NOTES ON HILBERT SCHEMES

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Introduction

These notes are for lectures on Hilbert schemes in the Summer School on *Introduction to explicit methods in algebraic geometry*, run September 3–7, 2007 at the University of Warwick as part of the Warwick EPSRC Symposium on Algebraic Geometry. They likely contain many errors, both mathematical and typographical; please send any you notice to me at the address `D.Maclagan@warwick.ac.uk`. Many thanks to those who have already done so.

I make no claim for comprehensiveness, and many important areas of this subject are not covered here. Some other references on Hilbert schemes include: *The Geometry of Schemes*, by Eisenbud and Harris [EH00], *Rational Curves on Algebraic Varieties*, by János Kollár [Kol96], the appendix by Iarrobino and Kleiman [IK99] to *Power Sums, Gorenstein Algebras, and Determinantal Loci* by Iarrobino and Kanev, the section on the Hilbert scheme of points in *Combinatorial Commutative Algebra* by Miller and Sturmfels [MS05], the paper *t, q-Catalan numbers and the Hilbert scheme* by Haiman [Hai98], and the paper *Multigraded Hilbert schemes* by Haiman and Sturmfels [HS04]. My treatment has been heavily influenced by these works.

1. Lecture 1: Introduction to the Hilbert scheme

The Hilbert scheme $\mathrm{Hilb}_P(\mathbb{P}^n)$ is a parameter space whose closed points correspond to subschemes of $\mathbb{P}^n$ with Hilbert polynomial P. The topology on $\mathrm{Hilb}_P(\mathbb{P}^n)$ gives a notion of when two subschemes are "close". Many other moduli spaces are constructed by realizing them as subschemes of the Hilbert scheme.

In this lecture we first review the basics of subschemes of $\mathbb{P}^n$ and Hilbert polynomials, then give the functorial definition of the Hilbert scheme.

1.1. **Algebraic preliminaries.** Let $S = \Bbbk[x_0, , \dots, x_n]$, where $\Bbbk$ is a commutative ring, and let its irrelevant ideal be $\mathfrak{m} = \langle x_0, \dots, x_n \rangle$. A homogeneous ideal $I \subseteq S$ not containing $\mathfrak{m}$ determines a closed subscheme of $\mathbb{P}^n_{\Bbbk}$ from the surjection $S \rightarrow S/I$ (see [Har77, Exercises II.2.14, II.3.12]).

In the opposite direction, given a subscheme $X \subset \mathbb{P}^n_{\Bbbk}$, the corresponding ideal sheaf $\mathcal{I}_X$ is the kernel of the map $\mathcal{O}_{\mathbb{P}^n_{\Bbbk}} \rightarrow \mathcal{O}_X$. The direct sum $I = \oplus_{l \geq 0} H^0(\mathbb{P}^n_{\Bbbk}, \mathcal{I}_X(l))$ is then a homogeneous ideal of S, because $S \cong \oplus_{l \geq 0} H^0(\mathbb{P}^n_{\Bbbk}, \mathcal{O}(l))$.

It is important to note that this correspondence between subschemes of $\mathbb{P}^n_{\Bbbk}$ and ideals of S is not a bijection. Essentially this is because the irrelevant ideal $\mathfrak{m}$ does not correspond to a subscheme of $\mathbb{P}^n_{\Bbbk}$. More specifically, two ideals I and J correspond to the same subscheme of $\mathbb{P}^n_{\Bbbk}$ if and only if the saturations of I and J with respect to $\mathfrak{m}$ coincide, so $(I:\mathfrak{m}^\infty) = (J:\mathfrak{m}^\infty)$, where $(I:\mathfrak{m}^\infty) := \langle f \in S : f\mathfrak{m}^k \subseteq I \text{ for some } k > 0\rangle$. Note that if $(I:\mathfrak{m}^\infty) = (J:\mathfrak{m}^\infty)$ then $I_k = J_k$ for $k \gg 0$. Note also that $I \subseteq (I:\mathfrak{m}^\infty)$. An ideal I is called *saturated* if $I = (I:\mathfrak{m}^\infty)$. The saturation of I is the largest ideal corresponding to the same subscheme as I, and there is a one-to-one correspondence between homogeneous saturated ideals of S and subschemes of $\mathbb{P}^n_{\Bbbk}$. Thus to parameterize subschemes of $\mathbb{P}^n_{\Bbbk}$, it suffices to parameterize homogeneous saturated ideals of S.

If $\Bbbk$ is an algebraically closed field, and the ideal I is prime, then the subscheme of $\mathbb{P}^n_{\Bbbk}$ determined by I is the variety $V(I)$ determined by I, which has closed points $\{x \in \mathbb{P}^n_{\Bbbk} : f(x) = 0 \text{ for all } f \in I\}$.

Example 1.1. Let $n = 3$, and let $I = \langle x_0x_3 - x_1x_2, x_0x_2 - x_1^2, x_1x_3 - x_2^2\rangle$. Then I is a prime ideal whose variety is the twisted cubic in $\mathbb{P}^3$. Let $I' = \langle x_2^2x_3 - x_1x_3^2, x_1x_2x_3 - x_0x_3^2, x_1^2x_3 - x_0x_2x_3, x_2^3 - x_0x_3^2, x_1x_2^2 - x_0x_2x_3, x_0x_2^2 - x_0x_1x_3, x_1^2x_2 - x_0x_1x_3, x_0x_1x_2 - x_0^2x_3, x_1^3 - x_0^2x_3, x_0x_1^2 - x_0^2x_2\rangle$. Then $(I' : \mathfrak{m}^\infty) = (I' : \mathfrak{m}^3) = I$, as $I' = I \cap \mathfrak{m}^3$. Thus I and I' determine the same subscheme of $\mathbb{P}^3$.

The *Hilbert polynomial* of a homogeneous ideal of S, or a subscheme of $\mathbb{P}^n_{\Bbbk}$, is an invariant of an ideal/subscheme that will determine the connected components of the Hilbert scheme. For simplicity, we assume that $\Bbbk$ is a field from now on. The Hilbert polynomial is determined from the *Hilbert function* of the ideal. This is the function $H_{S/I} : \mathbb{N} \to \mathbb{N}$ given by

$$H_{S/I}(t) = \dim_{\Bbbk}(S/I)_t,$$

where $(S/I)_t$ is the tth graded piece of the S-module (and thus vector space over $\Bbbk$) S/I. The key fact is that the function $H_{S/I}$ agrees with a polynomial $P_{S/I}$ for large t, so $H_{S/I}(t) = P_{S/I}(t)$ for $t \gg 0$. The polynomial $P_{S/I}(t)$ is called the *Hilbert polynomial* of S/I. If X is the subscheme of $\mathbb{P}^n$ corresponding to I, then $P_{S/I}(t) = \chi(\mathcal{O}_X(t))$.

There are many different proofs of the fact that $P_{S/I}$ exists. The one we give here, while not the shortest, contains some important ideas that we will return to often. The key idea is to first reduce to the case where I is a monomial ideal, and then give a combinatorial proof in the monomial case. This will be a repeated theme. The reduction to the monomial case uses the theory of Gröbner bases, and we now summarize the facts we need. Anyone unfamiliar with Gröbner bases is urged to spend an evening or two with the first few chapters of the classic [CLO07]. More geometric details are found in [Eis95, Chapter 15].

Fix $w \in \mathbb{N}^{n+1}$. For $f = \sum_{u\in\mathbb{N}^{n+1}} c_u x^u$, set the initial term of f to be $\mathrm{in}_w(f) = \sum c_u x^u$, where the sum is over those $u \in \mathbb{N}^{n+1}$ with $w \cdot u$ maximal among those u with $c_u \neq 0$. For example, if $n = 3$, $w = (1,0,0,1)$, and

$f = x_0x_3 - x_1x_2$, then $\mathrm{in}_w(f) = x_0x_3$. The *initial ideal* of I is the ideal $\mathrm{in}_w(I) := \langle \mathrm{in}_w(f) : f \in I\rangle$. We warn that $\mathrm{in}_w(I)$ is almost never generated by the initial terms of a minimal generating set for I. A set $\mathcal{G} = \{g_1, \dots, g_s\}$ of polynomials in I is a *Gröbner basis* for I with respect to w if $\mathrm{in}_w(I) = \langle \mathrm{in}_w(g_1), \dots, \mathrm{in}_w(g_s)\rangle$.

A geometric description of this is as follows. For $f = \sum_{u\in\mathbb{N}^{n+1}} c_u x^u$, set $\tilde{f} = t^{d(f)} \sum_{u\in\mathbb{N}^{n+1}} c_u x^u t^{-w\cdot u}$, where $d(f) = \max_{c_u\neq 0} w \cdot u$. By construction $\tilde{f} \in S[t]$, with at least one term containing no power of t. Set $I_t = \langle \tilde{f} : f \in I\rangle$. Again, I_t is almost never generated by $\{\tilde{f} : f$ is a minimal generator of $I\}$. The $S[t]$-module $S[t]/I_t$ is in fact a flat $\Bbbk[t]$-module. In geometric language this says that

$$\begin{array}{c} \mathrm{Spec}(S[t]/I_t) \subset A^{n+1} \times \mathbb{A}^1 \\ \big\downarrow \\ \mathbb{A}^1 \end{array}$$

is a flat family over $\mathbb{A}^1$. If I is homogeneous, we can replace $\mathrm{Spec}(S[t]/I_t)$ by $\mathrm{Proj}(S[t]/I_t) \subset \mathbb{P}^n \times \mathbb{A}^1$, where t has degree zero. By construction the fiber over $t = 1$ is $\mathrm{Spec}(S/I)$ (or $\mathrm{Proj}(S/I)$), and the fiber over $t = 0$ is $\mathrm{Spec}(S/\mathrm{in}_w(I))$ (respectively $\mathrm{Proj}(S/\mathrm{in}_w(I))$). This says that there is a flat degeneration from an ideal to its initial ideal, and thus from the corresponding affine or projective scheme to the one determined by the initial ideal. Such a degeneration is called a *Gröbner degeneration*.

Even more geometrically, consider the $(\Bbbk^*)^{n+1}$-action on $\mathbb{A}^{n+1}$ given by scaling the coordinates. This action extends to an action on ideals in S or equivalently to subschemes of $\mathbb{A}^{n+1}$. Consider the one-parameter subgroup of $(\Bbbk^*)^{n+1}$ which is the image of $\phi : \Bbbk^* \to (\Bbbk^*)^{n+1}$ where $\phi(t) = (t^{-w_1}, \dots, t^{-w_n})$. Then the subscheme corresponding to $\mathrm{in}_w(I)$ is $\lim_{t\to 0} \phi(t) \cdot X$, where X is the subscheme of $\mathbb{A}^{n+1}$ corresponding to I. When I is homogeneous we can replace affine space by projective space throughout.

For sufficiently generic w the ideal $\mathrm{in}_w(I)$ is generated by monomials. The initial ideal then coincides with one coming from a *monomial term order*. This is a total order on the monomials in S compatible with multiplication. The most common of these are the *lexicographic term order*, where $x^u \prec x^v$ if the first nonzero entry of $v-u$ is positive, or the *reverse lexicographic term order*, where $x^u \prec x^v$ if the degree of x^u is smaller than that of x^v or if the degrees are the same, and the last nonzero entry of $v - u$ is negative. When working with a term order, the initial term of a polynomial is the largest monomial occurring in the polynomial. A term order can be obtained from a vector $w \in \mathbb{N}^{n+1}$ by setting $x^u \prec x^v$ if $w \cdot u < w \cdot v$, breaking ties with the lexicographic order if necessary.

