Mathematical Thinking & the Brain

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§1 Introduction

Though our knowledge of brain activity is still rudimentary, it is becoming increasingly clear that our partial understanding of the phenomenon can give significant insight into mental processes; in particular our growing understanding of the brain itself highlights certain aspects of mathematical thinking. In considering the brain we must first decide which level (or levels) of structure we wish to study; we may concentrate on the molecular level, investigating the chemical changes involved, we may look at the neuronal level and consider their function, or we may desire a more global model of activity. Investigations into the chemistry show hopeful signs that may lead to a better understanding of the nature of memory [11], [12]. At the cellular level a great deal is known. For instance the neurons always fire in a specific direction [12, p.93], so that electrical mental activity is an ordered structure. Thoughts cannot be reversed in the simple way we reverse a film by running it backwards, the best we can do is to key into certain parts of the ordered schema to re-run a portion of it. Is it any wonder therefore that children have problems with reversible properties in mathematics? Within the brain the reversal of a property cannot be a mirror image, it requires the construction of an alternative route.

At these levels of brain structure we can therefore already see signs of useful information developing, but it seems that a more global pattern is essential to have much hope of encapsulating the more complex nature of mathematical thinking. Biological theories at this level are limited, but a mathematical model has been suggested by Zeeman [13], regarding the brain in terms of a dynamical system. This of course is, of course, a theoretical model based not on specific measurements, but rather on the nature of the electrical activity of the brain. It can be viewed in terms of resonance [8] which we shall outline in this paper. As with any model, it does not describe the actuality of the brain, however it suggests a plausible manner in which the brain may work at a global level. It concentrates on the qualitative nature of the thinking processes: the resonances and the abrupt changes in resonance (the latter being described in terms of catastrophe theory).

In this paper we shall concentrate on the notion of resonance and look at some specific examples of this phenomenon in mathematical thinking. In particular we shall be interested in the kind of thinking used in creating a mathematical proof. The mathematician does not arrive at his proof by a process

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of inexorable logic. At first he may be confused by the data, then out of a partial understanding of the problem may come strategies of attack which may at first only go part of the way, to be later refined into a more complete and acceptable version, The final polished form of proof is rarely a logical structure either, it concentrates on certain novel turning points and suppresses routine detail. So if the final edifice built by the mathematical mind is not an absolute logical structure, nor is the tortuous route by which the proof was obtained. It is stretching the imagination to think that the brain itself can be described in terms of purely logical circuitry. In this paper the term generic thought will be used to describe the notion of thinking within a resonance framework. Certain aspects of a problem may fit together and cause a mental resonance leading to an ordered mental schema in action (perhaps, but not necessarily, in terms of a mathematical algorithm); then if the result is incomplete, subsequent thought may lead to suitable refinements, and so on. The chief problem in describing the process is that certain fundamental mental actions are so fast that the individual cannot say how the total chain of activity happened in his mind, only that certain aspects have occurred. The examples given will exhibit this limitation. They consider students' answers to mathematical problems and show how a resonance interpretation of thought may be considered. After discussing the notion of generic thought we shall compare this with the idea of 'generalisation' and consider the appropriateness of basing teaching schemes on mathematical hierarchies.

§2 Resonance

René Thom has outlined the beginning of a theory of resonance in brain activity in his essay "Topologie et Signification" [8, p. 201]:

> Si nous considérons la totalité de nos activites cérébrales comme un système dynamique (selon le modèle de C. Zeeman [12], nous serons amenés à supposer qu'a tout champ moteur codifié en verbe correspond un mode propre, un attracteur A de la dynamique cérebrale subit un stimulus spécifiques, qui la met dans un état instable d'excitation; cet état évolue ensuite vers la stabilité par sa capture par l'attracteur A, dont l'excitation engendre par couplage aux motoneurones 1'execution motrice de l'ordre. … Lorsque, sous l'effet d'un message, la dynamique mentale ne presente pas d'attracteur qui la capture de manière solide, c'est qu'alors le message est sans grande signification. Comme le montre bien l'assimilation du phenomène de comprendre à une resonance dynamique, l'absence de signification d'un texte n'est jamais totale: il se forme toujours des rcsonances plus ou moins fluctuantes, mais qui ne peuvent attacher l'esprit. ...

Thus an external stimulus presented to the brain via its receptors sends it into an unstable state of excitation which evolves very quickly to a state of stability; this new state of stability causes the effectors to carry out the appropriate physical or mental actions.

