3G6 COMMUTATIVE ALGEBRA - HOMEWORK 6

NOT ASSESSED

Since this homework is not assessed, I may add to it. I will delete this sentence after making the last intentional edit (i.e. if this sentence has disappeared I will only edit to correct typos).

(1) Give two different minimal primary decompositions of the ideal $I = \langle x^2y^3, x^3y^2, x^4y \rangle \subset K[x, y]$.

(2) Compute a primary decomposition for $I = \langle x^2z + y^2z + z^3, x^4 + x^2y^2 - y^2z^2 - z^4 \rangle \subset \mathbb{Q}[x, y, z]$. Hint: Use colon ideals. The syntax in M2 is `I:f` for the ideal $(I : f)$, and `saturate(I,f)` for $(I : f^\infty)$.

(3) Show that if val is a valuation on a field K with value group $\Gamma = \operatorname{im} \operatorname{val} \cong \mathbb{Z}$, then there is a unique $\lambda > 0$ with the value group of $\lambda \operatorname{val}$ equal (as opposed to merely isomorphic) to $\mathbb{Z}$.

(4) Let $K = k(t)$ be the ring of rational functions in t, with the valuation given in lectures. Show that the residue field of K is k.

(5) Let $K = \{\sum_{i=N}^{\infty} a_i t^i : a_i \in \mathbb{C}, N \in \mathbb{Z}\}$.
   (a) Check that the natural addition and multiplication make this into a field. Here by "natural" I mean the extension of the addition and multiplication from the ring of power series. This is the field of Laurent series.
   (b) Check (as claimed in lectures) that the function given by $\operatorname{val}(\sum_{i=N}^{\infty} a_i t^i = N$ when $a_N \neq 0$ obeys the valuation axioms.
   (c) What is the value group of this valuation? What is the valuation ring? What is the residue field?

(6) We showed that if R is a DVR, then every element of $K = \operatorname{frac}(R)$ can be written uniquely as ut^n, where $\operatorname{val}(t) = 1$. Give this description for $f = \frac{3t^5 - t^3 + 7t^2}{3t^2 + 9t + 2} \in k(t)$.

(7) Show that if $I \subset R$ satisfies $P = \sqrt{I}$ is prime, then R/I has only one minimal prime.

(8) Write down an example of an ideal $I \subset K[x, y, z]$ with $|\operatorname{Ass}(R/I)| > 1$ but $\sqrt{I}$ prime.

(9) Show that the residue field $R/\mathfrak{m}$ of $\mathbb{Q}$ with the p-adic valuation is isomorphic to $\mathbb{Z}/p\mathbb{Z}$.