The monomials not lying in $\mathrm{in}_w(I)$, called *standard monomials*, form a vector-space basis for S/I. To see this, note that for every monomial $x^u \in \mathrm{in}_w(I)$ there is a polynomial $x^u - f_u \in I$ with $f_u = \sum_{x^v\notin \mathrm{in}_w(I)} c_v x^v$, which shows that the standard monomials span S/I. If there was a choice of c_v not all zero with $\sum_{v\notin \mathrm{in}_w(I)} c_v x^v = 0$ in S/I, then $g = \sum_{v\notin \mathrm{in}_w(I)} c_v x^v \in I$, so

$\mathrm{in}_w(g) \in \mathrm{in}_w(I)$. From this contradiction we see that the standard monomials are linearly independent in S/I, so form a basis.

Thus when I is homogeneous we have $H_{S/I} = H_{S/\mathrm{in}_w(I)}$. This shows that it suffices to prove the existence of the Hilbert polynomial for monomial ideals. To do this, we first note that the Hilbert function of the polynomial ring S is $H_S(t) = \binom{t+n}{n}$ for $t \geq 0$, which is a polynomial of degree n. This shows the existence of the Hilbert polynomial for polynomial rings. We next show that the standard monomials of a monomial ideal can be partitioned into translates of the monomials in smaller polynomial rings.

Definition 1.2. Let I be a monomial ideal. A Stanley decomposition for S/I is a finite decomposition of the standard monomials of I into disjoint sets of the form $(x^u, \sigma) = \{x^{u+v} : \mathrm{supp}(x^v) \subseteq \sigma\}$, where $\sigma \subseteq \{0, 1, \ldots, n\}$, and $\mathrm{supp}(x^v) = \{i : v_i > 0\}$.

Example 1.3. Let $I = \langle x_0^2 x_1 \rangle \subset \Bbbk[x_0, x_1]$. Then one Stanley decomposition for S/I is $\{(1, \{0\}), (x_1, \{1\}), (x_0 x_1, \{1\})\}$. Another Stanley decomposition is $\{(1, \{1\}), (x_0, \{0\}), (x_0 x_1, \emptyset), (x_0 x_1^2, \{1\})\}$.

Recall that if $f \in S$ and I is an ideal in S then $(I : f) = \{g \in S : gf \in I\}$, and $(I : f^\infty) = \{g \in S : gf^k \in I \text{ for some } k \geq 0\}$.

Lemma 1.4. *Let I be a monomial ideal. Then a Stanley decomposition for S/I exists.*

Proof. Let $k_i = \min\{k : (I : x_i^k) = (I : x_i^\infty)\}$, and let $k = \sum_{i=0}^n k_i$. The proof is by induction on n and k. When $n = 0$ we must have $I = \langle x_0^l \rangle$ for some l. Then $\cup_{j=0}^{l-1} (x_0^j, \emptyset)$ is a Stanley decomposition for S/I. If $k = 0$ then $I = P_\sigma = \langle x_i : i \notin \sigma \rangle$ is a monomial prime ideal, and $\{(1, \sigma)\}$ is a Stanley decomposition for S/I.

Consider the short exact sequence

$$0 \to S/(I : x_i) \to S/I \to S/(I, x_i) \to 0.$$

Note that $S/(I, x_i)$ is isomorphic to the quotient of a smaller polynomial ring, missing x_i, by a monomial ideal, so by induction a Stanley decomposition $\{(x^{u_j}, \sigma_j) : 1 \leq j \leq s\}$ for $S/(I, x_i)$ exists. Also, the invariant k_i for $(I : x_i)$ is smaller than that for I, while all other k_j are no larger, so again by induction a Stanley decomposition $\{(x^{v_j}, \tau_j) : 1 \leq j \leq t\}$ for $S/(I : x_i)$ exists. Then $\{(x^{u_j}, \sigma_j) : 1 \leq j \leq s\} \cup \{(x_i x^{v_j}, \tau_j) : 1 \leq j \leq t\}$ is a Stanley decomposition for S/I. $\square$

If $\{(x^{u_j}, \sigma_j) : 1 \leq j \leq s\}$ is a Stanley decomposition for S/I, then

$$H_{S/I}(t) = \sum_{j=1}^{s} H_{S_{\sigma_j}}(t - |u_j|),$$

where S_{σ_j} is the polynomial ring $\Bbbk[x_i : i \in \sigma_j]$, and $|u_j| = \sum_{i=0}^n (u_j)_i$. The fact that the Hilbert function of S/I eventually agrees with a polynomial thus follows from the fact that the Hilbert function of a polynomial ring agrees with a polynomial for nonnegative values.

Example 1.5. Let $S = \Bbbk[x_0, x_1, x_2, x_3]$, and let $I = \langle x_0x_3 - x_1x_2, x_0x_2 - x_1^2, x_1x_3 - x_2^2\rangle$. To compute the Hilbert polynomial of S/I, we first compute a Gröbner basis for I and thus compute an initial ideal. For $w = (1, 0, 0, 1)$ the given generating set is a Gröbner basis, so $J = \mathrm{in}_w(I) = \langle x_0x_3, x_0x_2, x_1x_3\rangle$. A Stanley decomposition for S/J is $\{(1, \{2, 3\}), (x_1, \{1, 2\}), (x_0, \{0, 1\})\}$, so for $t \geq 1$ we have $H_{S/I}(t) = H_{\Bbbk[x_2,x_3]}(t) + H_{\Bbbk[x_1,x_2]}(t-1) + H_{\Bbbk[x_0,x_1]}(t-1) = t + 1 + t + t = 3t + 1$. Thus $P_{S/I}(t) = 3t + 1$.

1.2. **Functorial definition.** It is natural to worry that there might be many ways of constructing a scheme (or variety) whose closed points correspond to subschemes of $\mathbb{P}^n$. This is taken care of by requiring that the Hilbert scheme be a *fine moduli space*, which thus carries a universal bundle.

To define a fine moduli space, we need the notions of a *representable functor* and of the *functor of points* of a scheme. The key idea is that a scheme is determined by its morphisms to other schemes.

Definition 1.6. Let X be a scheme. The functor h_X from the opposite of the category of schemes to the category of sets is given by

$$h_X(Y) = \mathrm{Mor}(Y, X),$$

and if $f : Y \to Z$ is a morphism of schemes, then

$$h_X(f) : \mathrm{Mor}(Z, X) \to \mathrm{Mor}(Y, X)$$

is the induced map of sets. The functor h_X is the *functor of points* of the scheme X.

Note that if $Y = \mathrm{Spec}(\Bbbk)$ for $\Bbbk$ a field then $h_X(Y)$ is the set of $\Bbbk$-valued points of X.

Definition 1.7. A functor $F : (\text{schemes})^\circ \to \text{sets}$ is *representable* if $F \cong h_X$ for some scheme X. The scheme X is unique if it exists. This follows from the categorical result known as Yoneda's lemma.

Lemma 1.8 (Yoneda's Lemma)**.** *Let $\mathcal{C}$ be a category and let X, X' be objects of $\mathcal{C}$.*

1. *If F is any contravariant functor from $\mathcal{C}$ to the category of sets, the natural transformations from* $\mathrm{Mor}(-, X)$ *to F are in natural correspondence with the elements of $F(X)$.*
2. *If the functors* $\mathrm{Mor}(-, X)$ *and* $\mathrm{Mor}(-, X')$ *from $\mathcal{C}$ to the category of sets are isomorphic, then $X \cong X'$. More generally, the maps of functors from* $\mathrm{Mor}(-, X)$ *to* $\mathrm{Mor}(-, X')$ *are the same as maps from X to X'; that is the functor*

$$h : \mathcal{C} \to \mathit{Fun}(\mathcal{C}^\circ, (\mathit{sets}))$$

sending X to h_X is an equivalence of $\mathcal{C}$ with a full subcategory of the category of functors.

The second part of this lemma immediately proves that if F is representable then the scheme representing it is unique.

The functor F is in fact determined by its values on affine schemes.

Lemma 1.9. *If R is a commutative ring, then a scheme over R is determined by the restriction of its functor of points to affine schemes over R. Specifically,*

$$h : (R\text{-}schemes) \to Fun((R\text{-}algebras), (sets))$$

is an equivalence of the category of R-schemes with a full subcategory of the category of functors.

Example 1.10. The Grassmann functor $G(d, n)$ takes a ring R to the set of rank d direct summands of R^n. This functor is represented by a scheme $G(d, n)$. One way to show this is to define $G(d, n)$ to be $\mathrm{Proj}(\mathbb{Z}[x_I]/I_{d,n})$, where the variables x_I are indexed by the $\binom{n}{d}$ sets of size d subsets of $[n] := \{1, \dots, n\}$, and $I_{d,n}$ is the ideal generated by the *Plücker relations*. See, for example, [EH00, Section III.2.7 and Exercise VI.18]. By looking at local charts one can show that this scheme does represent the functor $G(d, n)$. Note that if $\Bbbk$ is a field, then the set of $\Bbbk$-valued points of $G(d, n)$ is the set of closed points of the familiar Grassmannian of d-planes in affine n-space.

A major reason to work with representable functors is that the functorial language makes many proofs easier. Geometrically, this means that corresponding scheme is a *fine moduli space* for the moduli problem. Specifically, if F is a moduli functor, for example taking a scheme B to the set of families over B with all fibers having a prescribed form, a scheme X representing F is called a fine moduli space for this moduli problem.

Let Ψ be the isomorphism from F to h_X. Then a family over B with appropriate fibers is taken by Ψ to a morphism from B to X, so each such family gives a map to X. Conversely, let 1_X be the identity morphism from X to itself. This is taken by Ψ^{-1} to a family $U \to X$ whose fibers all have the prescribed property. The scheme U is called the *universal family* over X. If $G \to B$ is a family over B with appropriate fibers, then we can pull back the family U over X by the induced map of $\pi : B \to X$. The uniqueness implies that $U \times_X B \cong G$:

$$\begin{tikzcd} G \cong U \times_X B \arrow[r] \arrow[d] & U \arrow[d] \\ B \arrow[r] & X \end{tikzcd} .$$

Remark 1.11. We are lucky that the Hilbert functor, defined below, is representable. There are many naturally occurring moduli functors that are not representable. A prominent example of this is the moduli problem of parameterizing all curves of genus g, which does not have a fine moduli space. One partial solution is to ask for the existence of a *coarse moduli space*. A scheme Y is a coarse moduli space for a moduli functor F if there is a natural transformation Ψ_Y from F to $\mathrm{Mor}(-, Y)$ with the property that the map $\Psi_{\mathrm{Spec}(\mathbb{C})}$ from $F(\mathrm{Spec}(\mathbb{C}))$ to the $\mathbb{C}$-valued points of Y is a bijection of sets, and given another scheme Y' and a natural transformation $\Psi_{Y'} : F \to \mathrm{Mor}(-, Y')$, there is a unique morphism $\pi : Y \to Y'$ such that the induced map $\Pi : \mathrm{Mor}(-, Y) \to \mathrm{Mor}(-, Y')$ satisfies $\Psi_{Y'} = \Pi \circ \Psi_Y$. Such a Y is then unique up to canonical isomorphism. The disadvantage, though, is that we do not get a nice universal family as for a fine moduli space. Another possibility, beyond the reach of these notes, is use a stack description.

