Calculations and Canonical Elements

Part 2

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In this first part of this article we suggested that the concept of a "canonical element" was the missing link required to unite abstract notions based on equivalence relations with the more traditional computations of school mathematics. Part I discussed canonical elements and their mathematical implications: we now take a look at possible educational implications.

6. Piaget and Number

Let us begin our reassessment of various topics in the school syllabus that use equivalence relations by looking at the concept of cardinal number. Since Piaget's famous investigations into the development of the number concept in children [8] a number of projects in primary school (for instance [l, 5, 7] have begun their development of number by emphasising matching activities between sets. Some teachers have taken these ideas further and emphasised the global structure to such an extent that number bonds were neglected, leading to the unfortunate jibe that a child may know that 15+6 is the same as 6+15, but cannot calculate either. Fortunately the pendulum is swinging back again to proficiency with number bonds.

If we look at this controversy through the picture of equivalence classes and canonical elements, we find a hidden reason for Piaget's interpretation. For a mathematical interpretation of the cardinal number concept he had Cantor's description using bijective equivalence between sets. Mathematicians have a habit of emphasising aesthetic elegance (as in the case of an equivalence relation) and suppressing technicalities (such as canonical elements). Thus the mathematical theory that Piaget had at his disposal was fundamentally deficient. He also noted that children at a certain stage were able to count but did not grasp the idea of a correspondence between two sets with the same number of elements. In this way the role of counting became further diminished. In [8] on page 64, he states

At the point at which correspondence becomes quantifying, thereby giving rise to the beginnings of equivalence, counting aloud may, no doubt, hasten the process of evolution. Our only contention is that the process is not begun with numerals as such.

He realises that Cantor's description is inadequate, for on page 41 he states:

Yet although one-one correspondence is obviously the tool used by the mind in comparing two sets, it is not adequate . . .

He goes on to investigate the perceptual cues which may mislead the child from attaining true quantitative correspondence between sets.

Of course these observations should be taken seriously in teaching young children, but interpretations based on an inadequate theoretical framework may be seriously wide of the mark, especially when the missing link in the framework is the number concept of traditional mathematics upon which computation is based.

Freudenthal has launched an acid attack in [6, page 192] on Piaget's theory of the concept of number in child development. Concentrating on Cantor's definition of number through equivalence between sets, which he refers to disparagingly as "numerosity number", he devastatingly shows it to be mathematically and didactically insufficient. He goes on to emphasise the role of counting in a child's number concept.

Leaving aside the point that, in attacking "numerosity number", Freudenthal is not precisely criticising Piaget's concept, for Piaget himself has said that one-one correspondence is not adequate, we see that in looking at the number concept they are in fact emphasising different parts of the picture. Freudenthal points to the efficient tool of counting (using canonical elements) and Piaget concentrates on equivalence. A complete theory needs both.

7. Place Value

Dienes blocks (see [3]) have been used successfully to give a geometric idea of place value. A simple alternative (or adjunct), which is analogous, is the use of Egyptian number symbols. Cards marked with strokes, hoops and scrolls could be used like money. There is no reason why a card should not have, say, five strokes on it, then two such cards would be equivalent to a hoop card. Given a collection of cards (totalling less than 1 000), the canonical equivalent is found by the physical process of exchanging 10 strokes for a hoop, or 10 hoops for a scroll, and so on as necessary, until there are less than 10 of each scroll, hoop or stroke. (The teacher could have higher order Egyptian symbols on hand for numbers over 1 000, for instance a lotus flower for 1 000, and so on.)

An alternative to such symbols could be unifix cubes of different colours, say white for 1, blue for 10, red for 100, and other colours for larger powers of 10. The advantage of the Egyptian symbolism is the opportunity for stimulating the children's imagination by setting the mathematics within a project on life in ancient Egypt. Pieces of card with strokes, hoops, and scrolls marked on them also prove to be cheaper to produce rather than expensive structural apparatus. Handling any of these physical materials gives them an intuitive grasp of the theory of equivalence (through interchange of symbols) and canonical element (through reduction to standard form) without any fussy formalism.

8. Negative Numbers

On introducing the topic of negative numbers to children we have a choice between the equivalence class approach (through equivalence classes of ordered pairs), or the canonical element approach (through the number line with positive and negative numbers marked on it), or a blend of the two. Several modern approaches have adopted the more aesthetically pleasing approach of equivalence classes. We believe this to be based on an incomplete theoretical analysis which omits canonical elements.

Recent post-Piagetian research (for example [4]) shows that young children have strong rational powers within a real-life intellectual framework. There are a number of real-life situations in which negative quantities arise naturally. A thermometer in degrees centigrade in the winter involves temperatures below zero. (A subtle point here is that a vertical number line has a strong up-down directionality that is to be preferred to the weaker left-right directionality of a horizontal number line.) The more mathematically minded football enthusiast will also find negative numbers in the goal differences at the foot of a league table. The latter involves equivalence classes of ordered pairs (goals for and against), with the goal difference itself giving rise to positive and negative numbers. The time-honoured ideas of positive numbers for credits and negative numbers for debts also give an analogous real-life example of the use of negative numbers.