The biological nature of this activity is not fully understood, short term it may be an electrical process, but long term chemical changes in the brain are constantly modifying the possible configurations of the electrical activity. It may be, for instance, that short term memory is an electrical phenomenon and long term memory is caused by chemical change [4, p. 240]. This means that as the brain is enriched by sensory experience and mental activity its available configurations for electrical resonance change and respond in different ways to a given stimulus. Thus a stimulus which at first causes minimal resonance may through repetition cause a chemical change in the brain which eventually leads to resonance, remembering and response. Such conditioning lies behind stimulus-response theory, but it does not explain higher mathematical reasoning.

In mathematics we are concerned with more complex phenomena; at this level we may obtain certain insights into mental activity by considering it in terms of resonance. For instance it is possible in dynamical systems theory to superimpose two stable resonances leading to an unstable state which evolves to a third stable resonance [9?]. Thus resonance allows for a kind of creative response which is not just the result of learned stimuli.

R. R. Skemp [6] has given a different analogy for resonance by likening it to the resonating of the undamped strings of a piano when a sound is made. The piano strings respond with a given sound. In this analogy the resonance is limited by the physical configuration of the piano. The mind is more subtle because its resonance configuration changes with experience and time. Unlike a passive piano which only responds to the actions upon it, the mind is active, with much of this activity being subconscious. The process of long-term memory storage takes several hours, as can be shown by the common occurrence of loss of memory after concussion. Experimental evidence has indicated that memory fixation occurs over a period of thirty minutes to three hours [4, p. 239]. Problems which are attempted sometimes prove intractable at the time, yet a solution can leap to mind after a considerable period, indicating possible subconscious activity during at least part of the intervening time. University lecturers teaching difficult subjects such as analysis consider the need of a long period of study before the basic concepts are understood, again understanding can occur after a period of relaxation away from the topic (given some work on it in the first place!)

In considering the long-term activity of the brain, new sensory input, or brain activity itself, may change the possible resonance configurations. Given an established resonance, an alternative may develop, and during the period when two (or more) alternatives are possible, a slight change in sensory input may cause a dramatic change from one resonance to the other. The passage of time may lead to the dominance of a new resonance and the decay of the old. Such a model is a possible description of Piaget’s observed transitions from one stage of development to another [7].

In this paper we shall not concern ourselves with transitions and catastrophe theory, concentrating more on the resonance phenomenon. Resonance has certain simple implications; for instance with one or more factors resonating strongly and others not at all, attention is centred on resonating factors. With limited conscious capacity the brain may not consciously register non-resonating details. From this viewpoint we discriminate by virtue of the design of the brain, only noting the essential elements which resonate. (The musical analogy may be inappropriate here, since differences may cause brain resonance as well as similarities.)

Resonance may be viewed as an essential factor of human thought. It may allow us to recognise an object from a slightly different viewpoint. It may also explain how communication between two individuals is possible; similar, but not identical, experiences may be communicated as the two brains resonate on the important essentials and neglect the differences.

§3 Mathematical Examples

Example 1

Consider the following part of a university examination question:

Show that the matrix $\begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix}$ has two linearly independent eigenvectors .

The solution technique is straightforward, with one technical problem. When $\alpha \neq 0$ there are precisely *two* eigenvalues, $1+\alpha$, $1-\alpha$, and *two* eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ but for $\alpha = 0$ there is *one* eigenvalue 1 and *all* (non-zero) vectors are eigenvectors. For the eigenvalue $1+\alpha$ one finds the eigenvector $\begin{pmatrix} x \\ y \end{pmatrix}$ by solving

$$x + \alpha y = (1+\alpha)x$$
$$\alpha x + y = (1+\alpha)y$$

which simplifies to

$$\alpha y = \alpha x,$$
$$\alpha x = \alpha y.$$

Virtually all students reaching this point deduced from the equations that the solution was $x = y$, or $\begin{pmatrix} x \\ y \end{pmatrix} = x\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, giving the eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Only three out of 170 attempted to consider the case $\alpha = 0$, which can have *any* solution for x, y (and this despite the emphasis in the course of not dividing by zero in an equation such as $\alpha x = \alpha y$).

One could give various interpretations as to why this occurred, the most likely being that students consider α as a *general* scalar and did not think of it being a specific number, for instance $\alpha = 0$.

Regarding the thought processes as a resonance phenomenon, one can see the average student being drawn through the problem by the inexorable resonance of the technique, following an ordered mental schema, and failing to perceive the non-resonating singularity of $\alpha = 0$.

Example 2

In an ‘A-level’ question (for British students ending their school years aged 18), the following problem was posed:

> The inclined faces of a fixed triangular wedge make angles α and β with the horizontal. Two particles of equal weight, connected by a light inextensible string which passes over the vertex of the wedge, rest on two faces of the wedge. If both particles are in limiting equilibrium, show that
>
> $$\mu = \tan\tfrac{1}{2}|\alpha - \beta|$$
>
> where μ is the coefficient of friction between each particle and the face of the wedge.