We now describe the moduli problem defining the Hilbert scheme.

Definition 1.12. Fix a base scheme S. The Hilbert functor is the functor $h_P : (\text{schemes})^\circ \to (\text{sets})$ that associates to any scheme B over S the set of subschemes $Y \subseteq \mathbb{P}^n_B$ flat over B whose fibers over points of B have Hilbert polynomial P.

We will assume here for simplicity that $S = \mathrm{Spec}(\Bbbk)$ for $\Bbbk$ a field. Taking $S = \mathrm{Spec}(\mathbb{Z})$ allows great generality.

Theorem 1.13. *There is a scheme* $\mathrm{Hilb}_P(\mathbb{P}^n)$ *that represents* h_P.

We sketch the proof in the next lecture.

Remark 1.14. One can also consider the Hilbert scheme $\mathrm{Hilb}(X)$ where X is a projective scheme. Loosely, one embeds X into some projective space, and constructs $\mathrm{Hilb}(X)$ as a subscheme of $\mathrm{Hilb}(\mathbb{P}^N)$. One then shows that this construction also represents some functor, thus showing that it is independent of the choice of embedding into projective space. In Lecture 4 we will consider the Hilbert scheme of points in affine space, which is an analogous construction.

1.3. **Exercises 1.** The following are more exercises than any of you would want to do this week. So before beginning, look at your notes and decide which aspect you would like to understand better. Then read through *all* the exercises, before choosing which one to start with. Hopefully there is something for everyone!

(1) Let $S = \Bbbk[x_0, x_1, x_2, x_3]$, where $\Bbbk$ is a field. Compute the saturation of $I = \langle x_3^2, x_2x_3, x_1x_3, x_0x_3, x_1x_2, x_0^3\rangle$.
(2) (a) Show that $(I : \mathfrak{m}^\infty) = (I : \mathfrak{m}^k)$ for some $k > 0$, and that if $(I : \mathfrak{m}^k) = (I : \mathfrak{m}^{k+1})$ for some $k > 0$, then $(I : \mathfrak{m}^k) = (I : \mathfrak{m}^\infty)$. Here $(I : \mathfrak{m}^k) = \{f \in S : fg \in I \text{ for all } g \in \mathfrak{m}^k\}$.
    (b) Show that $(I : \mathfrak{m}^\infty) = \cap_{i=0}^n (I : x_i^\infty)$.
    (c) (For those who know more about Gröbner bases) Show that a generating set for $(I : x_i^\infty)$ is obtained by computing a Gröbner basis for I with respect to the reverse lexicographic order with x_i smallest, and then dividing out any power of x_i dividing an element. This explains how saturation can be computed in a computer algebra system.
(3) Show that the Hilbert polynomial of $\mathbb{P}^n$ is $P(t) = \binom{n+t}{n}$.
(4) Let $S = \Bbbk[x_0, x_1, x_2]$. Compute the Hilbert polynomial of S/I for the following ideals.
    (a) $I = \langle x_0^2, x_1x_2\rangle$,
    (b) $I = \langle x_0^3, x_0^2x_1^3, x_0x_1x_2, x_0x_2^4, x_1^4x_2^3, x_1^6, x_2^7\rangle$,

(c) $I = \langle x_1^2 - x_0x_2 \rangle$,
(d) $I = \langle 3x_0x_1 - 2x_1^2 - 3x_0x_2 + x_1x_2 + x_2^2, 9x_0^2 - 4x_1^2 - 18x_0x_2 + 8x_1x_2 + 5x_2^2, x_1^3 - 3x_1x_2^2 + 2x_2^3 \rangle$. (Hint: You'll probably want to use a computer algebra package).

(5) Show that I and $(I : \mathfrak{m}^\infty)$ have the same Hilbert polynomial.

(6) In this exercise we sketch a more straight-forward proof of the existence of the Hilbert polynomial, assuming the existence of finite free resolutions. We can extend the definition of the Hilbert function to arbitrary S-modules by setting $H_M(t) = \dim_{\Bbbk} M_t$.
(a) Show that the Hilbert function is additive on short exact sequences, in the sense that if
$$0 \to M' \to M \to M'' \to 0$$
is a short exact sequence of S-modules, then $H_M = H_{M'} + H_{M''}$.
(b) The S-module $S[a]$ is the polynomial ring S with the grading shifted, so $S[a]_t = S_{a+t}$. Thus the 1 of $S[a]$ has degree $-a$. Show that the Hilbert function of $S[a]$ agrees with a polynomial for $t \gg 0$.
(c) Deduce the existence of the Hilbert polynomial for any finitely generated S-module from the existence of a finite free resolution
$$0 \to \oplus_j S[-b_{nj}]^{\beta_{nj}} \to \cdots \to S[-b_{1j}]^{\beta_{1j}} \to S[-b_{0j}]^{\beta_{0j}} \to M \to 0.$$

(7) Prove Yoneda's lemma

(8) Give a direct argument that $h_{\mathrm{Spec}(\mathbb{Z})}$ is not isomorphic to any functor of points of a different scheme.

(9) Flesh out the details of the construction of the Grassmannian. What are the Plücker relations? Why does the given scheme represent the Grassmann functor?

2. Lecture 2: Construction

In this lecture we outline the construction of the Hilbert scheme. The proof comes in two parts. First, one constructs a scheme X whose closed points correspond to subschemes of $\mathbb{P}^n$. This is essentially combinatorial commutative algebra. One then shows that X satisfies the desired universal property, and thus represents the functor h_P. We will only give details on the first of these steps.

The scheme X is constructed as a subscheme of a Grassmannian. The key to the construction of the Hilbert scheme is the fact that there is a uniform degree $D = D(P)$ for which all ideals $I \subseteq S$ of Hilbert polynomial P are generated in degree at most D. This follows from Gotzmann's regularity theorem, which uses the notion of Castelnuovo-Mumford regularity. A good reference for the commutative algebra from this lecture is [BH93]. Recall that for a homogeneous ideal the degrees of the minimal generators, and also the degrees of minimal generators for higher syzygy modules, is well-defined. See [Eis95, Chapter 20] for details.

Definition 2.1. Given a homogeneous ideal $I \subseteq S$, we say S/I is k-regular if

$$H^i_{\mathfrak{m}}(S/I)_j = 0 \text{ for } j + i > k,$$

where $H^i_{\mathfrak{m}}$ is the ith local cohomology functor, or equivalently if $H^i_{\mathfrak{m}}(S/I)_{k-i+1} = 0$ for $i \geq 0$. Equivalently (in the case that I is saturated), in sheaf-theoretic language, $\mathcal{I}$ is k-regular if

$$H^i(\mathbb{P}^n, \mathcal{I}(k-i)) = 0,$$

for $i > 0$. By [EG84] we have the following reformulation in terms of free resolutions. Let

$$0 \to \oplus_i S[-\beta_{li}] \to \cdots \to \oplus_i S[-\beta_{1i}] \to S \to S/I \to 0$$

be the minimal free resolution of S/I, where $S[-\beta]$ is the polynomial ring S with the grading shifted so that 1 has degree β. Then S/I is k-regular if $k \geq \max_{ij}(\beta_{ij} - i)$.

Remark 2.2. Note that the β_{1i} are the degrees of the minimal generators of I, so in particular if S/I is k-regular, then I is generated in degree at most $k+1$. Also, if S/I is k-regular, then $H_{S/I}(k) = P_{S/I}(k)$. This follows from, for example, [BH93, Theorem 4.4.3], or the proof of the existence of the Hilbert polynomial using free resolutions given in Exercise 6.

There is a uniform bound on the regularity of all ideals with a given Hilbert polynomial, which we now present. We first detour with a definition that secretly motivates the following theorem, and will be useful in the following lecture.

Definition 2.3. The lexicographic, or dictionary, order on the monomials of degree d is defined by setting $x^u \prec_{lex} x^v$ if the first nonzero entry of $v - u$ is positive. A monomial ideal I is lexicographic if whenever $x^u \in I$ and $\deg(x^u) = \deg(x^v)$ with $x^u \prec_{lex} x^v$ we have $x^v \in I$.

The following proposition is essentially due to Macaulay.

Proposition 2.4. *There is exactly one lexicographic ideal with a given Hilbert function. There is exactly one saturated lexicographic ideal with a given Hilbert polynomial.*

Theorem 2.5. *Let P be a Hilbert polynomial, and write*

$$P(t) = \sum_{j=1}^{D} \binom{t + a_i - i + 1}{a_i},$$

where $a_1 \geq a_2 \geq \cdots \geq a_D \geq 0$. Then if I is a saturated ideal with Hilbert polynomial $P_{S/I}(t) = P$ then S/I is $D-1$-regular.

For a concise proof of Gotzmann's regularity theorem, see [BH93, Theorem 4.3.2]. See also [MS05, Theorem 5.2] for a proof that generalizes to the multigraded case. The number D is called the *Gotzmann number* of P.

Theorem 2.5 means that every saturated ideal with Hilbert polynomial P is generated in degree at most D. Note that $(I_{\geq D} : \mathfrak{m}^\infty) = (I : \mathfrak{m}^\infty)$, and if I is generated in degree at most D then $I_{\geq D}$ is generated in degree D. Thus we can consider ideals generated in degree D. Let $\mathcal{G}_D$ be the Grassmannian $\mathrm{Gr}(\binom{n+D}{n} - P(D), S_D)$. Saturated ideals I with Hilbert polynomial P correspond to closed points in $\mathcal{G}$, where an ideal I corresponds to the $\Bbbk$-subspace I_D of S_D. We now show that the closed points in $\mathrm{Gr}(N - P(D), N)$ corresponding to such ideals are the closed points of a subscheme $\mathcal{H}$ of $\mathcal{G}$.

This relies on Gotzmann's persistence theorem, which gives a criterion for the Hilbert function of an ideal to agree with its Hilbert polynomial. This relies on a curious numerical function from $\mathbb{N}$ to $\mathbb{N}$ depending on a parameter $d \in \mathbb{N}$.

Definition 2.6. Given $n, d \in \mathbb{N}$, we can write n uniquely as

$$n = \sum_{j=0}^{t} \binom{k_j}{d-j},$$

where $k_j > k_{j+1} \geq 0$. The Macaulay upper boundary of n with respect to d is then

$$n^{\langle d \rangle} = \sum_{j=0}^{t} \binom{k_j + 1}{d - j + 1}.$$

We note that this is very closely related to the description of the Hilbert polynomial in Theorem 2.5.

Theorem 2.7. *Let $k \in \mathbb{N}$ be such that all minimal generators of I are in degrees less than k. If $H_{S/I}(k+1) = H_{S/I}(k)^{\langle k \rangle}$, then $H_{S/I}(t+1) = H_{S/I}(t)^{\langle t \rangle}$ for all $t \geq k$.*

Note that in particular that the Hilbert polynomial satisfies $P_{S/I}(t+1) = P_{S/I}(t)^{\langle t \rangle}$ for all $t \gg 0$, so a particular corollary of Theorem 2.7 is that if an ideal J is generated in degrees at most D, and if $H_{S/J}(D) = P(D)$ and $H_{S/J}(D+1) = P(D+1)$ then $P_{S/J} = P$.

