrically in the calculus, we need to seek a description of an extended number line that includes not only the usual real numbers, but also infinitesimal points as well. A positive infinitesimal, if we could conceive such a thing, would be a point to the right of the origin, yet to the left of any positive real number. Of course we cannot draw an infinitesimal to the same scale as finite numbers and get a picture in which it is distinguishable from the origin, but it is not beyond the realms of imagination to conceive of such a phenomenon.

We put geometrical considerations aside temporarily and begin with algebra. If we wished to handle a positive infinitesimal ε algebraically, then we should need to add and subtract expressions involving ε, and form products and quotients, which means that we would need to build up rational expressions like

$$(3\varepsilon^2 + 2\varepsilon + 17)/(4\varepsilon^5 - \varepsilon^3).$$

Even these are not sufficient for the calculus; for instance, the sine of ε would require a power series

$\sin \varepsilon = \varepsilon - \varepsilon^3/3! + \varepsilon^5/5! - ....$

With such applications in mind, the superreal numbers $\mathfrak{R}$ are defined to consist of formal power series in an unspecified symbol ε, of the form

$$a_{-m}\varepsilon^{-m} + ... + a_{-1}\varepsilon^{-1} + a_0 + a_1\varepsilon + ... + a_n\varepsilon^n + ...$$

where the coefficients a_r are any real numbers. We include a finite number of negative powers of ε to allow for the formation of multiplicative inverses, so that, when these expressions are added term-by-term and multiplied in the usual way for power series, it may be shown that $\mathfrak{R}$ is a field (see [2]). The reader should also note that we do not concern ourselves with convergence, and that there is not intended to be any restriction on the values of the real coefficients a_r.

Now we specify an order on these symbols in such a way that $\mathfrak{R}$ is an ordered field and ε is an infinitesimal. This may be done by writing two elements α and β, in $\mathfrak{R}$ as

$$\alpha = \sum_{n=k}^{\infty} a_n \varepsilon^n \text{ and } \beta = \sum_{n=l}^{\infty} b_n \varepsilon^n \text{ (where } k, l \in \boldsymbol{Z} \text{ and } a_n, b_n \in \boldsymbol{R}\text{);}$$

then either $\alpha = \beta$, or $a_q \neq b_q$ for some q. In the latter case, let r be the smallest integer such that $a_r \neq b_r$, and define

$$\alpha < \beta, \Leftrightarrow a_r < b_r.$$

For instance,

$$1000\varepsilon < 3 - 5\varepsilon^2,$$

since $a_0 = 0$ is less than $b_0 = 3$; and

$$1/\varepsilon + 7 + \varepsilon + \varepsilon^2 + \varepsilon^3 + . . . < 1/\varepsilon + 7 + 3\varepsilon - 257\varepsilon^2,$$

since $a_{-1} = b_{-1} = 1$, $a_0 = b_0 = 7$, but $a_1 = 1$ is less than $b_1 = 3$.

It is a routine matter to verify that $\mathfrak{R}$ is an ordered field (which means checking axioms (i)–(iv) of §2). We shall freely use $\beta>\alpha$ instead of $\alpha<\beta$, and such equivalents as $\alpha\leq\beta$ for "$\alpha < \beta$ or $\alpha = \beta$".

Using this definition, we have

$$0 < \varepsilon < a \text{ for every positive real number } a,$$

so (using the definition of §2) ε is a positive infinitesimal in $\mathfrak{R}$.

Just as ε is 'very small', so terms involving 1/ε are 'very large'. Formally, an element α in $\mathfrak{R}$ is said to *be positive infinite* if

$$a < \alpha \text{ for all } a \in \boldsymbol{R}$$

and *negative infinite* if

$$\alpha < a \text{ for all } a \in \boldsymbol{R}.$$

For instance, $1/\varepsilon - 10^{100}$ is positive infinite and $-1/\varepsilon + 10^{100}$ is negative infinite.

On the other hand, if $a < \alpha < b$ for some $a,b \in \boldsymbol{R}$, then α is said to be *finite*.

4. Infinitesimal microscopes, windows and infinite telescopes

There are severe practical problems in attempting to draw superreal points as additional points on the normal real number line. Positive infinite points are too far off to the right to be drawn on a finite piece of paper, and negative infinite points too far off to the left. Even though we might make an attempt to represent a finite point

$$\alpha = a_0 + a_1\varepsilon + a^2\varepsilon^2 + \dots ,$$

it would look no different from the real number a_0 because the remainder of the expansion is infinitesimal. We call the first coefficient, a_0 of a finite superreal α the *standard part* of α and denote it by st α. It is easily manipulated algebraically to obtain

$$\mathrm{st}(\alpha + \beta) = \mathrm{st}\,\alpha + \mathrm{st}\,\beta,\ \mathrm{st}(a\beta) = \mathrm{st}\,a\ \mathrm{st}\,\beta,$$

and subtraction and division follow exactly the same pattern. (Technically st is a ring homomorphism from the ring of finite elements onto $\boldsymbol{R}$ with kernel I, the set of infinitesimals.)

Normal scale pictures, seeing only the standard parts of finite superreals lose all the infinitesimal detail and faraway infinite structure. To reveal these subtleties, we use the superreal map given by

$$\mu(x) = (x—\alpha)/\delta.$$

This moves the point α to the origin and multiplies by a scale factor $1/\delta$. By careful choice of α and δ we can see details which are not visible in ordinary pictures.

EXAMPLE 1. Let $\alpha = 2$, $\delta = \varepsilon$ then we may compute $\mu(x)$ for various superreal values of x to get results such as

$$\mu(3) = 1/\varepsilon,\ \ \mu(2 + \tfrac{1}{2}\varepsilon + 27\varepsilon^3) = \tfrac{1}{2} + 27\varepsilon^2,$$

$$\mu(2 + 2\varepsilon – 3\varepsilon^4) = 2—3\varepsilon^3,\ \ \mu(2 + 2\varepsilon + \varepsilon^2) = 2 + \varepsilon.$$

Now $\mu(3)$ is infinite, but the other images mentioned are all finite. In fact, $\mu(x)$ is finite precisely when $x = 2 + \delta$ where δ is infinitesimal, because

$$\mu(2 + a_1\varepsilon + a^2\varepsilon^2 + ...) = a_1 + a_2\varepsilon\ +\ \dots\ .$$

The map μ magnifies all distances from the point 2 by a factor $1/\varepsilon$, and this scale factor is infinite. It has the effect of expanding the scale to such an extent that points, such as 3, which are a finite non-zero distance away from 2 are mapped way off to infinity, whilst points an infinitesimal distance away from 2 are spread out to cover the whole finite part of the number line. If we draw what we can, we get a picture like Fig. 5.