With such examples available it seems a much more reasonable idea to approach negative numbers through a (vertical?) number line. This could be linked to the equivalence relation idea through an embodiment such as the football goal difference, marking the goal difference on the number line. “Defining” negative numbers as equivalence classes of ordered pairs of positive numbers is an unnecessary complication.

Any of the above approaches gives a natural definition of addition of positive and negative numbers. For instance a goal difference of +2 (say 3 for and 1 against) added to a goal difference of -5 (say 2 for and 7 against) adds up to 5 for and 8 against, a goal difference of -3. Thus +2+-5=-3. Subtraction is not much more difficult. For example if a team has 10 goals for and 6 against (goal difference +4) and the result of a match lost by 2 goals (1 for and 3 against) is declared void and removed from the total, we arrive at +4 – -2=+6, because taking “1 for and 3 against” from “10 for and 6 against” leaves “9 for and 3 against” — a goal difference of +6.

A variant of this, the “Post Office game”, in which the postman delivers cheques and bills and also takes away cheques and bills, also gives an embodiment of the rule that “taking away a minus” is the same as a “plus”. Taking away a bill for £3, which the recipient was prepared to pay, is like giving him £3 to use which was previously put aside to pay the bill.

It is when we try to explain *multiplication* of negative numbers that we encounter real difficulties. The formal approach through equivalence classes of ordered pairs alone is of little use, for how do we motivate the product:

$(m, n) \times (p, q) = (mp + nq, mq + np)$?

This is the formula we arrive at by thinking of (m, n) as $m - n$ and translating

$(m - n) \times (p - q) = (mp + nq) - (mq + np)$.

The formal equivalence class approach at this level does not simplify matters, it makes them more complicated.

To extend multiplication to negative numbers in a natural way we must find naturally arising situations in which both numbers concerned may be negative. Now in early school experience, multiplication is introduced through repeated addition, so that 4×3 (thought of as "four lots of three") is given as

4×3=3+3+3+3.

In such an approach it is easy to take four lots of -3 to get

4×-3 = -3+ -3+ -3+ -3 = -12.

To allow the first number in a product to be negative, however, requires an extension of the concept where, for instance, -4 × 3 means "take away four lots of three". In such an interpretation, -4x-3 means "take away four lots of -3" which becomes "take away -12" and this is +12.

Such an embodiment succeeds in the goal difference model and the Post Office game. Suppose a team has a certain goal difference and three games each of goal difference -2 are declared void. Removing these games from the total, it will be seen that taking away 3 times -2 has the same result as adding +6. Similarly, in the Post Office game, removing three bills for £2 has the same effect as adding £6 because removing the three bills has the effect of releasing £6 which had been put aside to pay the bills.

One cannot pretend that these are ideal ways of introducing multiplication of negative numbers, even though they are better than the formal equivalence class approach. The only way in which multiplication of negative numbers can be given a natural interpretation is when both numbers concerned can be directed. Such an example might be through

distance = velocity × time.

Taking "time before" as negative, and "velocity in reverse" as negative, we find that a car travelling in reverse at 3 kilometres an hour was, at a time 4 hours ago 12 kilometres in front of its present position. Such a concept may be a little too complicated to use when negative numbers are first introduced, but later it may prove a valuable strengthening of the notion that the product of two negative numbers is positive.

9. Fractions

As with negative numbers, fractions may be introduced through equivalence classes of ordered pairs of (positive) integers, as in example 4 in Section 2. For the same reasons as were rehearsed for negative numbers, an approach using only equivalence classes and not canonical elements is incomplete. Elsewhere [13], it has been discussed at length that a logical development need not be the most appropriate for teaching purposes and the argument is even stronger when the logical analysis is deficient. Suffice it to say that a balanced approach to fractions through fractions in lowest terms (canonical elements) and equivalent fractions gives a more coherent blend of calculations and general theory.

10. Real Numbers

The most comprehensive number system (not going as far as the complex numbers) is the field of real numbers. In school this is represented by the number line, though at university real numbers may actually be constructed set-theoretically from the rationale. We have suggested elsewhere [11, 12] that it is developmentally inappropriate to build up formally from the natural numbers through the integers, rationals and then via a completion process to the real numbers themselves. In secondary school however, real number concepts, at least on an intuitive level, are required for geometrical purposes and for the calculus. Here we shall address ourselves to the problem: "what is a real number?" and to answer this by expressing the number as a decimal.