Let D be the Gotzmann number of P, and let $\mathcal{H}_P$ be the scheme $\{(L, M) \in \mathcal{G}_D \times \mathcal{G}_{D+1} : x_i L \subseteq M \text{ for all } i\}$ with the natural induced closed subscheme structure on $\mathcal{G}_D \times \mathcal{G}_{D+1}$.

Theorem 2.8. *The scheme $\mathcal{H}_P$ represents the functor h_P.*

Remark 2.9. One can also describe $\mathrm{Hilb}_P(\mathbb{P}^n)$ directly as a subscheme of $\mathcal{G}_D$, by writing $S_1 L \subset S_{D+1}$ in terms of the coordinates on $\mathcal{G}_D$, and demanding that $\dim_{\Bbbk}(S_1 L) \leq \binom{n+D+1}{n} - P(D+1)$, which gives determinantal equations. This works because Macaulay's theorem guarantees the reverse inequality, so such L must actually have $\dim_{\Bbbk}(S_1 L) = \binom{n+D}{n} - P(D+1)$. See also [HS04] and [DB82].

2.1. **Exercises 2.**

(1) Compute the Gotzmann number for the Hilbert polynomial of $I = \langle x_0^2, x_1 x_2 \rangle \subset \Bbbk[x_0, x_1, x_2]$. Repeat for the Hilbert polynomial of $I = \langle x_0 x_3, x_0 x_2, x_1 x_2 \rangle \subset \Bbbk[x_0, x_1, x_2, x_3]$.

(2) Let $I = \langle x_0x_3 - x_1x_2, x_0x_2 - x_1^2, x_1x_3 - x_2^2\rangle$. Compute saturated lexicographic ideal with the same Hilbert polynomial as I. Verify that it is generated in degrees at most the Gotzmann number of $P_{S/I}$.
(3) Let $S = \Bbbk[x_0, x_1, x_2]$, and let $P(t) = 2$. Write down equations describing $\mathrm{Hilb}_P(\mathbb{P}^2)$.
(4) Describe the equations for the Hilbert scheme $\mathrm{Hilb}_{3t+1}(\mathbb{P}^3)$.
(5) Assuming Macaulay's theorem (Proposition 2.4), show that every Hilbert polynomial can be written in the form of Theorem 2.5. Hint: What do Stanley decompositions of lexicographic ideals look like? Show that such a decomposition is unique. Hint: This is a purely numerical property. Fix a large t, and look at the corresponding decomposition of $P(t)$. Can you identify a_1 in this case?
(6) Show that there is no bound on the regularity of S/I with Hilbert polynomial P if I is not assumed to be saturated.
(7) This question outlines a proof of Theorem 2.5.
   (a) Note first that it suffices to show that the bound given in Theorem 2.5 bounds the regularity of saturated *monomial* ideals with Hilbert polynomial P. (Hint: Gröbner degeneration and upper semicontinuity - skip this part if you don't know these words).
   (b) A *Stanley filtration* for S/I, where I is a monomial ideal, is a Stanley decomposition $\{(x^{u_i}, \sigma_i) : 1 \leq i \leq s\}$ with the extra property that $\{(x^{u_i}, \sigma_i) : 1 \leq i \leq j\}$ is a Stanley decomposition for $S/(I, x^{u_{j+1}}, \dots, x^{u_s})$ for $1 \leq j \leq s$. Show that if I is a monomial ideal then a Stanley filtration for S/I exists.
   (c) What can you say about regularity in short exact sequences? Deduce that if $\{(x^{u_i}, \sigma_i) : 1 \leq i \leq s\}$ is a Stanley filtration for S/I, where I is a saturated ideal, then $\mathrm{reg}(S/I) \leq \max_i |u_i|$, where the sum is over all i with $|\sigma_i| > 0$.
   (d) Show that there is always a Stanley decomposition $\{(x^{u_i}, \sigma_i) : 1 \leq i \leq s\}$ for S/I with $|u_i| \leq i - 1$, and $\max\{i : \sigma_i \neq \emptyset\} \leq D$. Conclude Gotzmann's theorem.
   (e) It is an open question whether the upper bound on the regularity of S/I given in (7c) is ever not sharp for some Stanley filtration, and (weaker) whether there always a Stanley decomposition for which the maximum $|u_i|$ is at most the regularity.

3. Lecture 3: Connectedness and Pathologies

3.1. **Connectedness.** Little is known about the global structure of the Hilbert scheme. The one uniform fact that is known is Hartshorne's theorem that the Hilbert scheme is always connected. We now outline the proof of this result. The key idea of the proof is to first reduce to showing that all monomial ideals in $\mathrm{Hilb}_P(\mathbb{P}^n)$ live on the same connected component. This can be done, for example, by using a *Gröbner degeneration*. This reduces connectedness to a more combinatorial problem, as monomial ideals are essentially combinatorial objects, and there are only a finite number of monomial ideals in the Hilbert scheme. We make repeated use of the elementary topological fact that if $f : X \to Y$ is a map from an irreducible variety to a scheme Y, and $x_1, x_2 \in X$, then $f(x_1)$ and $f(x_2)$ live on the same connected component of Y. In particular, we construct a sequence of maps from $\mathbb{A}^1$ to $\mathrm{Hilb}_P(\mathbb{P}^n)$ sending 0 to one monomial ideal and 1 to another. In this fashion we can walk between monomial ideals staying within the same connected component of the Hilbert scheme. The connectedness is proven by showing that we can walk from any monomial ideal to the lex-segment ideal.

Remark 3.1. A curious aspect of this proof is that one does not need to know the construction of the Hilbert scheme to show that it is connected. Indeed, this can be thought of as a relative result "if the Hilbert scheme exists, it must be connected". To quote Hartshorne [Har66], "It also appears that the Hilbert scheme is never actually needed in the proof".

We reduce the size of the resulting combinatorial problem of dealing with all monomial ideals by restricting to the smaller set of *Borel-fixed* ideals.

Definition 3.2. Let B be the Borel-subgroup of $\mathrm{GL}(n+1, \Bbbk)$ consisting of upper-triangular matrices. The group B acts on S by sending $b \cdot x_i = \sum_{j=0}^{n} b_{ji} x_j$. Since the action preserves the grading of the ring, we get an induced action on the set of homogeneous ideals, and thus on $\mathrm{Hilb}_P(\mathbb{P}^m)$. An ideal I is *Borel-fixed* if $b \cdot I = I$ for all $b \in B$.

Since the torus $(\Bbbk^*)^n$ is a subgroup of B, it is straightforward to check that a Borel-fixed ideal must be monomial. We have the following characterization (see [Eis95, Theorem 15.23]) of Borel-fixed ideals when the characteristic of the field $\Bbbk$ is zero. A slightly more complicated result holds when the characteristic is positive.

Proposition 3.3. *An ideal I is Borel-fixed if and only if*

(1) *I is monomial, and*
(2) *whenever $x^u \in I$, $i < j$, and x_j divides x^u, we have $x_i x^u / x_j \in I$.*

We now show that every ideal lives in the same connected component as a Borel-fixed ideal. This uses the notion of a *generic initial ideal.*

Definition/Proposition 3.4. Fix a generic vector $w \in \mathbb{N}^{n+1}$. There is a Zariski-open set $U \subseteq \mathrm{GL}(n+1, \Bbbk)$ and a monomial ideal J for which $\mathrm{in}_w(g \cdot I) = J$ for all $g \in U$. The ideal J is called the *generic initial ideal* of I with respect to w, and is written $J = \mathrm{gin}_w(I)$. The generic initial ideal can also be defined with respect to a monomial term order $\prec$. The most commonly used here is the reverse lexicographic term order (revlex).

Proposition 3.5. *Every ideal I with Hilbert polynomial P lives on the same connected component of* $\mathrm{Hilb}_P(\mathbb{P}^n)$ *as a Borel-fixed ideal.*

Proof. By [BS87] the revlex gin is Borel-fixed, so it suffices to check that I lives on the same connected component as each of its gins. Fix the reverse lexicographic term order $\prec$ and let $g \in \mathrm{GL}(n+1, \Bbbk)$ lie in the Zariski open set U for computing the revlex gin. Pick a map ψ from $\mathbb{A}^1$ to $\mathrm{GL}(n+1, \Bbbk)$ that has g and the identity in its image. This shows that I and gI lie in the same connected component. Then $\mathrm{in}_{\prec}(gI)$ lies in the same irreducible component as gI by the Gröbner degeneration, so I lies in the same connected component as the Borel-fixed ideal $\mathrm{gin}_{\prec}(I)$. □

This proof can be refined to show that I actually lives in the same irreducible component as a Borel-fixed ideal. Proposition 3.5 reduces the problem of showing connectedness to showing that all Borel-fixed ideals live on the same connected component, by showing we can "walk" from each one to the lex-segment ideal.

We now sketch a proof of the fact that the Hilbert scheme is connected. We follow the paper of Alyson Reeves [Ree95], who analyzed Hartshorne's approach to prove the stronger result that the radius of the component-graph of the Hilbert schemes is at most $d+1$, where d is the degree of the Hilbert polynomial (and thus the dimension of the subschemes of $\mathbb{P}^n_{\Bbbk}$ being parameterized). We restrict here the base field $\Bbbk$ being algebraically closed of characteristic zero.

Definition 3.6. The component graph of the Hilbert scheme $\mathrm{Hilb}_P(\mathbb{P}^n)$ has vertices the irreducible components of $\mathrm{Hilb}_P(\mathbb{P}^n)$, and an edge connecting two vertices if and only if the corresponding components intersect. The radius of the Hilbert scheme is the minimum over all vertices v of the component graph of the maximum distance from v to any other vertex.

By a result of Reeves and Stillman [RS97] the saturated lexicographic ideal is a smooth point of $\mathrm{Hilb}_P(\mathbb{P}^n)$, so lies on exactly one irreducible component, which we call the *lexicographic component*.

Theorem 3.7. [Ree95] *Let d be the degree of the Hilbert polynomial P. Then the distance from any vertex to the vertex of the lexicographic component is at most $d+1$, so the radius of the Hilbert scheme $\mathrm{Hilb}_P(\mathbb{P}^n)$ is at most $d+1$.*

The key idea of the proof is to do a construction known as the *distraction* which takes a Borel-fixed ideal to another Borel-fixed ideal that is closer to the lexicographic ideal.

Definition 3.8. Let $I \subset S$ be a monomial ideal. The *polarization* of I is the following monomial ideal $p(I)$ in the polynomial ring $\Bbbk[z_{ij} : 0 \leq i \leq n, j \geq 0]$ in infinitely many variables:

$$p(I) = \langle \prod_{i=0}^{n} \prod_{j=1}^{u_i} z_{ij} : x^u \text{ is a minimal generator of } I \rangle.$$

Note that $p(I)$ is a squarefree monomial ideal, and thus radical.

Define $\sigma : \Bbbk[z_{ij}] \to S$ by $\sigma(z_{ij}) = x_i - \alpha_{ij} x_n$, where $\alpha_{ij} \in \Bbbk$. The *distraction* of I is then $\sigma(p(I))$.

Lemma 3.9. *For sufficiently generic choice of α_{ij} the distraction $\sigma(p(I))$ has the same Hilbert function as I. In fact, there is a Gröbner degeneration from $\sigma(p(I))$ to I.*

We note that the second sentence of the lemma follows from the first, as it is straightforward to observe that the lexicographic initial ideal of $\sigma(p(I))$ contains I, so if they have the same Hilbert function they must be equal.