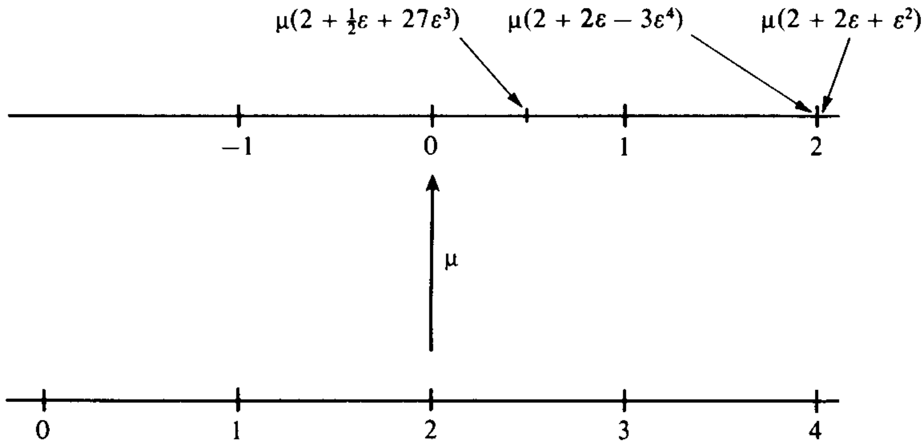

FIGURE 5.

A useful notational device is to drop the symbol μ in the image. This is a regular map-making technique, whereby a place on a map is denoted by the name of the actual location. In the context of the superreals the device has a great liberating effect; it enables us to imagine that μ is a magnification by the infinite factor 1/δ, revealing the infinitesimal detail near the point 2. Of course, the integer points –1, –2, … on the upper line in Fig. 5 are also the images μ(2–ε), μ(2), μ(2+ε), μ(2+2ε), …so we must rename these as well. The result is Fig. 6.

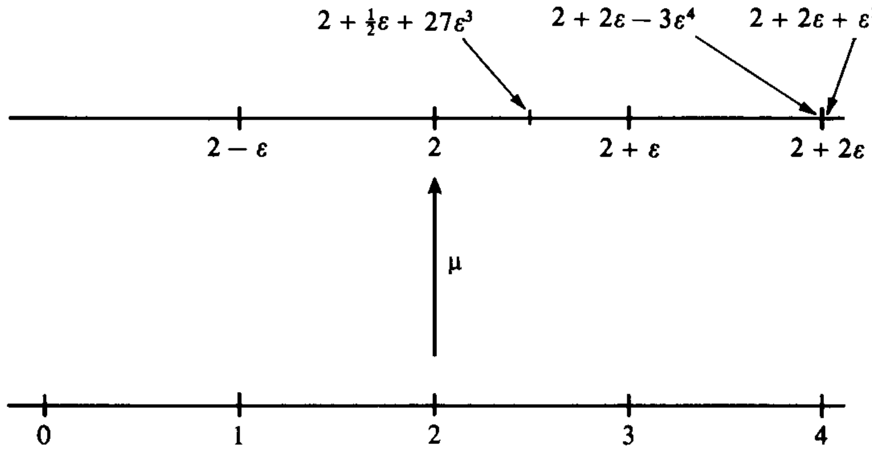

FIGURE 6.

We have also intimated in the diagram that $2 + 2\varepsilon - 3\varepsilon^4$ is to the left of $2 + 2\varepsilon$, and that $2 + 2\varepsilon + \varepsilon^2$ is to the right. In a practical picture we find that the magnification by the factor 1/δ is insufficient to reveal the distinction between these points, because the differences are 'higher order infinitesimals'. To reveal these more subtle details we will require an even greater magnification.

The description of the various magnifications possible is greatly facilitated by introducing a new concept, the *order* of a superreal number (not to be confused with the order relation '<'). The order o(α) of a non-zero superreal number α is the suffix of the first non-zero coefficient in the expression

$$\alpha = a_k\varepsilon^k + a_{k+1}\varepsilon^k + \ldots ,$$

namely (supposing $a_k \neq 0$)

$$o(\alpha) = k.$$

For instance,

$$o(\varepsilon) = 1,\ o(\varepsilon^2 + 17\varepsilon^4) = 2,\ o(\varepsilon^{-9} + 3 + \varepsilon) = -9.$$

(The order of zero, o(0), is not defined in this way, though it is often formally defined to be $+\infty$. Technically, the order is called a *valuation* because it has the properties

(i) $o(\alpha\beta) = o(\alpha)o(\beta)$,

(ii) $o(\alpha + \beta) \leq \min\{o(\alpha), o(\beta)\}$.

These properties hold even for o(0), provided that the usual arithmetic conventions are observed for operating with $+\infty$.)

Using this definition, infinitesimals are those elements of strictly positive order, whilst infinite elements are those with strictly negative order Elements of order zero are simply finite elements with non-zero standard part.

We can now grade superreal numbers into relative orders of size; for each positive integer n we say that α is an nth *order infinitesimal* if $o(\alpha) = n$, and an nth *order infinite element* if $o(\alpha) = -n$. For instance, $\varepsilon^3 + 2\varepsilon^4$ is a third order infinitesimal and $\varepsilon^{-9} + 3 + \varepsilon$ is a ninth order infinite element.

If $o(\delta) = n$, then the image of $x = a + \theta$ under the map μ mentioned earlier is finite if

$$\mu(x) = \theta/\delta$$

has non-negative order. But

$$o(\theta/\delta) = o(\theta) - o(\delta)$$

so $o(\theta/\delta)$ is non-negative precisely when

$$o(\theta) \geq n.$$

In a practical drawing of the magnification μ, we can therefore only draw the images of points in the set

$$I(\alpha,n) = \{\alpha+\theta \in \mathfrak{R} \mid o(\theta) \geq n\}.$$

At the same time we can only draw the standard parts of the image points. With these practical limitations in mind we reach the central concept in the pictorial representation of superreal numbers:

DEFINITION. For a positive superreal δ of order n and any superreal α the *optical δ-lens* aimed at α is the map $\nu_\delta : I(\alpha, n) \to \mathfrak{R}$ given by

$$\nu_\delta(x) = \mathrm{st}((x - \alpha)/\delta).$$

(In this definition the choice of δ is restricted to positive values, because negative ones give the same pictures, but with the order reversed as a mirror image.) The domain $I(\alpha, n)$ of ν_δ , is called *the field of view.* It is an interval, with centre α, consisting of all the superreal points which differ from α by an element of order at least *n.* If $n = 1$, as in Example 1, then the field of view is precisely the set of superreals infinitesimally close to α; but we have the option of choosing *n* to be any integer. If *n* is positive, then we call the optical δ-lens an *infinitesimal microscope of order n,* if *n* is zero we call it a *window,* and if $n = -m$ *is* negative, it is termed an *infinite telescope of order m.* (The term "microscope" was introduced into non-standard analysis by Stroyan (see, for example, [4]). However, the original definition did not include taking standard parts, and the term "telescope" was used by Stroyan where we would use the terminology "window pointed at an infinite element α".)

The flexibility of choice of α and δ allows us to visualise a whole variety of different situations. Not only may we expand an infinitesimal field of view by choosing $o(\delta)= 1$, we may also look at higher order detail as well. On the other hand, by choosing α or δ infinite, we may view infinite points through microscopes, windows or telescopes, as appropriate.

The action of a δ-lens expands or contracts the field of view to the whole of the real line; the taking of standard parts loses detail of higher order. Meanwhile details of lower order are outside the field of view and cannot be seen either! In this way a microscope, window or telescope concentrates precisely on the details of chosen order *n* in the field of view.