Such a question, involving a modulus sign (as in $\tan\frac{1}{2}|\alpha - \beta|$) had not been set in recent years, though the modulus sign is part of the syllabus. A complete solution depends on considering the cases $\alpha \geq \beta$ and $\beta \geq \alpha$. separately. Many students drew a picture with α, β similar in size and failed to see this subtlety. Out of a sample of 100 students, seventeen obtained $\tan\frac{1}{2}(\alpha - \beta)$ from their diagram. This looks similar to the answer $\tan\frac{1}{2}|\alpha - \beta|$ and is the same for $\alpha \geq \beta$. Only one of these seventeen went on to consider the case $\alpha \leq \beta$ to obtain a complete solution. However five others drew a different diagram which led to the solution $\tan\frac{1}{2}(\beta - \alpha)$. This is also correct (for $\beta \geq \alpha$), but β, α now occur in a different order from that in $\tan\frac{1}{2}|\alpha - \beta|$ and the solution looks visibly wrong. Four of these five students immediately used an ad hoc argument to introduce the modulus and reconcile their solution – a much higher proportion than in the other case where the solution looks more acceptable. Apart from the one correct solution mentioned earlier and these four, the remaining 95 made no reference to the modulus sign whatsoever, almost as if it was not there.

Many, of course, were doubtless worried by it, but knew they would only lose a few marks if the rest of the question were correctly handled; the time factor truncated their attempts and they concentrated on the aspect of the question that they could handle. As an amusing additional remark, the examiner failed to consider the singular case $\alpha = \beta$.

Example 3

To show that mental resonance need not be purely in terms of mathematical algorithms, consider the individual case of an eight year old child faced with the following question:

> “Guessing Answers”: 19+12 is nearly 20+10 which equals 30. A good guess for the answer of 19+12 would be 30. Try to make a good guess for the answer of these. The first is done for you.
>
> 1. 21+8 is nearly 20+10 which equals 30.

2. 31+7	3. 19+21	4. 27+11	5. 42+9	6. 11+12
7.17+27	8. 32-11	9. 43–16	10. 38–19	

The child’s solutions were as follows:

2.	31+7 is nearly	29+10 which equals	39
3.	19+21	31+10	41
4.	27+11	29+10	39
5.	42+9	51+1	52
6.	11+12	23+1	24
7.	17+27	44+1	45
8.	32–11	10+10	20
9.	43–16	18+10	28
10.	39–19	10+10	20

The first comment to make is that this is rather a stupid question. Thirty seven pages earlier in the book was the last reference to “guessing to the nearest number ending in 0”, and this child had clearly forgotten. So he approached the question in his own way. Having admitted this, can we diagnose what mental activity caused him to write down the answers he did?

Within the terms of Thom’s theory we must be prepared for two different mental resonances causing instability and settling down as an entirely new resonance. Not only may the child not know why he came to a certain conclusion, but the observer should not expect to be able to deduce his reasoning logically from his observations. Even had we been able to get the child to “think out loud” as he did the problems, this might have interfered with his thinking, causing possibly a different outcome, it would have also been a very rudimentary analysis in his own terms. Discussing his solution afterwards, taking each question in turn, the first light dawned on the questioner in the solution of question 5 where the child mentioned that he knew that 42+9 was 51, so he added on 1 “to make it a guess”. It was only realised much later by the observer that all the answers with one exception were “the correct answer plus 1”. Questions 8, 9, 10 were interesting in that the child kept to the format

“? + ? is nearly something plus something”

instead of subtracting. Perhaps his strategy was to keep to the rhythm of the worked solution and change the answer by one to make sure it was a “guess” (after all a “guess” can’t be correct, otherwise it isn’t a guess!)

§4. Generic Thought

It is difficult to give a precise definition of a high level concept like that of ‘generic thought’. Perhaps we could begin by saying that *all* thought is generic but that the term ‘generic’ is used when it is viewed as a resonance activity. Sensory input causes an excitation of the brain which quickly stabilises through resonance activity and produces a chain of effective mental or physical action, It is the chain of mental action, a mental schema in the sense of Piaget, that we call generic thought. Of course the chain reaction leads to further mental excitation which itself may be temporarily unstable and lead to a totally different stable resonance. This may manifest itself as an observable discontinuity in thought. To give more precision I would prefer to confine the term “generic” to the connected chains of mental activity rather than the sudden discontinuities, but in practice such a distinction may not be possible in view of our inability to monitor brain activity accurately. If such measurement were possible we would almost certainly find that connected chains of thought contained many discontinuities anyway, but that we were so accustomed to them that we no longer sensed the dramatic nature of the discontinuities in the same manner that occurred the first time. For the moment then we shall use the term ‘generic thought’ to describe the whole process including the connected schemas and discontinuities.