The plan to show connectedness is then to start with an arbitrary ideal, take the gin, take the distraction of the gin with a sufficiently general choice of α_{ij}, take its revlex gin, and then repeat, taking distractions and then revlex gins. All ideals obtained in this fashion are clearly in the same connected component of the Hilbert scheme, so it suffices to check that after a finite number of steps we obtain the lexicographic ideal. This finite number will be bounded by $d+1$, proving Theorem 3.7.

This is accomplished by analyzing the irreducible components of $\sigma(p(I)$. For sufficiently generic α_{ij} these are linear subspaces of $\mathbb{P}^n$.

Definition 3.10. An irreducible component of $\sigma(p(I))$ is in *lexicographic position* if it is an irreducible component of $\sigma(p(L))$, where L is the lexicographic ideal with the same Hilbert polynomial as I.

To prove the $d+1$ bound, the key of Reeves' argument is:

Proposition 3.11. *Let I be a saturated Borel-fixed ideal such that all irreducible components of $\sigma(p(I))$ of dimension at least $i+1$ are in lexicographic position. Let J be the saturation of $\mathrm{gin}_{revlex}(\sigma(p(I)))$. Then $\sigma(p(J))$ has all irreducible components of dimension at least i in lexicographic position.*

This proves the radius bound, since any saturated Borel-fixed ideal with Hilbert polynomial P has all components of $\sigma(p(I))$ of dimension at most d, so trivially satisfies the hypotheses of the proposition for $i=d$. Thus after $d+1$ iterations of the $(\mathrm{gin}_{revlex}(\sigma(p(I))) : \mathfrak{m}^\infty)$ procedure we have an ideal whose distraction has all components in lexicographic position. This means its distraction equals the distraction of the lexicographic ideal (as containment would imply a smaller Hilbert polynomial), and thus that the ideal equals the lexicographic ideal.

Remark 3.12. We note that there are many other proof of this result. Reeves' proof outlined above assumes that the base scheme is a field of characteristic zero. The characteristic assumption was removed in the Keith Pardue's thesis [KP94]. A substantially different proof with the characteristic assumption, appears in the work of Peeva and Stillman [PS05], which is also related to the work of Daniel Mall [Mal00]. See [Fum05] for extensions of Mall's work.

3.2. **Pathologies.** It is somewhat expected that connectedness is the only positive geometric property to be shared by all Hilbert schemes. This belief is expressed in the book [HM98] by the following law.

Law 3.13 (Murphy's Law for Hilbert schemes [HM98])**.** *There is no geometric possibility so horrible that it cannot be found generically on some component of some Hilbert scheme.*

An early piece of evidence for this is the following result of Mumford [Mum62].

Theorem 3.14. *Let $P(t) = 14t - 23$. Then $\mathrm{Hilb}_P(\mathbb{P}^3_k)$ has an irreducible component that is nonreduced. This component parameterizes curves of degree 14 and genus 24 contained in a cubic surface that are linearly equivalent in S to $4H + 2L$ where H is a hyperplane section and L is a line in S.*

The Law has been made precise for singularities by the following result of Ravi Vakil. Define an equivalence relation on pointed schemes generated by: If $(X, p) \to (Y, q)$ is a smooth morphism, then $(X, p) \sim (Y, q)$. We call the equivalence classes *singularity types*, and will call pointed schemes *singularities*.

Theorem 3.15. [Vak06] *Every singularity of finite type over $\mathbb{Z}$ appears in some Hilbert scheme. This can be taken to be a Hilbert scheme of surfaces in $\mathbb{P}^4$.*

Vakil's result extends to many other moduli spaces, such as stable maps to projective space and the Chow variety, showing that this bad behaviour is almost ubiquitous. The key idea of the proof is again combinatorial, and we outline it here.

An *incidence scheme of points and lines in* $\mathbb{P}^2$ is a locally closed subscheme of $(\mathbb{P}^2)^m \times (\mathbb{P}^{2*})^n = \{p_1, \dots, p_m, l_1, \dots, l_n\}$ parameterizing $m \geq 4$ marked point and n marked lines as follows.

(1) $p_1 = [1 : 0 : 0], p_2 = [0 : 1 : 0], p_3 = [0 : 0 : 1], p_4 = [1 : 1 : 1]$.
(2) For each pair (p_i, l_j) either p_i is required to lie on l_j, or p_i is required not to lie on l_j.
(3) The marked points p_i are required to be distinct, and the marked lines are required to be distinct.
(4) Given any two marked lines, there is a marked point required to be on both of them.
(5) Each marked line contains at least three marked points.

Theorem 3.16 (Mnëv's Universality Theorem)**.** *Every singularity type of finite type over $\mathbb{Z}$ appears on the some incidence scheme.*

We note that this theorem is constructive, so given a singularity type we can construct an incidence scheme with that singularity type. The idea of the proof of Theorem 3.15 is then to fix a particular type of singularity over $\mathbb{Z}$, and construct a smooth morphism from the incidence scheme containing that singularity to a particular Hilbert scheme.

3.3. **Exercises 3.**

(1) In this exercise you will construct two Hilbert schemes that "do what you expect", unlike the pathologies we discussed.

   (a) Let $P(t) = t+1$ be the Hilbert polynomial of a line. What is $\mathrm{Hilb}_P(\mathbb{P}^3)$?
   (b) Let $P(t) = 2t+1$ be the Hilbert polynomial of a conic. What is $\mathrm{Hilb}_P(\mathbb{P}^2)$?
   (c) Explain geometrically why you expect these answers.

(2) Let $P(t) = d$ be a constant polynomial. List the saturated monomial ideals in $\mathrm{Hilb}_P(\mathbb{P}^1)$. Which of these are Borel-fixed? What does this tell you about $\mathrm{Hilb}_P(\mathbb{P}^1)$? What is $\mathrm{Hilb}_P(\mathbb{P}^1)$?

(3) Show that an ideal I is fixed by the $T \cong (\Bbbk^*)^n$ action on $\mathrm{Hilb}_P(\mathbb{P}^n)$ if and only if I is monomial. Conclude that Borel-fixed ideals are monomial. Finish the proof of the "only if" direction of Proposition 3.3 by showing that the conditions of the second part of that proposition are satisfied by a Borel-fixed ideal.

(4) Show that the lexicographic ideal is Borel-fixed.

(5) Check that the saturation of a Borel-fixed ideal is Borel-fixed.

(6) Let $\prec$ be the reverse-lexicographic term order. Compute by hand the generic initial ideal of $I = \langle x_0x_1\rangle \subset \Bbbk[x_0, x_1]$ with respect to $\prec$. What is the corresponding set $U \subset \mathrm{GL}(2, \Bbbk)$?

(7) It is essentially never possible to compute the generic initial ideal exactly by adding extra variables for the elements of $\mathrm{GL}(n+1, \Bbbk)$, as the complexity of Gröbner basis calculations increases with the number of variables. In practice one chooses a "random" element $g \in \mathrm{GL}(n+1, \Bbbk)$, and computes $\mathrm{in}_w(g \cdot I)$. Do this for $I = \langle x_0x_3 - x_1x_2, x_0x_2 - x_1^2, x_1x_3 - x_2^2\rangle \subset \Bbbk[x_0, x_1, x_2, x_3]$ using a computer algebra system, and verify that the result is Borel fixed. Warning: Random g chosen with small entries may well not be sufficiently generic. One implemented algorithm to compute gins is to try 50 different g, and take the most common answer.

(8) Let $P(t) = 3$. List all Borel-fixed ideals in $\mathrm{Hilb}^P(\mathbb{P}^2)$. Compute the distraction of each. What does Reeves' walk do on this Hilbert scheme?

4. Lecture 4: Hilbert schemes of points on surfaces

We saw in the previous lecture that we should generally expect the Hilbert scheme to be as nasty as we can imagine. This result does not, however, cover absolutely all Hilbert schemes, and there are still some that are nice (meaning smooth and irreducible). In particular, Hilbert schemes of points on smooth surfaces are always smooth and irreducible, by a result of Fogarty [Fog68]. By a Hilbert scheme of points we mean one where the Hilbert polynomial is that of a finitely collection of points, so $P(t) = d$ for all t, where d is a constant.

We will consider the related local picture of the Hilbert scheme of d points in the affine plane $\mathrm{Hilb}^d(\mathbb{A}^2)$. This parameterizes all artinian ideals I in the polynomial ring $S = \Bbbk[x, y]$ with $\dim_{\Bbbk}(S/I) = d$. Note that these ideals need no longer be homogeneous. In this lecture we will show that $\mathrm{Hilb}^d(\mathbb{A}^2)$ is smooth and irreducible, following the proof of Haiman [Hai98]. It is important to emphasize that this is only scratching the surface of what is known about Hilbert schemes of points on smooth surfaces, and we could spend more than the entire week on this topic alone. In particular, we will not touch on the description of the Betti numbers of $\mathrm{Hilb}^d(\mathbb{A}^2)$ for all d, and the Heisenberg algebra action on the homology of $\mathrm{Hilb}^d(A^2)$. See [Nak99] for an introduction to these topics.

The proof that $\mathrm{Hilb}^d(\mathbb{A}^2)$ is smooth and irreducible has four steps:

(1) Show that $\mathrm{Hilb}^d(\mathbb{A}^2)$ is connected by showing that every ideal lives in the same irreducible component as a monomial ideal, and all monomial ideals live in the same irreducible component (the "good component" of the Hilbert scheme of points).
(2) Reduce to showing that all monomial ideals on $\mathrm{Hilb}^d(\mathbb{A}^2)$ are smooth points.
(3) Show that the dimension of the good component of $\mathrm{Hilb}^d(\mathbb{A}^2)$ is at least $2d$.
(4) Give a combinatorial description of the tangent space to a monomial ideal in $\mathrm{Hilb}^d(\mathbb{A}^2)$, and show that the dimension of this space is at most $2d$.

Together these steps show that $\mathrm{Hilb}^d(\mathbb{A}^2)$ is smooth and connected, and thus smooth and irreducible. The last of these steps has the most content. Most of this plan generalizes to $\mathrm{Hilb}^d(\mathbb{A}^n)$, with the exception of the last step showing an upper bound on the dimension of the tangent space. We can thus use this outline to give examples of smooth and singular $\mathrm{Hilb}^d(\mathbb{A}^n)$ for $n > 2$.

4.1. **Step 1: Connectedness.** Let I be an ideal in $\mathrm{Hilb}^d(\mathbb{A}^2)$, so $\dim_{\Bbbk}(S/I) = d$. For any term order $\prec$, the ideal $J = \mathrm{in}_{\prec}(I)$ satisfies $\dim_{\Bbbk}(S/J) = \dim_{\Bbbk}(S/I) = d$, and lives on the same irreducible component of $\mathrm{Hilb}^d(\mathbb{A}^2)$ as I, since the Gröbner degeneration from I to J is a flat family, so gives rise to a map from $\mathbb{A}^1$ to $\mathrm{Hilb}^d(\mathbb{A}^2)$, the image of which must lie in one irreducible component. Thus every ideal lives on the same irreducible component of $\mathrm{Hilb}^d(\mathbb{A}^2)$ as a monomial ideal.