In this process the order of δ matters more than the particular choice of itself, for it is the order that determines the field of view. Were we to use another positive element ρ of order *n,* then we would have the same field of view $I(\alpha, n)$ and the image of $\alpha + \theta$ in this field would be

$$\nu_\rho(x) = \mathrm{st}((x-\alpha)/\rho).$$

But

$$\nu_\delta(\alpha+\theta) = \mathrm{st}(\theta/\delta).$$

$$= \mathrm{st}((\theta\rho)/(\delta\rho))$$

$$\mathrm{st}(\rho/\delta)\mathrm{st}(\theta/\rho)$$

$$= \lambda\nu_\rho(\alpha+\theta)$$

where $\lambda = \mathrm{st}(\rho/\delta)$. Now θ and ρ are both positive and of the same order, so θ/ρ is positive and of order zero, which means that λ is a strictly positive real number. Thus varying the positive scale factor $1/\delta$, whilst keeping the order of δ constant, only has the effect of varying the image by a positive real scale factor.

At the same time, if we were to look through a δ-lens aimed at another point $a + \theta_0$ in the same field of view, then the image we would get would be

$$\mathrm{st}\,((x-\alpha-\theta_0)/\delta) = \mathrm{st}\,((x-\alpha)/\delta) - \kappa$$

where κ is the real number $\text{st}(\theta_0/\delta)$. This is simply a translation of the δ-lens aimed at α, where now the image of $\alpha+\theta_0$ is the origin instead of the image of α.

Thus by varying the positive nth order element δ and the element α within the field of view, we obtain the usual devices of changing the scale and shifting the origin in a diagram.

Since the element δ plays a non-essential role beyond determining the order and the real scale factor, we shall often omit the suffix in the symbol ν_δ and simply refer to the lens ν of order n (or microscope, window or telescope, as appropriate). To simplify matters it is usually convenient to take $\delta=\varepsilon^n$.

These techniques extend easily to looking at configurations in the superreal plane; we simply apply the scale factor to each coordinate.

DEFINITION. An nth *order optical lens* aimed at (α,β) in the superreal plane is a map ν given by the formula

$$\nu((x, y)) = (\text{st}\,((x-\alpha)/\delta), \text{st}\,((y-\beta)/\delta))$$

where δ is a positive superreal of order n.

The domain of ν is the set of superreal points

$$D = \{(\alpha+\theta, \beta+\phi) \in \Re^2 \mid o(\theta) \geq n, o(\phi) \geq \nu\},$$

and the range is the whole of the real plane. The domain D is again called the field of view; it consists of all points in the superreal plane whose coordinates differ from those of (α,β) by elements of order at least n.

We can rephrase this latter condition in a more convenient manner by extending the usual distance function to the superreal plane; defining the distance from (x, y) to (α,β) to be

$$\surd\{(x-\alpha)^2+(y-\beta)^2\}$$

then it is a relatively simple matter to show that D is the set of points whose distance from (α, β) is of order at least ν.

To be able to do this, however, we must first of all check that it is actually possible to take the square root required in the distance function, because not all positive superreals have a square root. For instance, a square root of ε would satisfy $\omega^2=\varepsilon$, so

$$o(\varepsilon) = o(\omega^2) = o(\omega) + o(\omega),$$

which would require $o(\omega)) = \frac{1}{2}$, contrary to the fact that the order of a non zero superreal must be an integer. Although ε has no superreal square root, a non-zero element of the form $\kappa^2 + \rho^2$ is clearly positive and of even order, say $2m$. Writing

$$\kappa^2 + \rho^2 = a_{2m}\varepsilon^{2m} + a_{2m+1}\varepsilon^{2m+1} + \ldots \text{ (where } a_{2m} \text{ is positive)}$$

$$= a_{2m}\varepsilon^{2m}(1+\delta)$$

where δ is infinitesimal, we may find a square root of $\kappa^2 + \rho^2$ in the form

$$\surd(a_{2m})\varepsilon^m(1+\delta)^{1/2}$$

where the root of $1 + \delta$ may be computed by using a binomial expansion. It is then an easy matter to verify that the order of $\kappa^2+\rho^2$ is at least n precisely when the orders of κ and ρ are at least n. This is all we have to check concerning our assertion about the field of view of an nth order optical lens.

The terminology for microscopes, windows and telescopes can be carried over in the obvious manner to lenses looking at the superreal plane. They are much more interesting than the one-dimensional case because they can be used for looking at plane configurations; in particular, we will be able to put graphs under an infinitesimal microscope.

5. Extending analytic functions

If we aim a microscope at a point (α, β) in the plane, then all the points in the field of view will be infinitesimally close to (α, β). In particular, there will be no other real points in view. To be able to look at graphs and to see worthwhile pictures, we must first extend real functions to take on superreal values.

The extension of functions with power series expansions is a straightforward matter. A function given by

$$f(x+h) = a_0 + a_1h + \ldots + a_nh^n + \ldots$$

for all real h satisfying $|h| < R$ is said to be analytic for $|h| < R$, and for an infinitesimal δ we can define

$$f(x+\delta) = a_0 + a_1\delta + \ldots + a_n\delta^n + \ldots$$

To compute $f(x+\delta)$ as a superreal power series in ε, we simply substitute

$$\delta = b_1\varepsilon + b_2\varepsilon^2 + \ldots$$

into the given expression.

For instance, we define

$$\sin \delta = \delta - \delta^3/3! + \ldots + (-1)^n\delta^{2n+1}/(2n+1)! + \ldots$$

and

$$e^{x+\delta} = e^x + e^x\delta/1! + \ldots + e^x\delta^n/n! + \ldots .$$

The computation of $\sin\delta$ is carried out by computing each coefficient in turn:

$$\sin\delta = (b_1\varepsilon + b_2\varepsilon^2 + b_3\varepsilon^3 + \ldots) - (b_1\varepsilon + b_2\varepsilon^2 + \ldots)^3/3! + \ldots$$

$$= b_1\varepsilon + b_2\varepsilon^2 + (b_3 - b_1{}^3/6)\varepsilon^3 + \ldots$$

Such a process is possible for any analytic function. In general, if $f: D \rightarrow \boldsymbol{R}$ is analytic on an open interval D of the real numbers, which means that it has a power series expansion

$$f(x+h)=\sum_{n=0}^{\infty}a_nh^n \quad \text{for } |h|<1$$

at each x in D, then f extends to a function f: $D^{\#} \to \Re$ where

$$D^{\#}=\{x\in\Re \mid \text{st } x\in D\}$$

and

$$f(x+\delta)=\sum_{n=0}^{\infty}a_n\delta^n \quad \text{for } x\in D \text{ and } \delta \text{ infinitesimal} .$$

In this definition, f(x) = a_0, so f(x+δ) – f(x) is infinitesimal.

This is reminiscent of the historical description of a continuous function, that an infinitesimal change in the variable x only causes an infinitesimal change in the value of f(x). Another way of saying the same thing is

st(f(x+δ)) = f(x) for real x and infinitesimal δ.