Identifying real numbers as points on a theoretical number line (see [11]) we can take a real number α on the number line and sandwich it between integers $m \leq \alpha < m + 1$ (Fig. 4):

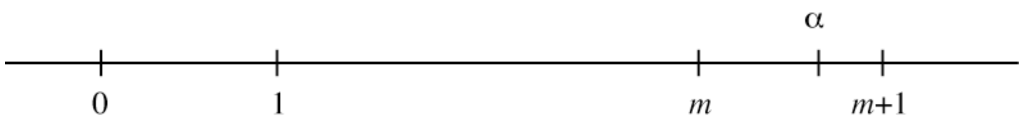

Figure 4

Then we can divide the interval from m to $m + 1$ into tenths to find

$$m + a_1/10 \leq \alpha < m + a_1/10 + 1/10$$

and then into hundredths to get the digit a_2 in the second decimal place:

$$m + a_1/10 + a_2/100 \leq \alpha < m + a_1/10 + a_2/100 + 1/100$$

and so on.

Writing a_0 instead of m, for each real number α we get a decimal expansion

$$a_0 \cdot a_1 a_2 \ldots a_n \ldots$$

where truncation to n decimal places gives an approximation to a within 1.10^{n}:

$$a_0 \cdot a_1 a_2 \ldots a_n \le \alpha < a_0 \cdot a_1 a_2 \ldots a_n + 1/10^n$$

If we write

$$s_n = a_0 \cdot a_1 a_2 \ldots a_n$$

as the truncation of the expansion to n decimal places, then we note that

$$\lim_{n \to \infty} s_n = \alpha.$$

(A fuller discussion on limits of sequences in general and limits of decimals in particular may be found in [11, Chapter 2].)

We therefore use the notation

$$a_0 \cdot a_1 a_2 \ldots a_n \ldots$$

which yields a decimal expansion for every real number α.

This, however, involves a technical problem. It may happen that some real numbers have two different decimal expansions. For instance, if $\alpha = 1$, then $\alpha = 1.000\ldots$ and $\alpha = 0.999\ldots 9\ldots$.

Seventy per cent of a sample of first year university students did not believe that these two expansions represented the same real number [10]. But if we let

$s_n = 0 \cdot 999 \ldots 9$ (n decimal places), then

$s_n = 1 - 1/10^n$ and

$$\begin{aligned} \lim_{n \to \infty} s_n &= \lim_{n \to \infty} (1 - 1/10^n) \\ &= 1 - 0 \\ &= 1 \end{aligned}$$

using standard results about limits.

How can we formalise this problem to give a coherent explanation of when a decimal expansion is unique and when it is not?

We define a decimal expansion d to be an expression

$$a_0 \cdot a_1 a_2 \ldots a_n \ldots$$

where a_0 is an integer and each $a_n (n \ge 1)$ is an integer between 0 and 9. (We can formalise this even further by talking about the sequence (a_n) of decimal places, the sequence (d_n) of truncated decimals where

$$d_n = a_0 \cdot a_1 a_2 \ldots a_n$$

but that would be taking matters to extremes.)

We will say that two decimal expansions c and d are equivalent, and write $c \sim d$, if they represent the same real number. We find (see [11, Chapter 2] for details) that each equivalence class contains either one element or two. It contains two precisely when one ends in a recurring sequence of zeros:

$$a_0 \cdot a_1 a_2 \ldots a_N 000\ldots$$

where $a_N \neq 0$ and $a_n = 0$ $(n > N)$, and the other is

$$b_0 \cdot b_1 b_2 \ldots b_N 999\ldots$$

where $a_n = b_n$ $(n < N)$, $b_n = a_n - 1$, $b_n = 9$ $(n > N)$. Thus a real number like

$1/3 = 0 \cdot 333\ldots.$

has a unique expression, but $0 \cdot 1066$ has two, namely $0 \cdot 1066000...0...$ and $0 \cdot 1065999...9....$. This is an interesting example of an equivalence relation in which the equivalence classes come in different sizes. Our choice of canonical element in this case is the given element when there is only one (that choice is forced on us!) and (usually) the expansion ending in repeated zeros when there are two. In the latter case we usually simplify the notation by omitting the superfluous zeros.

11. Conclusion

We have demonstrated amply the role that canonical elements play in the theory of equivalence classes, and how they arise naturally in many different cases of traditional computations. We could go further and give more examples. For instance in dealing with vectors that are considered equivalent if they have the same length and direction, there is a canonical choice in each equivalence class whose tail is at the origin. This leads to a description of free vectors and bound vectors (the latter being the canonical elements) within the conceptual framework we have described. More advanced concepts in group theory and ring theory have the same conceptual framework when we consider quotient groups and quotient rings. Here we get a generalisation of modular arithmetic. The elements in the quotient structure are equivalence classes and the operations are induced on the equivalence classes from operations on the elements in those classes. Once more we find that the computations on the classes can be carried out technically by working with canonical choices of elements.

In this way we find the notion of canonical element serving a useful purpose in many different mathematical contexts. In putting a central case for canonical elements in the theory of equivalence classes we are not advocating the use of the concept by name at all levels of teaching. The approach must be suitable for the learner at all times. The value to the teacher of this concept is to realise the theoretical importance of canonical elements in traditional computations and to avoid the schism between the emasculated formality of equivalence relations and the cumbersome theoretical difficulty of traditional calculations.

Mathematicians need both, and they cannot unify them without the missing link of the canonical element.

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