To show that all monomial ideals live in the same connected component, we will show that they all live in the same connected component as the ideal $J = \langle x, y^d \rangle$. Let $I = \langle y^{v_1}, \dots, x^{u_i}y^{v_i}, \dots, x^{u_l} \rangle$ be a monomial ideal in $\mathrm{Hilb}^d(\mathbb{A}^2)$, where we set $u_1 = v_l = 0$. Let $x^a y^b$ be a *socle element* for S/I with $b \neq 0$, so $x^{a+1}y^b$ and $x^a y^{b+1}$ both live in I. If no such element exists, we must have $I = J$. Consider the ideal $I' = \langle x^{u_i}y^{v_i} : 1 \leq i \leq l-1 \rangle + \langle x^a y^b - x^{u_l} \rangle$. One can check (using Buchberger's S-pair criterion) that $I = \mathrm{in}_w(I')$ for any $w \in \mathbb{N}^2$ with $aw_1 + bw_2 < u_l w_1$. Let $w' = N(b, u_l - a) - w$ for $N \gg 0$, and set $I_2 = \mathrm{in}_{w'}(I')$. We can repeat this construction for $I_3, I_4, \dots$. The exponent of the minimal generator of the form x^c increases at each step, since $x^{u_l} \notin I_2$ in the formulation above, so this procedure must terminate at some ideal I_j, at which point we must have $I_j = J$. Since each step is a pair of Gröbner degenerations, we conclude that I lives in the same connected component as J. We note that this argument can be strengthened to show that in fact all monomial ideals live in the same *irreducible component*. Since this argument shows that every ideal $I \subset S$ with $\dim_{\Bbbk}(S/I) = d$ lives in the same connected component of $\mathrm{Hilb}^d(\mathbb{A}^2)$ as J, we conclude that $\mathrm{Hilb}^d(\mathbb{A}^2)$ is connected.

4.2. **Step 2: Smoothness - reduction to the monomial case.** To show that it suffices to check that every monomial ideal in $\mathrm{Hilb}^d(\mathbb{A}^2)$ is a smooth point, we consider the $(\Bbbk^*)^2$ torus action on $\mathrm{Hilb}^d(\mathbb{A}^2)$. The $(\Bbbk^*)^2$-action on $S = \Bbbk[x, y]$ induces a $(\Bbbk^*)^2$-action on the set of ideals $I \subset S$ with $\dim_{\Bbbk}(S/I) = d$. The fixed points of this action are the monomial ideals of colength d. To see this, note first that since the $(\Bbbk^*)^2$ action scales each monomial in S, monomial ideals are fixed by this action. For the other inclusion, fix $w \in \mathbb{N}^2$, and let $\phi_w : \Bbbk^* \to (\Bbbk^*)^2$ be given by $\phi_w(t) = (t^{w_1}, t^{w_2})$, so $\phi_w(t)x = t^{-w_1}x$ and $\phi_w(t)y = t^{-w_2}y$. Then $\lim_{t\to 0} \phi_w(t)I = \mathrm{in}_w(I)$ for any ideal I. If w is chosen sufficiently generically then $\mathrm{in}_w(I)$ is a monomial ideal, so for I not a monomial ideal we have $\lim_{t\to 0} \phi(t)I \neq I$, so I is not fixed by the $(\Bbbk^*)^2$ action. Thus the monomial ideals are the fixed point of the $(K^*)^2$ action on $\mathrm{Hilb}^d(\mathbb{A}^2)$.

The singular locus of $\mathrm{Hilb}^d(\mathbb{A}^2)$ fixed by the $(\Bbbk^*)^2$-action, so consists of a union of $(\Bbbk^*)^2$-orbits. In addition it is Zariski-closed, so if it is nonempty, the singular locus must contain a $(\Bbbk^*)^2$-fixed point, and thus a monomial ideal. Thus showing that every monomial ideal in $\mathrm{Hilb}^d(\mathbb{A}^2)$ is a smooth point shows that $\mathrm{Hilb}^d(\mathbb{A}^2)$ is smooth.

4.3. **Step 3: Smoothness - lower bound on dimension.** We next show that $\dim(\mathrm{Hilb}^d(\mathbb{A}^2)) \geq 2d$. This is done using the *Hilbert-Chow* morphism. The 0-*cycle* of an element $I \in \mathrm{Hilb}^d(\mathbb{A}^2)$ is $\sum_i c_i p_i$, where the p_i are the points of $\mathbb{A}^2$ occurring in the support of the subscheme of $\mathbb{A}^2$ determined by the ideal I, and c_i is the multiplicity of point p_i, so $\sum_i c_i p_i = d$. The set of 0-cycles of $\mathbb{A}^2$ is parameterized by the points of $(\mathbb{A}^2)^d/S_d$, where S_d is the symmetric group on d elements permuting the coordinates of $(\mathbb{A}^2)^d$. The *Hilbert-Chow morphism* is the surjective morphism

$$\mathrm{Hilb}^d(\mathbb{A}^2) \to (\mathbb{A}^2)^d/S_d$$

which takes an ideal I to its 0-cycle. Since $(\mathbb{A}^2)^d/S_d$ is $2d$-dimensional, this shows that $\mathrm{Hilb}^d(\mathbb{A}^2)$ is at least $2d$-dimensional.

4.4. **Step 4: Smoothness - combinatorial construction of the tangent space.** The previous steps reduce the problem of showing that $\mathrm{Hilb}^d(\mathbb{A}^2)$ is smooth to showing that the tangent space to $\mathrm{Hilb}^d(\mathbb{A}^2)$ at a monomial ideal is at most $2d$-dimensional. We do this by giving a combinatorial description of the tangent space to a monomial ideal and giving a spanning set of size $2d$ for this space.

Given a monomial ideal $M \in \mathrm{Hilb}^d(\mathbb{A}^2)$, let $\mathcal{U}_M$ be the subscheme of $\mathrm{Hilb}^d(\mathbb{A}^2)$ whose closed points consist of those ideals I for which the standard monomials of M form a $\Bbbk$-basis for S/I. Technically this set describes a subfunctor of the Hilbert functor that is represented by $\mathcal{U}_M$. Since the standard monomials of an initial ideal of I form a $\Bbbk$-basis for S/I, the set of all $\mathcal{U}_M$ cover $\mathrm{Hilb}^d(\mathbb{A}^2)$.

The scheme $\mathcal{U}_M$ is affine, and we now describe its defining equations. Let $\mathcal{S} = \{x^{u_1}, \ldots, x^{u_d}\}$ be the set of standard monomials of M. Let $\mathcal{B}$ be the set of monomials $x^u \in M$ for which $x^u/x_i \notin M$ for some i with $u_i > 0$. The set $\mathcal{B}$ is the *border* of M. Set $b = |\mathcal{B}|$. Let I be an ideal in $\mathcal{U}_M$. Since $\mathcal{S}$ is a basis for S/I, for each $x^u \in \mathcal{B}$, there is a unique polynomial $f_u \in I$ of the form

$$f_u = x^u - \sum_{v \in \mathcal{S}} \gamma_v^u x^v,$$

where $\gamma_v^u \in \Bbbk$. Let $R = \Bbbk[z_v^u : x^u \in \mathcal{B}, x^v \in \mathcal{S}]$ be the coordinate ring of $\mathbb{A}^{bd}$. Each point in $\mathcal{U}_M$ gives a point in $\mathbb{A}^{bd}$ by taking the ideal to the vector γ_v^u.

Form the $d \times d$ multiplication matrix X_i with rows and columns indexed by $\mathcal{S}$ with the (x^u, x^v)th entry equal to $z_u^{v+e_i}$ if $x_i x^v \in M$, equal to 1 if $x^u = x_i x^v \notin M$, and equal to zero otherwise. Then the fact that x_1 and x_2 commute mean that we must have $X_1 X_2 - X_2 X_1 = 0$. This is also sufficient to guarantee that the ideal generated by the f_u for specific values of the z_u^v has colength d. So the ideal $\mathcal{I}_M$ generated by the entries of the commutator $X_1 X_2 - X_2 X_1$ defines the affine scheme $\mathcal{U}_M \subset \mathbb{A}^{bd}$.

Example 4.1. For convenience we set $S = \Bbbk[x, y]$ here, and use the notation z_y^x for $z_{(0,1)}^{(1,0)}$. Let $d = 3$, and let $M = \langle x^2, xy, y^2 \rangle$. Then the matrices X_1 and X_2 are

$$X_1 = \begin{pmatrix} 0 & z_1^{x^2} & z_1^{xy} \\ 1 & z_x^{x^2} & z_x^{xy} \\ 0 & z_y^{x^2} & z_y^{xy} \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & z_1^{xy} & z_1^{y^2} \\ 0 & z_x^{xy} & z_x^{y^2} \\ 1 & z_y^{xy} & z_y^{y^2} \end{pmatrix},$$

so $X_1 X_2 - X_2 X_1$ equals

$$\begin{pmatrix} 0 & z_1^{x^2} z_x^{xy} + z_1^{xy} z_y^{xy} - z_1^{xy} z_x^{x^2} - z_1^{y^2} z_y^{x^2} & z_1^{x^2} z_x^2 + z_1^{xy} z_y^{y^2} - z_1^{xy} z_x^{xy} - z_1^{y^2} z_y^{xy} \\ 0 & z_1^{xy} + z_x^{xy} z_y^{xy} - z_y^{x^2} z_x^{y^2} & z_1^{y^2} + z_x^{x^2} z_x^{y^2} + z_x^{xy} z_y^{y^2} - (z_x^{xy})^2 - z_x^{y^2} z_y^{xy} \\ 0 & z_y^{x^2} z_x^{xy} + z_{y^2}^{xy} - z_1^{x^2} - z_y^{xy} z_x^{x^2} - z^{y^2} - y z_y^{x^2} & z_y^{x^2} z_x^{y^2} - z_1^{xy} - z_x^{xy} z_y^{xy} \end{pmatrix}.$$

Thus $\mathcal{U}_M$ is the subscheme of $\mathbb{A}^9$ defined by the ideal $\langle z_1^{x^2} z_x^{xy} + z_1^{xy} z_y^{xy} - z_1^{xy} z_x^{x^2} - z_1^{y^2} z_y^{x^2}, z_1^{x^2} z_x^2 + z_1^{xy} z_y^{y^2} - z_1^{xy} z_x^{xy} - z_1^{y^2} z_y^{xy}, z_1^{xy} + z_x^{xy} z_y^{xy} - z_y^{x^2} z_x^{y^2}, z_1^{y^2} + z_x^{x^2} z_x^{y^2} + z_x^{xy} z_y^{y^2} - (z_x^{xy})^2 - z_x^{y^2} z_y^{xy}, z_y^{x^2} z_x^{xy} + z_{y^2}^{xy} - z_1^{x^2} - z_y^{xy} z_x^{x^2} - z^{y^2} - y z_y^{x^2}, z_y^{x^2} z_x^{y^2} - z_1^{xy} - z_x^{xy} z_y^{xy} \rangle \subset \Bbbk[z_1^{x^2}, z_x^{x^2}, z_y^{x^2}, z_1^{xy}, z_x^{xy}, z_y^{xy}, z_1^{y^2}, z_x^{y^2}, z_y^{y^2}]$.