Extensions are also possible infinitesimally close to poles of analytic functions. Recall that a real function satisfying

$$f(x+h)=\sum_{n=-m}^{\infty}a_nh^n \quad \text{for } 0<|h|<R$$

where $a_{-m} \neq 0$, is said to have a pole of order m at x. For a non-zero infinitesimal we can define

$$f(x+\delta)=\sum_{n=-m}^{\infty}a_n\delta^n .$$

A particularly simple case is a rational function f(x) = p(x)/q(x) which has poles where q has zeros. We may compute f(α) for every superreal α except these poles by simple substitution,

f(α) = p(α)/q(α).

6. Looking at graphs

We are now in a position to look at graphs through lenses of various orders, and we begin with a number of examples.

EXAMPLE 2. The graph of f(x)=1/x viewed through a first order microscope pointed at (1,1). Here the field of view consists of all points infinitesimally close to (1, 1). For any infinitesimal θ,

$$f(1+\theta)=\theta/(1+\theta)$$

$$=1-\theta+\theta^2-\ldots$$

Hence for a point (1 + θ, 1/(1 + θ)) on the graph,

$$\nu_\varepsilon(1+\theta, 1/(1+\theta)) = \nu_\varepsilon(1+\theta, 1+\theta-\theta^2+\ldots)$$
$$= (\mathrm{st}(\theta/\varepsilon), \mathrm{st}((-\theta+\theta^2-\ldots)/\varepsilon))$$
$$= (\lambda, -\lambda)$$

where $\lambda = \mathrm{st}(\theta/\varepsilon)$ is a real number. We can draw this as in Fig. 7, following the established convention of dropping the symbol ν_ε from its image, so that we can imagine ν_ε as a first order magnification of the points an infinitesimal distance from (1, 1).

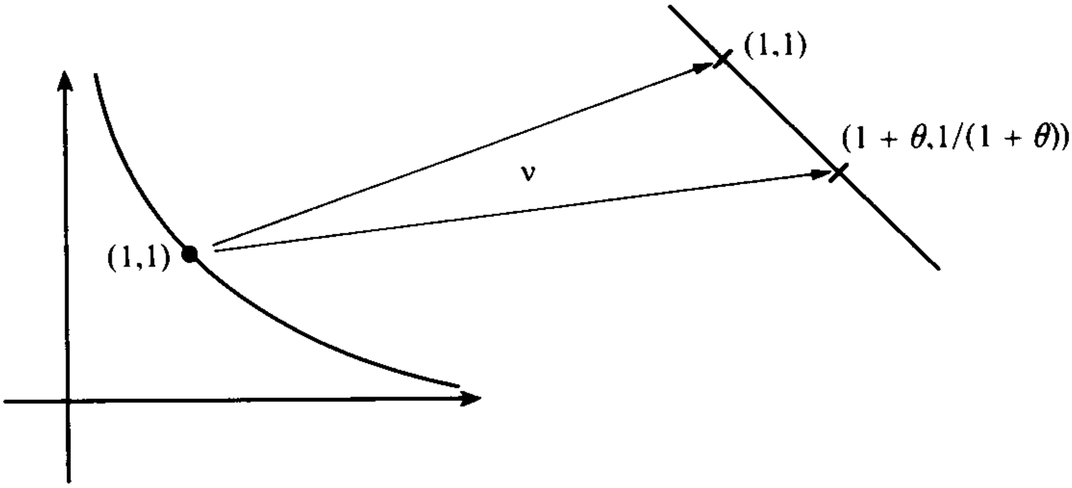

FIGURE 7.

Notice that the optical picture of an infinitesimal portion of the graph is a straight line. This will shortly be seen to be typical.

EXAMPLE 3. The graph of $f(x)=1/x$ viewed through an ε-microscope aimed at an infinite point $(1/\varepsilon, 0)$ on the axis. For an infinitesimal θ, we have

$$f\left(\frac{1}{\varepsilon}+\theta\right) = 1\Big/\left(\frac{1}{\varepsilon}+\theta\right)$$
$$= \varepsilon/(1+\varepsilon\theta)$$
$$= \varepsilon - \varepsilon^2\theta + \varepsilon^3\theta^2 - \ldots$$

So

$$\nu_\varepsilon(1/\varepsilon+\theta, f(1/\varepsilon+\theta)) = \nu_\varepsilon(1/\varepsilon+\theta, \varepsilon-\varepsilon^2\theta+\varepsilon^3\theta^2-\ldots)$$
$$= (\mathrm{st}(\theta/\varepsilon), \mathrm{st}((\varepsilon-\varepsilon^2\theta+\varepsilon^3\theta^2-\ldots)/\varepsilon))$$
$$= (\lambda, 1)$$

where λ is the real number $\mathrm{st}(\theta/\varepsilon)$.

The optical picture is as in Fig. 8, where the distances from $(1/\varepsilon, 0)$ to $(1/\varepsilon, \varepsilon)$ and from $(1/\varepsilon, \varepsilon)$ to $(1/\varepsilon + \theta, 1/(1/\varepsilon + \theta))$ have been scaled up to the finite lengths 1 and λ, respectively.

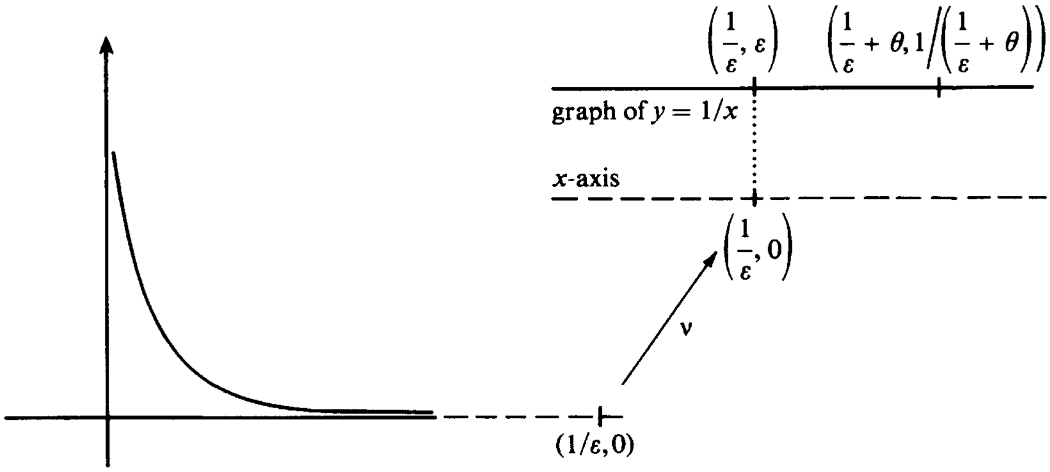

FIGURE 8.

EXAMPLE 4. If we view part of the infinite portion of the graph of y=1/x through a window (say a δ-lens where δ=1, aimed at a point (1/ε, 0) on the x-axis) then we find, for any *finite* δ,

$$\begin{aligned} v_1\left(\frac{1}{\varepsilon}+\theta, \mathrm{f}\left(\frac{1}{\varepsilon}+\theta\right)\right) &= v_1\left(\frac{1}{\varepsilon}+\theta, \varepsilon-\varepsilon^2\theta+\varepsilon^3\theta^2-\ldots\right) \\ &= (\mathrm{st}(\theta/1), st((\varepsilon-\varepsilon^2\theta+\varepsilon^3\theta^2-\ldots)/1) \\ &= (\lambda, 0), \end{aligned}$$

where λ is the real number st θ.

