3G6 COMMUTATIVE ALGEBRA - HOMEWORK 2

DUE TUESDAY 9 FEBRUARY, 2PM

Hand in the problems in Section B *only* to the boxes outside the undergraduate office. You are encouraged to work together on the problems, but your written work should be your own.

Throughout this sheet, K is a field.

A : Warm-up problems

(1) Show that any monomial ideal $I \subset K[x_1, \ldots, x_n]$ has a *minimal* monomial generating set.

(2) Show that $IJ \subseteq I \cap J$. Give an example to show that these can be different.

(3) Let $R = K[[x]]$ be the *ring of formal power series* in one variable with coefficients in a field K. This consists of elements $\sum_{i \geq 0} a_i x^i$ with $a_i \in K$, where addition and multiplication are as for (convergent) power series with coefficients in $\mathbb{R}$ (as in 1st/2nd year Analysis). Check that $K[[x]]$ is a ring.

(4) The sum of two ideals is $I + J = \{i + j : i \in I, j \in J\}$. Check that $I + J$ is an ideal.

(5) (Reid, Exercise 1.6) Prove or give a counterexample:
   (a) The intersection of two prime ideals is prime;
   (b) The ideal $P_1 + P_2$ is prime when P_1, P_2 are prime;
   (c) If $\phi \colon R \to S$ is a ring homomorphism, and M is a maximal ideal of S, then $\phi^{-1}(M)$ is a maximal ideal of R.

B: Exercises

(1) Fix a term order $\prec$ on the polynomial ring $K[x_1, \ldots, x_n]$. A set $\mathcal{G} = \{g_1, \ldots, g_s\}$ is a *reduced* Gröbner basis for an ideal I with respect to a term order $\prec$ if $\mathcal{G}$ is a Gröbner basis, $\{\mathrm{in}_\prec(g_1), \ldots, \mathrm{in}_\prec(g_s)\}$ is an irredundant (no repeats) minimal generating set for $\mathrm{in}_\prec(I)$, and for each g_i, the coefficient of $\mathrm{in}_\prec(g_i)$ is 1, and no term of g_i other than its initial term is divisible by $\mathrm{in}_\prec(g_j)$ for any $1 \leq j \leq s$. Show that for any ideal I and any term order $\prec$ there is a unique reduced Gröbner basis for I with respect to $\prec$. Hint: Question B1 from the last HW.

(2) Give an algorithm to describe when two ideals $I = \langle f_1, \dots, f_r \rangle$ and $J = \langle g_1, \dots, g_s \rangle$ in $K[x_1, \dots, x_n]$ are the same. Carry out your algorithm for $I = \langle x^3 + 3x^2z + y^3 + z^3, x^3 - y^3 + z^3 \rangle$ and $J = \langle 8y^9 + 27y^3x^6 - 27x^9, 3zx^2 + 2y^3, 4zy^6 - 9y^3x^4 + 9x^7, 2z^2y^3 + 3y^3x^2 - 3x^5, z^3 + y^3 + x^3 + 3x^2z \rangle$ in $K[x, y, z]$ (attach printouts if you use a computer).

(3) Show that if $I \subseteq K[x_1, \dots, x_n]$ is a radical ideal, then I is prime if and only if there do not exist ideals $J_1, J_2 \neq I$ with $I = J_1 \cap J_2$.

(4) Let $R = K[[x]]$ be the ring of formal power series with coefficients in a field K (as in Question A3).
   (a) Let $f = \sum_{i \geq 0} a_i x^i \in R$. Show that if $a_0 \neq 0$ then f is a unit (i.e., has a multiplicative inverse).
   (b) Describe $\operatorname{Spec}(R)$.

(5) Let $\phi : R \to S$ be a ring homomorphism. Show that if P is a prime ideal in S, then $\phi^{-1}(P)$ is a prime ideal in R. Is the induced map of sets $\phi^*\colon \operatorname{Spec}(S) \to \operatorname{Spec}(R)$ injective? Is it surjective?

(6) Let $R = \mathbb{C}[x]/\langle x^2 \rangle$. Describe $\operatorname{Spec}(R)$.

C: Extensions

(1) Let $f_1, \dots, f_r \in K[x_1, \dots, x_n]$ have the property that there are only a finite number of solutions to $f_1(x) = f_2(x) = \cdots = f_r(x) = 0$. Let $I = \langle f_1, \dots, f_r \rangle$, and let $\mathcal{G}$ be the reduced Gröbner basis for I with respect to the lexicographic term order with $x_1 > \cdots > x_n$. Show that $\mathcal{G}$ contains a polynomial in $K[x_n]$. Describe how you can use this to (numerically, at least) find all solutions to these equations.

(2) What are the prime ideals in $\mathbb{Z}[x]$?

(3) Fix $d \in \mathbb{Z}$. Let $\mathbb{Z}[\sqrt{d}] = \{a + b\sqrt{d} : a, b \in \mathbb{Z}\}$. Describe the prime ideals in $\mathbb{Z}[\sqrt{d}]$.

(4) Let $R \subset \mathbb{R}[[x]]$ be the ring of *convergent* powerseries. Check that R is a ring. What can you say about $\operatorname{Spec}(R)$? What can you say about power series in more variables?

(5) (Open Question) Let $R = K[x_{ij}, y_{ij} : 1 \leq i, j \leq n]$. Let $X = (x_{ij})$ and $Y = (y_{ij})$ be $n \times n$ matrices (with entries in R, and consider the matrix $XY - YX$. This has ijth entry $\sum_{k=1}^{n} (x_{ik}y_{kj} - x_{kj}y_{ik})$. Let I be the ideal in R generated by these n^2 polynomials. Is I prime? (This is known only for very small values of n; $n = 5$ may still be open!).