The generators for $\mathcal{I}_M$ are of the form

$$f_q^p = \sum_{x^u \in \mathcal{S}} z_{x^u}^{x_1 x^p} z_{x^q}^{x_2 x^u} - \sum_{x^v \in \mathcal{S}} z_{x^v}^{x_2 x^p} z_{x^q}^{x_1 x^v},$$

where $p, q \in \mathbb{N}^2$ with $x^p = x_1^{p_1} x_2^{p_2}$ and $x^q = x_1^{q_1} x_2^{q_2} \in \mathcal{S}$. Note that $f_q^p = 0$ when both $x_1 x^p, x_2 x^p \notin M$. The dimension of the cotangent space of $\mathcal{U}_M$ at

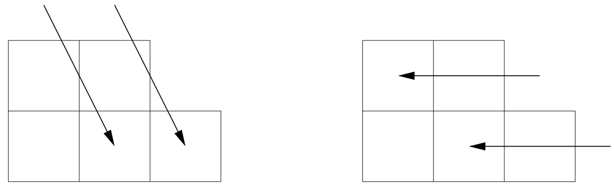

FIGURE 1. Two equivalent arrows

the origin of $\mathbb{A}^{bd}$ (representing the point $M \in \mathrm{Hilb}^d(\mathbb{A}^2)$) is given by

$$\dim_{\Bbbk} \mathfrak{m}/\mathfrak{m}^2,$$

where $\mathfrak{m} = \langle z_v^u : x^u \in \mathcal{B}, x^v \in \mathcal{S}\rangle$. This is equal to

$$\dim_{\Bbbk}\langle z_v^u\rangle/\langle (z_v^u)^2 + \mathcal{I}_M\rangle = \dim_{\Bbbk}\langle z_v^u\rangle/\langle (z_v^u)^2 + \tilde{f}_q^p : x^u \in \mathcal{B}, x^v, x^p, x^q \in \mathcal{S}\rangle,$$

where $\tilde{f}_q^p$ is the degree one part of f_q^p. Now $\tilde{f}_q^p = z_{x^{q-e_2}}^{x^p x_1} + z_{x^q}^{x^p x_1 x_2} - z_{x^{q-e_1}}^{x^p x_2} - z_{x^q}^{x^p x_1 x_2}$, where the first term only shows up if x_2 divides x^q and $x^p x_1 \in M$, the second only shows up if $x^p x_1 \notin M$, the third only shows up if x_1 divides x^q and $x^p x_2 \in M$, and the last term only shows up if $x^p x_2 \notin M$. We thus see that $\tilde{f}_q^p$ has either one or two variables, and the variables z_d^c occurring have $c - d$ constant. If $\tilde{f}_q^p = z_d^c - z_{d'}^{c'}$ then $c' = c \pm e_i$.

We draw the variable z_v^u as an arrow in $\mathbb{Z}^2$ with tail u and head v. The degree one part of the polynomial ring has $\Bbbk$-basis the set of all arrows with tail in $\mathcal{B}$ and head in $\mathcal{S}$. The relations coming from $\tilde{f}_q^p$ are that two arrows are equivalent if one can be obtained from the other by moving the first horizontally or vertically keeping the tail in M and the head in $\mathcal{S}$. This is illustrated in Figure 1

There is a distinguished representative for each nonzero equivalence class of arrows consisting of an arrow z_v^u where either $x^u/x_1 \notin M$ and $x_2 x^v \in M$, or the equivalent statement with 1 and 2 reversed. Given such an arrow, let $\phi(z_v^u) = \gcd(x^u, x^v) \in \mathcal{S}$. Note that the map from distinguished arrows to $\mathcal{S}$ is two-to-one, so we conclude that the number of distinguished arrows is at most $2|\mathcal{S}| = 2d$. This shows that the dimension of the tangent space to a monomial ideal is at most $2d$. Combined with the lower bound from the previous subsection, we conclude that the dimension of the tangent space is the dimension of $\mathrm{Hilb}^d(\mathbb{A}^2)$, so $\mathrm{Hilb}^d(\mathbb{A}^2)$ is smooth.

4.5. **Exercises 4.**

(1) List all monomial ideals I in $\Bbbk[x, y]$ with $\dim_{\Bbbk}(\Bbbk[x, y]/I) = 4$. Verify that for each of these the tangent space to $\mathrm{Hilb}^4(\mathbb{A}^2)$ at I is 8-dimensional. Repeat if desired with 4 replaced by 5.

(2) Compute the equations for $\mathcal{U}_M$ for each of the monomial ideals M from your answer to the previous question.

(3) Show that every monomial ideal in $\mathrm{Hilb}^d(\mathbb{A}^2)$ lives in the same *irreducible* component.

(4) When $d = 2$, the variety $(\mathbb{A}^2)^2/S_2$ is the quotient of affine space by an abelian group, and is thus a toric variety. Can you describe it?

(5) In fact, $\mathrm{Hilb}^2(\mathbb{A}^2)$ is a toric variety. Can you describe it?
(6) The combinatorial description of the tangent space to a monomial ideal generalizes to artinian monomial ideals in more variables, so to points of $\mathrm{Hilb}^d(\mathbb{A}^n)$. Write down the definition of this when $n = 3$.
(7) Use your answer to Question 6 to show that $\mathrm{Hilb}^4(\mathbb{A}^3)$ is singular.
(8) (Hard) The smallest d for which $\mathrm{Hilb}^d(\mathbb{A}^3)$ is reducible is not known, though $8 \leq d \leq 78$ by work of Iarrobino. Cartwright, Erman, Velasco and Viray have shown (work in progress) that $\mathrm{Hilb}^8(\mathbb{A}^4)$ is reducible, with another 24 dimensional component as well as the good component of dimension 32. The intersection of these two components is 24-dimensional. Show that the ideal $\langle x_1^2, x_2^2, x_3^2, x_4^2, x_1x_2, x_3x_4, x_2x_3 - x_1x_4\rangle \subset \Bbbk[x_1, x_2, x_3, x_4]$ does not lie on the main component. To give the straightforward proof of this you will need to learn how to compute the tangent space to any point on the Hilbert scheme. The ideal $\langle x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3^2, x_3x_4, x_4^2\rangle$ lies on both components. Can you show this directly?

5. Lecture 5: Multigraded Hilbert schemes

The multigraded Hilbert scheme, introduced by Haiman and Sturmfels [HS04] is a moduli space parameterizing all ideals in a polynomial ring with a given multigraded Hilbert function. One of the original motivations for its construction was as a common generalization of the Hilbert scheme of points in affine space and the toric Hilbert scheme (defined below). The "classical" Hilbert scheme described in the first three lectures is also a special case of a multigraded Hilbert scheme.

Throughout this lecture we will use the following notation. Let $S = \Bbbk[x_1, \dots, x_n]$, where $\Bbbk$ is an arbitrary commutative ring. We fix an abelian group A and a homomorphism $\deg\colon \mathbb{Z}^n \to A$. We may assume that deg is surjective (by replacing A by the image of deg). The homomorphism deg induces a grading of S by A, by setting $\deg(x_i) = \deg(\mathbf{e}_i)$, where $\mathbf{e}_i$ is the ith standard basis vector of $\mathbb{Z}^n$, so $S = \oplus_{a \in A} S_a$, where S_a is the degree a part of S, and $S_a \cdot S_{a'} \subseteq S_{a+a'}$. We call this a *multigrading* of S. Let A^+ be the semigroup of elements $a \in A$ for which $S_a \neq 0$.

Example 5.1.
(1) Let $S = \Bbbk[x_1, \dots, x_n]$, and let $\deg(x_i) = 1$ for all i. Then $A = \mathbb{Z}$, and $A^+ = \mathbb{N}$. This is the standard grading of the polynomial ring.
(2) Let $S = \Bbbk[x_1, x_2, x_3, x_4]$, and let $\deg(x_1) = (1, 0)$, $\deg(x_2) = (1, 1)$, $\deg(x_3) = (1, 2)$, $\deg(x_4) = (1, 3)$. Then $A \cong \mathbb{Z}^2$, and A^+ is the subsemigroup consisting of those elements $(a, b) \in \mathbb{Z}^2$ with $a \geq 3b \geq 0$. The degree $(3, 4)$ part of S, for example, has basis $\{x_1x_2x_4, x_1x_3^2, x_2^2x_3\}$.
(3) Let $S = \Bbbk[x_1, x_2]$, and $\deg(x_1) = 1$, $\deg(x_2) = -1$. Then $A = A^+ = \mathbb{Z}$. The graded pieces of S are infinite-dimensional in this case. For example, S_0 has basis $\{x^iy^i : i \geq 0\}$.
(4) Let $S = \Bbbk[x_1, x_2]$, and $\deg(x_1) = 1 \mod 3$, $\deg(x_2) = 2 \mod 3$. Then $A = A^+ \cong \mathbb{Z}/3\mathbb{Z}$. The graded pieces of S are again infinite-dimensional.

Definition 5.2. An ideal $I \subseteq S$ is *admissible* if $(S/I)_a = S_a/I_a$ is a locally free $\Bbbk$-module of finite rank for all $a \in A$. The multigraded Hilbert function of an admissible ideal I is

$$h_I : A \to \mathbb{N}, \;\; h_I(a) = \mathrm{rk}_{\Bbbk}(S/I)_a.$$

Note that $h_I(a) = 0$ for $a \notin A^+$.

Example 5.3.
(1) Let $S = \Bbbk[x_1, x_2]$ with $\deg(x_1) = 1$ and $\deg(x_2) = -1$. Then $I = \langle x_1x_2 \rangle$ is admissible, and gives $h(a) = 1$ for all $a \in \mathbb{Z}$. By constrast the ideal $J = \langle x_1^2 \rangle$ is not admissible, as $\mathrm{rk}_{\Bbbk}(S/J)_0$ is infinite.
(2) Let $S = \Bbbk[x_1, x_2]$ with $\deg(x_1) = 1 \mod 3$ and $\deg(x_2) = 2 \mod 3$. Then $I = \langle x_1^2, x_1x_2, x_2^2 \rangle$ is admissible, with $h(a) = 1$ for all $a \in A$.

Given the multigraded Hilbert function h_I of an admissible ideal I, the multigraded Hilbert scheme H_S^h parameterizes all ideal of S with multigraded Hilbert function h_I. More formally, it represents the following functor.

Definition 5.4. The Hilbert functor $H_S^h : (\Bbbk-\text{algebras}) \to (\text{sets})$ is defined as follows. For a $\Bbbk$-algebra R, $H_S^h(R)$ is the set of homogeneous ideals $I \subseteq R \otimes_{\Bbbk} S$ such that $(R \otimes S_a)/I_a$ is a locally free R-module of rank $h(a)$ for each $a \in A$.

Theorem 5.5 (Theorem 1.1 of [HS04]). *There is a quasiprojective scheme H_S^h over $\Bbbk$ that represents H_S^h.*

Before giving some idea of the construction of H_S^h we first observe that the Hilbert schemes considered in the previous sections are examples of multigraded Hilbert schemes.

Example 5.6. If $A = 0$ is the trivial group, and $h : A \to \mathbb{Z}$ is given by $h(0) = d$ for some $d > 0$ then H_S^h is the Hilbert scheme $\mathrm{Hilb}^d(\mathbb{A}^n)$ of d points in affine n-space.

Example 5.7. Fix a Hilbert polynomial P, and let $A = \mathbb{Z}$ with $\deg(x_i) = 1$ for $1 \leq i \leq n$. Let D be the Gotzmann number of P. Define $h : A \to \mathbb{N}$ by setting $h(a) = 0$ for $a < D$, and $h(a) = P(a)$ for $a \geq D$. Then $H_S^h = \mathrm{Hilb}_P(\mathbb{P}^{n-1})$. To see this, note that ideals in S with Hilbert function h are in bijection with subschemes of $\mathbb{P}^{n-1}$ with Hilbert polynomial P.