This means that, looking through a window at infinite points on the graph, we see the same picture as the x-axis. This too is typical, in the sense that looking at an asymptote at infinity is the same through a window as looking at the graph, as we shall see later.

EXAMPLE 5. As a case of looking through a telescope, let us view the graph of f(x) = 1/x through an optical (1/ε)-telescope aimed at the origin. The field of view is the set of points (x, y) where x, y are both, at worst, infinite elements of first order. In particular, a point (a, 1/a) on the graph is in the field of view if

$$\mathrm{o}(a) \geq 1 \text{ and } \mathrm{o}(1/a) \geq -1,$$

so that

$$-1 \leq \mathrm{o}(a) < 1.$$

For o(a) = –1 we have

$$a = a_{-1}\varepsilon^{-1} + a_0 + a_1\varepsilon + \ldots \text{ (where } a_{-1} \neq 0),$$

Hence

$$\begin{aligned} 1/a &= (a_{-1}\varepsilon^{-1} + a_0 + a_1\varepsilon + \ldots)^{-1} \\ &= (\varepsilon/a_{-1})(1 + (a_0/a_{-1})\varepsilon + (a_1/a_{-1})\varepsilon^2 + \ldots)^{-1} \end{aligned}$$

$= (\varepsilon/a_{-1})(1 - (a_0/a_{-1})\varepsilon + \rho)$ where $o(\rho) \geq 2$.

Hence

$$\nu(a, 1/a) = (\mathrm{st}(a\varepsilon), \mathrm{st}(\varepsilon/a))$$
$$= (a_{-1}, 0).$$

A similar calculation for $o(a) = 1$, where

$$a = a_1\varepsilon + \ldots \text{ (where } a_1 \neq 0),$$

gives

$$\nu(a, 1/a) = (0, 1/a_1)$$

Finally for $o(a) = 0$ we have

$$a = a_0 + a_1\varepsilon + \ldots \text{ (where } a_0 \neq 0),$$

so

$$1/a = (a_0 + a_1\varepsilon + \ldots)^{-1}$$
$$= (1/a_0)(1 + (a_1/a_0)\varepsilon + \ldots)^{-1}$$
$$= (1/a_0) - (a_1/a_0^2)\varepsilon + \ldots$$

and

$$\nu(a, 1/a) = (\mathrm{st}(a\varepsilon), \mathrm{st}(\varepsilon/a)) = (0, 0).$$

Thus viewing the graph of $f(x) = 1/x$ through a $(1/\varepsilon)$-telescope pointed at (0, 0) gives a picture as in Fig. 9. Notice that the whole of the finite part of the graph is collapsed onto the origin whilst the (first order) infinite parts are mapped onto the axes.

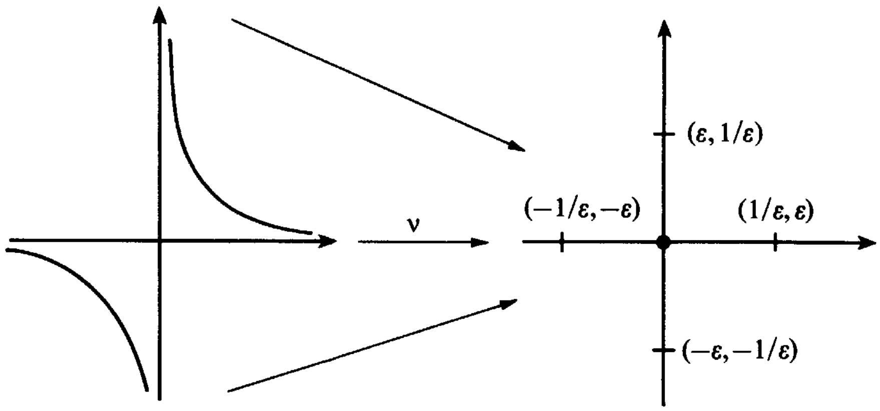

FIGURE 9.

It is sometimes convenient to use a lens which applies a different scale factor to each coordinate. Such a lens aimed at a point (α, β) would be of the form

$$(x, y) = (\mathrm{st}((x-\alpha)/\delta), \mathrm{st}((x-\beta)/\rho)).$$

When the factors δ and ρ have different orders, the lens is said to be *astigmatic*. An example where an astigmatic lens proves useful is in viewing an elemental strip of infinitesimal width beneath the graph of an analytic function. The height of such a strip over a point x is $f(x)$ and (unless $f(x) = 0$) we cannot choose a δ-lens whose field of view includes the finite height and infinitesimal width, yet is able to magnify the width to visible proportions. The solution is to use an infinite scaling factor on the x-coordinates whilst retaining a finite scaling factor for the y-coordinates. For instance, if the width of the strip is θ, we could take $\delta = \theta$ and $\rho = 1$ in the formula for an astigmatic lens. Then aiming the lens at, say, $(x, 0)$, we find

$$\nu(x, f(x)) = (0, f(x))$$

and, for a real number λ between 0 and 1,

$$\nu(x + \lambda\theta, f(x + \lambda\theta)) = (\mathrm{st}\lambda, \mathrm{st}(x + \lambda\theta)) = (\lambda, f(x)).$$

Thus the elemental strip has its width magnified by the infinite factor $1/\theta$ and, through the astigmatic lens, is seen to be a rectangle (Fig. 10).

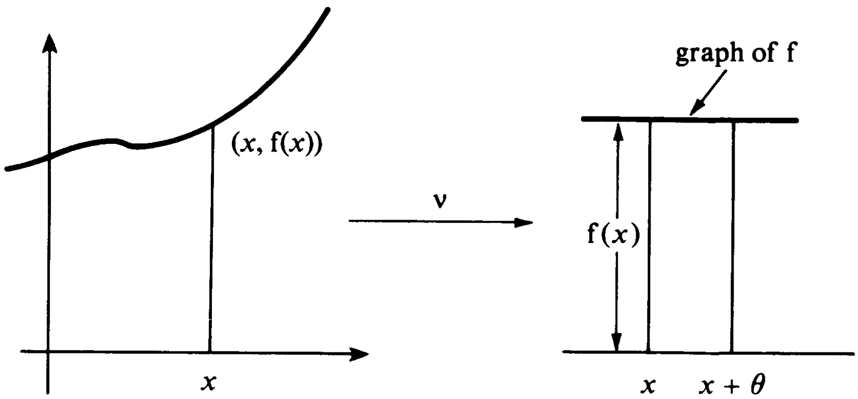

FIGURE 10.