The concept of generic thought helps to give a rational insight into how mathematicians often get the general grasp of the principles involved in a mathematical argument whilst (initially) omitting details. At this stage the essential features resonate. Logically the missing details may render a proof invalid, but in practice the details can often be tidied up to give an acceptable proof. Such a style of thought is often found in Lakatos [3]. It is a human activity often at variance with the formal idea of a step-by-step logical proof.

Generic thought is a natural consequence of the resonance of mental activity, which causes certain resonating ideas to come to the fore, whilst others are ignored. Having followed a chain of mental activity (together with possible discontinuities), the preliminary, or partial, proof may be considered unsatisfactory and the resultant instability can lead to further chains of thought which may (or may not) lead to improvements.

In the first example of the last section, considering α to be any real number is generic in a mathematical sense, whilst $\alpha=0$ is a singular case. This ties up with the use of the term '‘generic’ in algebraic geometry. The other examples are not generic in the same mathematical sense, but the phenomena are clearly analogous.

§5. Generalisation and Learning Hierarchies

Much attention is focused in mathematical education these days on the topic of generalisation and the ability to abstract mathematical properties from particular examples. The notion of generalisation in this sense is quite different from that of generic thought, though the latter often involves thinking in ‘general terms’. The difference is that generic thinking is construed as mental resonance in the electrochemical activity of the brain whereas generalisation is a mathematical classification of concept hierarchy. The distinction between these is absolutely fundamental in mathematical education, for learning hierarchies are sometimes falsely equated with mathematical hierarchies when the mathematical hierarchy may not correspond with the actual order of difficulty for a particular individual. It may not be, for example that a particular individual needs to abstract a generalisation from a number of different exemplars, though another individual may find this helpful,

The work of Dienes [11], establishes various principles of teaching in Piaget's concrete operational stage. For instance in his mathematical and perceptual variability principles he advocates the use of a number of exemplars from which to abstract a higher order concept. Skemp [5] also talks of the use of several exemplars from which to abstract a higher order concept, but he also speaks of 'noise' (properties in the exemplars different from those to be abstracted). With many examples and much noise, abstraction may be inhibited. Noise may be interpreted as interference in the electrical circuits which inhibits resonance. Thus there is a delicate balance between the exemplars used and the noise involved.

Krutetskii has observed [2, page 335] that more capable older pupils are able to generalise from a single example. He also notes an interesting case (p.296) which is worth citing:

> A capable pupil would do a problem of a definite type and in two or three months he would be given a problem of the same type (but not the one he had done earlier) .... Often a "feeling of familiarity" would come to the pupil: he would believe he had done this problem already (not one of the same type, but the same one).

The implications of this observation in terms of resonance and generic thought need not be dwelt upon.

If we pass to the research mathematician, we may find that in a particular case he is far happier with a high level generalisation than with a lower order example containing excessive distracting factors, (for instance he may well prefer linear maps in Banach spaces to 3x3 real matrices), The advocates of an "example–generalisation" hierarchy would say that this high level generalisation is now his 'example', but that simply does not hold water when it is clear that he finds it easier manipulating the curtailed simplicity of the generalisation than the excessive detail of a supposedly lower level example.

What is common to the child and the research mathematician will not be seen through a mathematical classification in terms of "example–generalisation", but through a mental classification: what electrochemical configurations are available in the brain to be able to process the information and to solve the problem at hand.

What is common between the child and the researcher is the nature of brain activity, which may be interpreted as resonances, continuous schemas, instabilities and discontinuities of thought. We may also briefly mention the thesis that in a highly structured subject like mathematics one needs to master subordinate tasks before preceding to higher order tasks, or, in other words, to understand all the parts before preceding to the whole. This arises through equating logical structure with learning hierarchy. Practice tells us this is not always so (see also [9] p.98); indeed a grasp of many of the parts may lead to sufficient resonance to put the other parts into context.

§6 Conclusion

In this paper we have suggested a way in which mathematical thinking may be related at a qualitative level with brain activity in terms of resonance and generic thought. This allows for the strengths and weaknesses of brain activity in a model theoretic sense which is not found in logical or computer-based simulations. The paper is itself an example of generic thought; it yields a possible approximation to a theory with all the consequent weaknesses of mental activity which may need radical correction with the passage of time. Nevertheless, mathematics is an activity of the human brain, so to understand the nature of mathematical thinking we must eventually have an insight into the workings of the brain itself. As yet we lack a complete understanding, but in the meantime a qualitative model in terms of resonance gives an alternative paradigm within which we may be able to perceive some of the subtle qualities of thought which seem to elude us at present.

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