Example 5.8. Let the grading given by A be *positive*, in the sense that 1 is the only monomial in S of degree zero, and torsion free. See [MS05, Theorem 8.6] for many equivalent definitions of the notion of a positive grading. Define $h : A \to \mathbb{N}$ by $h(a) = 1$ for $a \in A^+$ and $h(a) = 0$ otherwise. Then H_S^h is by definition the *toric Hilbert scheme H_A* of A. The toric Hilbert scheme was introduced by Peeva and Stillman [PS02] based on earlier work of Sturmfels [Stu94]. See also [PS00], [MT02], [MT03], [SST02] for more work on the toric Hilbert scheme.

Our assumptions on the grading by A mean that we can identify $A \cong \mathbb{Z}^d$ for some $d > 0$, and set $\mathcal{A} \subset \mathbb{Z}^d$ to be the collection of n vectors $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$, where $\mathbf{a}_i = \deg(\mathbf{e}_i)$. The toric Hilbert scheme has a distinguished irreducible component containing the *toric ideal* of $\mathcal{A}$. This is the ideal $I_{\mathcal{A}} = \langle x^u - x^v : \deg(x^u) = \deg(x^v) \rangle$ defining the semigroup algebra $\Bbbk[t^{\mathbf{a}_1}, \ldots, t^{\mathbf{a}_n}]$. The name "toric ideal" comes from the fact that Spec of this semigroup algebra is a not-necessarily-normal affine toric variety. The other closed points on the irreducible component containing $I_{\mathcal{A}}$ are of the form $\lambda\, \mathrm{in}_w(I_{\mathcal{A}})$, where $w \in \mathbb{R}^n$ is a weight vector, and $\lambda\, \mathrm{in}_w(I_{\mathcal{A}})$ is the corresponding scaled initial ideal of $I_{\mathcal{A}}$.

When $n - d \leq 2$ the toric Hilbert scheme H_A is smooth and irreducible; see [PS02], [MT03]. For higher codimension this is rarely the case. For example, when $d = 1, n = 4$, and $\mathcal{A} = \{1, 3, 4, 7\}$ the scheme H_A is reducible. This is most easily seen by exhibiting monomial ideals with Hilbert function h that are not initial ideals of $I_{\mathcal{A}}$; see [Stu96, Theorem 10.4]. Unlike the classical Hilbert scheme, the toric Hilbert scheme need not be connected. In [San05] Santos constructed a several configurations $\mathcal{A}$ for which H_A is disconnected. The smallest of these has $d = 6$ and $n = 26$, and H_A has at least thirteen connected components.

Example 5.9. When A is a finite group, and $h : A \to \mathbb{N}$ is given by $h(a) = 1$ for all $a \in A$, then H_S^h is the A-Hilbert scheme of Nakamura [Nak01]. When $n = 2$ or $n = 3$ and $\deg(x_1 x_2 x_3) = 0$ this scheme is smooth and irreducible [Kid01], [Nak01], but H_S^h is often otherwise reducible. See, for example, [Cra05, Example 4.12].

Example 5.10. When $n = 2$, so $S = \Bbbk[x_1, x_2]$, then for any group A and any $h : A \to \mathbb{N}$ the multigraded Hilbert scheme H_S^h is smooth and irreducible. This generalizes the fact that $\mathrm{Hilb}^d(\mathbb{A}^2)$ is smooth and irreducible (though uses that fact in the proof). This is work in progress with Greg Smith, and settles a conjecture of Haiman and Sturmfels (see [HS04, Example 1.3] and [MS05, Conjecture 18.46]).

We now sketch the construction of the multigraded Hilbert scheme. The idea is again to construct the H_S^h as a closed subscheme of a Grassmannian. The trick is to first find the multigraded analogue of the Gotzmann number from Lecture 2.

Definition 5.11. A finite set $D \subset A$ is *supportive* for a multigraded Hilbert function h if

(1) Every monomial ideal with Hilbert function h is generated by monomials of degree belonging to D.
(2) Every monomial ideal I generated in degrees in D satisfies: if $H_{S/I}(a) = h(a)$ for all $a \in D$, then $H_{S/I}(a) \leq h(a)$ for all $a \in A$.

The set D is *very supportive* if the first condition above holds, and in addition

(1) Every monomial ideal I generated in degrees in D satisfies: if $H_{S/I}(a) = h(a)$ for all $a \in D$, then $H_{S/I}(a) = h(a)$ for all $a \in A$.
(2) For every monomial ideal I with $H_{S/I} = h$, the syzygy module of I is generated by syzygies $x^u x^{v'} = x^v x^{u'} = \mathrm{lcm}(x^u, x^v)$ among generators x^u, x^v of I such that $\deg(\mathrm{lcm}(x^u, x^v)) \in D$.

Let D be the Gotzmann number of a Hilbert polynomial P. Gotzmann's regularity and persistence theorems (Theorems 2.5 and 2.7) imply that in the standard graded case where $h(a) = P(a)$ for $a \geq D$, and $h(a) = 0$ for $a < D$, then $\{D\}$ is supportive for h, and $\{D, D+1\}$ is very supportive.

Proposition 5.12. *There is a finite set $D \subset A$ of degrees that is very supportive for any multigraded Hilbert function h.*

The key to prove Proposition 5.12 is to first note that there are only a finite number of monomial ideals in S with Hilbert function h. This follows from the fact (see [Mac01]) that in any infinite collection of monomial ideals there must be two with one contained in the other. This implies that there is a finite set of degrees D_1 in which all homogeneous ideals with multigraded Hilbert function h are generated, and which contains generators for the syzygies of all monomial ideals with multigraded Hilbert function h. We then take the bigger set D_2 on which monomial ideals generated in degrees in D_1 with Hilbert function h on D_2 have Hilbert function H everywhere (which exists, since there are only finitely many monomial ideals generated in degrees D_1). We then repeat with D_1 replaced by D_2 to construct a et D_2. One can show that this procedure must terminate, which gives the finite set D.

We now define a more general Hilbert functor $H^h_{S_D}$, where D is a finite subset of A and S_D is the $\Bbbk$-module consisting of those homogeneous pieces of S whose degrees lie in D.

Definition 5.13. Let $D \subset A$ be a finite set, and let $h : D \to \mathbb{N}$ be a function. The Hilbert functor $H^h_{S_D} : (\Bbbk - \text{modules}) \to (\text{sets})$ is defined as follows. For a $\Bbbk$-module R, $H^h_S(R)$ is the set of S_D-submodules $I \subseteq R \otimes_{\Bbbk} S$ such that $(R \otimes S_a)/I_a$ is a locally free R-module of rank $h(a)$ for each $a \in A$.

To construct the multigraded Hilbert scheme, we first show that the functor $H^h_{S_D}$ is represented by a quasiprojective scheme over $\Bbbk$. This is similar in philosophy to the construction of the classical Hilbert scheme $\mathrm{Hilb}_P(\mathbb{P}^n)$. The idea is to take the Grassmannian $\mathcal{G}$ of locally free $\Bbbk$-submodules of S_D with corank $h(a)$ in each degree $a \in D$. The precise definition of $\mathcal{G}$ is slightly technical, since S_D may not be a finitely generated $\Bbbk$-module. One then checks that $H^h_{S_D}$ is a closed subscheme of $\mathcal{G}$, which in particular shows that the corresponding functor is representable.

The advantage of (very) supportive sets is then seen from the following proposition.

Proposition 5.14. *If D is a supportive set then the natural morphism $H^h_S \to H^h_{S_D}$ is a closed embedding. If the set D is very supportive then this morphism is an isomorphism.*

Remark 5.15. We note that the paper [HS04] actually provides a broader framework for constructing Hilbert schemes, in that the entire polynomial ring is not needed. One place where this is useful is when constructing the Hilbert scheme of subschemes of a toric variety. By [Cox95] subschemes of a toric variety correspond to particular homogeneous ideals in a multigraded polynomial ring. The multigraded Hilbert polynomial of such an ideal only restricts the Hilbert function in a subset of the possible degrees, so one uses the partial polynomial ring framework of [HS04] to construct the Hilbert scheme of all subschemes of a toric variety with a given multigraded Hilbert polynomial. See [MS05] for details.

5.1. **Open questions.** We close with a sampling of open questions. The bias here is to structural questions. There are more in [HS04].

(1) Is the A-Hilbert scheme always connected? Or can one modify Santos' example (or hopefully find a smaller one) for a disconnected toric Hilbert scheme to give a disconnected A-Hilbert scheme?
(2) Is the toric Hilbert scheme always connected in codimension three ($n - d = 3$)? This is motivated by the closely related work on the bistellar flip graph of triangulations of point configurations by Azaola and Santos [AS00]. Analogously, is the A-Hilbert scheme connected when $n = 3$?

(3) Give an effective bound on location of a very supportive set $D \subset A$ for a multigraded Hilbert function h. See [MS05] for an algorithm to construct such a bounding set; a better bound is desired.
(4) Give good necessary and sufficient conditions for a set $D \subset A$ to be very supportive for a multigraded Hilbert function h.
(5) In [MS05] a multigraded version of Gotzmann's regularity theorem is given (see Exercise 7). Give a multigraded version of Gotzmann's persistence theorem.

5.2. **Exercises 5.**

(1) Let $S = \Bbbk[x_1, x_2, x_3]$ be graded by $\deg(x_1) = 3$, $\deg(x_2) = 2$, $\deg(x_3) = 1$. Show that not every ideal in S has the same Hilbert function as a lex-segment ideal.
(2) Let $S = \Bbbk[x_1, x_2]$ have the standard grading (so $A = \mathbb{Z}$), and let $h = H_{S/I}$ for $I = \langle x_1^3, x_2^3 \rangle$. Find a supportive set for h, and a very supportive set.
(3) Let $S = \Bbbk[x_1, x_2]$ be graded by $\deg(x_1) = 1$, $\deg(x_2) = -1$. List all monomial ideals with the same multigraded Hilbert function as $I = \langle x^2y^2 \rangle$.
(4) Let $S = \Bbbk[x_1, x_2]$ be graded by $\deg(x_1) = 1$, $\deg(x_2) = 2$. Let $h(0) = 1$, $h(1) = 2$, $h(2) = 2$, $h(3) = 2$, $h(4) = 2$, $h(5) = 1$, and $h(6) = 1$. List all monomial ideals with the same multigraded Hilbert function as I. Compute a supportive set and a very supportive set for h.
(5) Let $S = \Bbbk[x_1, x_2, x_3]$ be graded by $\deg(x_1) = (1, 0)$, $\deg(x_2) = (1, 1)$, and $\deg(x_3) = (1, 2)$ and let $I = \langle x_1^3 \rangle$. List all monomial ideals with the same multigraded Hilbert function as I.
(6) When $\mathcal{A} = \{1, 3, 4, 7\}$ the toric Hilbert scheme is reducible. In particular, the ideal $J = \langle x_1^3, x_1x_2, x_2^2, x_2x_3, x_1x_4, x_1^2x_3^2, x_1x_3^4, x_2x_4^3, x_2x_3^3, x_4^4 \rangle$ does not lie on the distinguished component. Verify this.

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