Now that we have flexible methods of looking at graphs of analytic functions, let us consider the graph of a general power series, which may have a pole at the origin,

$$f(x) = \sum_{n=-m}^{\infty} a_n x^n .$$

If this has positive radius of convergence R, then we have an analytic function, defined for $0 < |x| < R$ in general, and for $|x| < R$ when the origin is not a pole. Of course, a power series like $\sum_{n=0}^{\infty} n!x^n$ only convergent for $x = 0$, which means that an expression like

$$\frac{1}{x} + \sum_{n=0}^{\infty} n!x^n$$

is not defined for any real x. But it *is* defined when x is a non-zero infinitesimal. (It is for this reason that we do not need to concern ourselves with questions of convergence of superreal expansions, thus avoiding convergence technicalities in the definition of $\Re$; we can then handle superreal extensions of *all* power series, not just those with positive real radius of convergence.)

We may consider the graph of f for (at least) non-zero infinitesimal values of x and view it through appropriately chosen lenses.

The line $x = \varepsilon$ meets the graph at the point (ε, f(ε)), which may be seen by aiming an ε-microscope at it. As f varies through all such power series we therefore get a correspondence between the function

$$\mathrm{f}(x) = \sum_{n=-m}^{\infty} a_n x^n$$

and the supperreal number

$$\mathrm{f}(\varepsilon) = \sum_{n=-m}^{\infty} a_n \varepsilon^n$$

which is the ordinate where $x = \varepsilon$ meets the graph. (Technically, this is an isomorphism of ordered fields between the field of such functions and the superreal numbers.) If we restrict our attention to the subfield $\boldsymbol{R}(x)$ of rational functions described in §2, then we derive an isomorphism between $\boldsymbol{R}(x)$ and the subfield $\boldsymbol{R}(\varepsilon)$ of $\Re$ consisting of all rational expression in ε, in which the function $y = x$ corresponds to the infinitesimal superreal number ε.

The nineteenth century mathematician Cauchy defined an infinitesimal quantity as a variable whose numerical value decreases indefinitely in such a way as to converge to zero. He therefore visualised an infinitesimal as a *function* which converges to zero at the origin. The value of the modern definition is that an infinitesimal can be visualised now as a *point* on the extended superreal line $\Re$, with the link between functions and points being given by the isomorphism just described. What is remarkable about this link is the fact that an analytic function near the origin is described totally by knowing the single point where its graph meets the line $x = \varepsilon$!

7. The differential triangle of Leibniz

Let $\mathrm{f}{:}D \to \boldsymbol{R}$ be an analytic function with extension $\mathrm{f}{:}D^{\#} \to \Re$. We view the graph of f through an optical microscope aimed at (x,f(x)). If o(θ) ≤ o(δ), then

$$\begin{aligned} v_\delta\big(x+\theta, \mathrm{f}(x+\theta)\big) &= \left(\mathrm{st}(\theta/\delta),\, st\left(\sum_{n=1}^{\infty} a_n\theta^n/\delta\right)\right) \\ &= (\lambda, a_1\lambda), \end{aligned}$$

where λ is the real number st (θ/δ).

Allowing λ to vary, we see that, on looking through a microscope, an infinitesimal portion of the graph of an analytic function is optically a straight line with gradient a_1 (Fig. 11).

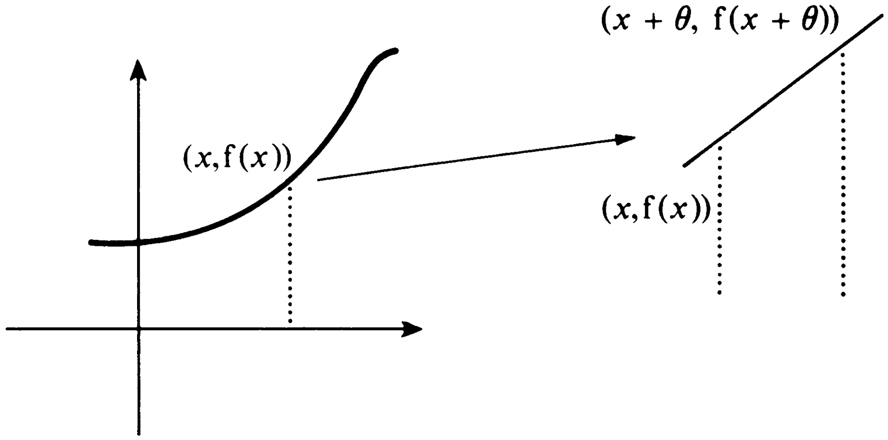

FIGURE 11.

This is to be expected, for in essence we are looking at terms of a certain order and 'neglecting' those of higher order, leaving only the linear approximation.

We can define the *derivative* of an analytic function in infinitesimal terms as

$$f'(x) = \mathrm{st}\left(\frac{f(x+\theta)-f(x)}{\theta}\right) \text{ (where } \theta \text{ is a non-zero infinitesimal).}$$

Thus we have

$$f'(x) = a_1.$$

This discussion may be summed up as:

THEOREM 1. *If* f:$D\to\boldsymbol{R}$ *is analytic,* $x \in D$, *and* δ *is a positive infinitesimal, then, through an optical microscope pointed at* (*x*, f(*x*)), *we have*

$$v_\delta(x+\theta,\ f(x+\theta)) = (\lambda,\ f'(x)\lambda),$$

where $\lambda = \mathrm{st}(\theta/\delta)$.

We say that two points (x_1,y_1), (x_2,y_2) are *optically indistinguishable* through v_δ if $v_\delta(x_1,y_1) = v_\delta(x_2,y_2)$

We can then say that two graphs f, g through (*x*, *y*) are optically indistinguishable if

$$v_\delta(x+\theta, f(x+\theta)) = v_\delta(x+\theta, g(x+\theta))$$

for $o(\theta) \geq o(\delta)$.

If $y = g(x)$ is the tangent to the graph through (x_0, y_0) we have

$$g(x) = y_0 + f'(x)(x - x_0)$$

and, of course,

$$\nu_\delta(x_0+\theta, g(x_0+\theta)) = (\mathrm{st}(\theta/\delta), \mathrm{st}(\mathrm{f}'(x)\theta/\delta))$$
$$= (\lambda, f'(x)\lambda).$$

Hence the tangent is optically indistinguishable from the graph, or, in other words, an infinitesimal portion of the graph looks the same as an infinitesimal portion of the tangent through an optical microscope.

This notion is essentially found in the *Lectiones geometriae* of Barrow (1670), where he performed computations and rejected second order (or greater) infinitesimals: “If the arc is assumed indefinitely small, we may safely substitute instead of it the small bit of the tangent” (in Child’s translation [5]).

In 1684 [6] Leibniz introduced the notation which has survived to the present day, taking dx to be any infinitesimal and defining

$$dy = \mathrm{f}'(x)\,dx$$

If we let $\delta x = dx$ and

$$\delta y = \mathrm{f}(x+\delta x) - \mathrm{f}(x)$$

then $(x + \delta x, y + \delta y)$ is a point on the graph, $(x + dx, y + dy)$ is a point on the tangent and these are optically indistinguishable seen through ν_{dx}.

We recall from the end of §4 that any positive element of even order, such as $dx^2 + dy^2$, will have a square root in $\mathfrak{R}$ and we define

$$ds = \sqrt{(dx^2+dy^2)}$$

We may then look through an optical dx-microscope (as in Fig. 12) to see the differential triangle of Leibniz. The gradient of the graph is the gradient of the tangent, namely dy/dx, and the infinitesimal segment of the graph is optically a straight line of length ds.

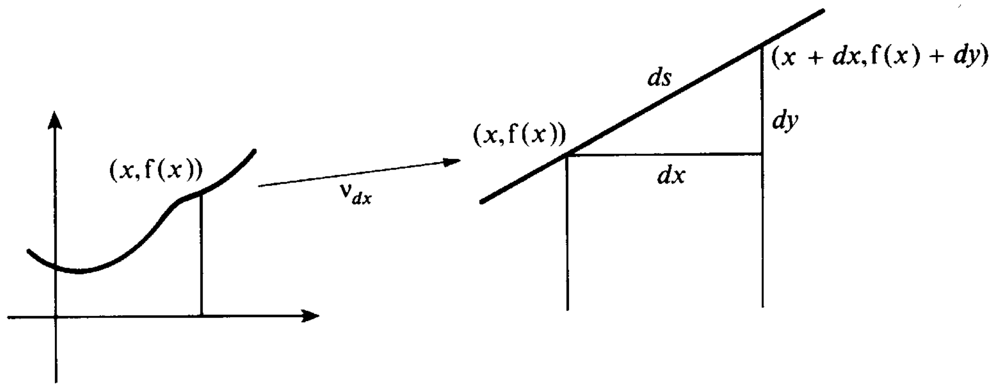

FIGURE 12.

8. Integration

To compute the area beneath a graph, we begin by looking at an elemental strip between x and $x + \delta$, where δ is an nth order infinitesimal. We have developed two different methods of doing this, one through an astigmatic lens looking at the whole strip (Fig. 10), the other through a microscope which has only the top of the strip in the field of view (Figs. 11, 12). In the first case we see the strip as a rectangle of area f(x)θ plus terms of order higher than n. Denoting the area under the graph from $x = u$ to $x = v$ by $A_f(u, v)$, then this means that

$$A_f(x, x+\theta) - \theta f(x)$$

is of order exceeding n; or, upon dividing through by the element θ of order n,

$$A_f(x, x+\theta)/\theta - f(x)$$

is infinitesimal.

An alternative calculation can be made using the evidence of Fig. 11. Since the area at the top of the elemental strip is a triangle to within terms of order higher than n, we find the total area of the elemental strip as that of a trapezium,

$$A_f(x, x+\theta) = \tfrac{1}{2}(f(x) + f(x+\theta))\theta + \rho,$$

where the error term ρ satisfies o(ρ) > n and does not register optically in a microscope of order n.

But

$$f(x+\theta) = f(x) + \delta$$

where δ is infinitesimal, so

$$A_f(x, x+\theta) = \tfrac{1}{2}(f(x) + f(x+\theta))\theta + \rho$$
$$= f(x) + \omega,$$

where $\omega = \tfrac{1}{2}\delta\theta + \rho$ is of order exceeding n. Once again we find that

$A_f(x, x+\theta)/\theta - f(x)$ is infinitesimal.

We now define an *area function* A_f for an analytic function f: $D \to \boldsymbol{R}$ (where D *is* an open interval) to be a superreal function $A_f(u,v)$ of two variables u, v in $D^{\#}$, such that

(i) $A_f(u, v) + A_f(v, w) = A_f(u, w)$ (where $u, v, w \in D^{\#}$);

(ii) if $x \in D$ and θ is a non-zero infinitesimal, then
$A_f(x, x+\theta)/\theta - f(x)$ is infinitesimal.

The first of these simply states that the area function must be additive in the usual way, conveniently expressed in a superreal formulation which proves to be exactly what we shall need. The second is suggested by our computations.

From this definition we easily deduce:

THE FUNDAMENTAL THEOREM OF CALCULUS. *If* f: $D \to \boldsymbol{R}$ *is analytic on the open interval* D, A_f *is an area function for* f, *and for some* $a \in D$ *we define*

$$F(x) = A_f(a, x) \text{ (for } x \in D^{\#}\text{)},$$

then

$$\mathrm{F}' = \mathrm{f}\,.$$

Conversely, if the analytic function F *satisfies* $\mathrm{F}' = \mathrm{f}$, *then*

$$\mathrm{A}_\mathrm{f}(a, b) = \mathrm{F}(b) - \mathrm{F}(a) \text{ is an area function.}$$

PROOF. Given an area function A_f for f, and defining $\mathrm{F}(x) = \mathrm{A}_\mathrm{f}(a, x)$, then (i) implies

$$\mathrm{F}(x+\theta) - \mathrm{F}(x) = \mathrm{A}_\mathrm{f}(x, x+\theta),$$

so (ii) gives

$$\frac{\mathrm{F}(x+\theta) - \mathrm{F}(x)}{\theta} - \mathrm{f}(x) \text{ is infinitesimal,}$$

and, taking standard parts, we find

$$\mathrm{F}'(x) - \mathrm{f}(x) = 0, \text{ as required.}$$

Conversely if $\mathrm{F}' = \mathrm{f}$, then defining

$$\mathrm{A}_\mathrm{f}(a,b) = \mathrm{F}(b) - \mathrm{F}(a)$$

we trivially find that (i) is satisfied, and

$$\frac{\mathrm{A}_\mathrm{f}(x, x+\theta)}{\theta} - \mathrm{f}(x) = \frac{\mathrm{F}(x+\theta) - \mathrm{F}(x)}{\theta} - \mathrm{f}(x),$$

so, taking standard parts,

$$\mathrm{st}\left(\frac{\mathrm{A}_\mathrm{f}(x, x+\theta)}{\theta} - \mathrm{f}(x)\right) = \mathrm{F}'(x) - \mathrm{f}(x) = 0,$$

which implies

$$\frac{\mathrm{A}_\mathrm{f}(x, x+\theta)}{\theta} - \mathrm{f}(x) \text{ is infinitesimal.}$$

This gives (ii) and completes the proof.

We remark that Riemann sums are not essential in this theory, since their chief function is to establish that an antiderivative actually exists. If

$$\mathrm{f}(x) = \sum_{n=0}^{\infty} a_n x^n$$

is analytic, then

$$\mathrm{F}(x) = \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}$$

satisfies $\mathrm{F}' = \mathrm{f}$, so an antiderivative of an analytic function can be found by explicit computation.

In Leibniz's notation we denote $A_f(a, b)$ by $\int_a^b f(x)\,dx$. Such a notation arose because Leibniz [7] regarded the area as an infinite sum of strips height f(x) and width dx. If we view such a strip through an astigmatic optical lens, as in Fig. 10, then we do indeed find that each elemental strip is optically a rectangle height f(x) and width dx. The fundamental theorem tells us that the sum of such strips is given by

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F' = f$.

9. Arc length

To compute lengths of curves, it is convenient to consider the more general case of a curve given parametrically by

$$\varphi(t) = (f(t), g(t)) \quad (a \le t \le b)$$

where f, g are analytic functions. If $f(t) = t$ then this reduces to the graph of $y = g(x)$.

We have

$$\varphi(t+h) = (f(t+h), g(t+h)) \quad (a \le t \le b)$$

and, looking through a δ-microscope pointed at (f(t), g(t)), we get

$$\begin{aligned} v_\delta(\varphi(t+\theta)) &= (\mathrm{st}(\{f(t+\theta) - f(t)\}/\delta, \mathrm{st}(\{g(t+\theta) - g(t)\}/\delta) \\ &= (f'(t)\,\mathrm{st}(\theta/\delta), g'(t)\,\mathrm{st}(\theta/\delta)) \\ &= (f'(t)\lambda, g'(t)\lambda), \end{aligned}$$

where $\lambda = \mathrm{st}(\theta/\delta)$. As usual, the graph is optically straight. We also find that

$$v_\delta(f(t) + \theta f'(t), g(t) + \theta g'(t)) = (f'(t)\lambda, g'(t)\lambda),$$

so $\varphi(t+\theta)$ is optically indistinguishable from $(f(t) + \theta f'(t), g(t) + \theta g'(t))$

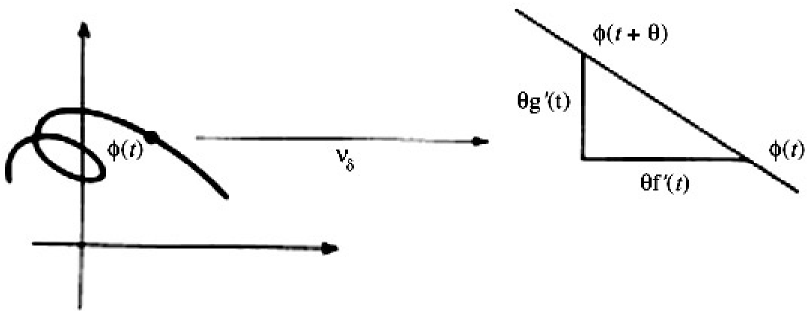

FIGURE 13.

It is reasonable to request that the length of the infinitesimal piece of curve, from $\varphi(t)$ to $\varphi(t+\theta)$, which is optically straight to order $n = o(\delta)$, is of length

$$\{\theta^2 f'(t)^2 + \theta^2 g'(t)^2\}^{1/2}$$

to within terms of even higher order.

By putting $\theta = \delta$, this requires that the length of arc $\mathfrak{l}(t, t+\theta)$, from $\varphi(t)$ to $t+\theta$) is

$$\mathfrak{l}(t, t+\theta) = \{f'(t)^2 + g'(t)^2\}^{1/2}\,\theta + \text{terms of degree higher than } \theta,$$

or, equivalently, that

$$\mathfrak{l}(t, t+\theta)/\theta - \{f'(t)^2 + g'(t)^2\}^{1/2} \text{ is infinitesimal.}$$

We therefore axiomatise the notion of length of an arc by defining

$$[a,b]^{\#} = \{t \in \Re \mid a \le t \le b\}$$

and then:

DEFINITION. The *length function* for an analytic curve $\varphi(t) = (f(t), g(t))$ $(a \le t \le b)$ is the superreal function of two variables $\mathfrak{l}(u, v)$ defined for $u,v \in D^{\#}$ such that

(i) $\mathfrak{l}(u, v) + \mathfrak{l}(v, w) = \mathfrak{l}(u, w)$ $(u,v,w \in [a, b]^{\#})$;

(ii) if $x \in [a, b]$, $x+\theta \in [a, b]^{\#}$, where θ is a non-zero infinitesimal, then $\mathfrak{l}(t, t+\theta)/\theta = \sqrt{f'(t)^2 + g'(t)^2}$ is infinitesimal.

From (i), if $L(t) = \mathfrak{l}(a,t)$, then $L(t+\theta) - L(t) = \mathfrak{l}(t, t+\theta)$,

so (ii) implies

$$\{L(t+\theta) - L(t)\}/\theta - \sqrt{f'(t)^2 + g'(t)^2}$$

is infinitesimal, and taking standard parts,

$$L'(t) = \sqrt{f'(t)^2 + g'(t)^2}\,.$$

The fundamental theorem gives

$$\mathfrak{l}(a, b) = L(b) = \int_a^b \sqrt{f'(t)^2 + g'(t)^2}\,dt\,.$$

Conversely, if $\mathfrak{l}$ is given by this equation then it easily follows that it sahsfies axioms (i), (ii) for a length function.

10. Asymptotes

So far all the applications have concerned infinitesimals—now we close with a brief use of infinite elements in looking at asymptotes. An example will make this clear. We consider the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

and look at the portion $f(t) = at$, $g(t) = b\sqrt{(t^2 - 1)}$ $(t \ge 1)$.

We may extend this function to (positive) infinite values of t, for if ω is positive infinite, then $\delta = 1/\omega$ is a positive infinitesimal and

$$\begin{aligned} f(\omega) &= a\omega, \\ g(\omega) &= b\surd(\omega^2 - 1) \\ &= \frac{b}{\delta}\left(1-\delta^2\right)^{1/2} \\ &= \frac{b}{\delta}\left(1-\tfrac{1}{2}\delta^2 + \ldots\right) \\ &= b\omega + \varepsilon \end{aligned}$$

where ε is infinitesimal.

If we point a 1-window at ($a\omega_0$, $b\omega_0$), then for (x, y) in the field of view,

$$\nu_1(x,y) = (\text{st}(x - a\omega_0),\ \text{st}\,(y - b\omega_0)).$$

Putting $\omega=\omega_0+k$, where k is finite, we find

$$\begin{aligned} \nu_1(\text{f}(\omega), \text{g}(\omega)) &= (\text{st}(\text{f}(\omega) - a\omega_0),\ \text{st}(\text{g}(\omega) - b\omega_0) \\ &= (\text{st}(ak),\ \text{st}(bk+\varepsilon)) \\ &= (a\lambda, b\lambda) \end{aligned}$$

where $\lambda = \text{st}k$.

A similar computation gives $\nu_1(a\omega,b\omega) = (a\lambda,b\lambda)$, which means that through the optical window ν_1, the point $(\text{f}(\omega),\text{g}(\omega))$ on the graph is optically indistinguishable from $(a\omega, b\omega)$ on the straight line

$$\frac{x}{a} = \frac{y}{b}$$

We may thus define two analytic curves to be *asymptotic* if they are optically indistinguishable when viewed through any window aimed at an infinite point. In practice this may involve a change in parametrisation of one of the curves to bring them into line. For instance, if

$$\phi(\text{t}) = (t^2, 2at),\ \psi(t) = (4t^2 + t^{-1},\ 4at),$$

then changing the parametrisation of the second curve to

$$\Psi(t) = \psi(\tfrac{1}{2}t) = (t^2 + 2t^{-1},\ 2at),$$

an optical 1-window aimed at $\phi(\omega)$ where ω is positive infinite gives

$$\nu_1(\Psi(w)) = (\text{st}(2\omega^{-1}), \text{st}(0)) = (0,0)$$

Thus $\phi(\omega)$ and $\Psi(\omega)$ are optically indistinguishable through v, and the curves are asymptotic.

In this way we see that, just as infinitesimals facilitate the calculus, so do infinite elements prove useful in the theory of asymptotes.

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