252 COMBINATORIAL OPTIMIZATION

HOMEWORK 5, MARCH 2014

These exercises are at a mixture of levels. There may be typos or mistakes; please let me know as soon as you find one. I'm happy to discuss your solution attempts in my office hours (Monday 12pm).

Hand in Questions 1, 2, 3, 5 and 6 to the box outside the undergraduate office by Thursday, 13th March, at 2pm. You are strongly encouraged to do the other questions at this time as well, though you should not hand them in. You also are encouraged to work on these problems in groups, though your final write-up should be your own.

Warning: It is easy to make adding mistakes when doing these problems, and also when designing them. All optimal solutions for these problems should be integral, so if you get rational numbers in your answers check your work, and then let me know if you think there is a typo.

(1) Recall that $\bar{c}_i$ is defined by $\bar{c}_i = c_i - c_I^T B_I^{-1} A_i$. Show that $\bar{c}_i = 0$ for $i \in I$.

(2) Let $\mathbf{x}$ be a basic feasible solution corresponding to a set $I \subset \{1, \dots, n\}$. Let $\mathbf{d}$ be the ith basic direction for some $i \notin I$. Show that if there is $\theta > 0$ with $\mathbf{x} + \theta\mathbf{d} \geq 0$ then $\{\mathbf{x} + \theta\mathbf{d} : \theta \geq 0, \mathbf{x} + \theta\mathbf{d} \geq 0\}$ is an edge of the feasible region.

(3) Consider the linear program: Minimize $-x - 2y$ subject to $(x, y) \in \mathbb{R}^2, x \geq 0, y \geq 0, x \leq 1, y \leq 1$.
   (a) Draw the feasible region for this LP.
   (b) Put this LP in standard form.
   (c) Verify that the basic solution with $x_1 = x_2 = 0$ is a basic feasible solution.
   (d) Run the simplex algorithm with this basic feasible solution to find an optimal solution to the standard-form LP.
   (e) Find the corresponding solution to the original LP, and indicate the process of the simplex algorithm on your diagram of the feasible region.

(4) Repeat Question 3 for the linear programs
   (a) Minimize $-x$ subject to $(x, y) \in \mathbb{R}^2, x \geq 0, y \geq 0, y \geq x - 1, y \leq x + 1, x + y \leq 3$.

(b) Minimize $-x$ subject to $(x, y) \in \mathbb{R}^2, x \geq 0, y \geq 0, 2y \leq x + 2, y \leq x + 1, y \geq x - 1, y \geq 2x - 2$
(c) Minimize $-x - y$ subject to $(x, y) \in \mathbb{R}^2, x \geq 0, y \geq 0, 2x + y \geq 2, x + 2y \geq 2$. In this case find an initial basic feasible solution with $x = y = 1$.
(d) Minimize $-x - y - z$ subject to $(x, y, z \in \mathbb{R}^3, x \geq 0, y \geq 0, z \geq 0, x \leq 1, y \leq 1, z \leq 1, x + y + z \leq 2$.
(e) Maximize z subject to $(x, y, z) \in \mathbb{R}^3$, $y \leq x + z, y + x \geq z, y + x + z \leq 2, x - y + z \leq 2, z \geq 0$. In this case find an initial basic feasible solution with $x = y = z = 0$.

(5) Consider the linear program: Minimize $(1, 2, 3, 4) \cdot \mathbf{x}$ for $\mathbf{x} \in \mathbb{R}^4$ with

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & 6 & 7 & 8 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}, \quad \mathbf{x} \geq 0.$$

Find an initial basic feasible solution for this linear program using the "clever trick" method covered in lectures. When you run the simplex algorithm, use the pivoting rule of choosing the first i with $\bar{c}_i < 0$.

(6) Let $P = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0\}$, where A is a $d \times n$ matrix of rank d. Suppose that all basic feasible solutions are nondegenerate. Let $\mathbf{x} \in P$ have exactly d positive entries. Show that $\mathbf{x}$ is a basic feasible solution. Give an example to show that this is false with the nondegeneracy assumption removed.

(7) Let $P = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = b, \mathbf{x} \geq 0\}$, where A is a $d \times n$ matrix of rank d. For each of the followingc statements, decide if it is true or false. If true, prove. If false, provide a counterexample. An optimal solution is a vector $\mathbf{y} \in P$ with $\mathbf{c} \cdot \mathbf{y} \leq \mathbf{c} \cdot \mathbf{x}$ for all $\mathbf{x} \in P$.
(a) If $n = d+1$, then P has at most two basic feasible solutions.
(b) The set of all optimal solutions is bounded.
(c) At every optimal solution, no more than d variables can be positive.
(d) If there is more than one optimal solution, then there are uncountably many optimal solutions.
(e) If there is more than one optimal solution, then there exists at least two basic feasible solutions that are optimal.

(8) Suppose that at a particular iteration of the simplex method we replace I by $I' = \{I \setminus \{i\}\} \cup \{j\}$, which was chosen to have $\bar{c}_j < 0$. Let $\mathbf{d}$ be the jth basic feasible direction from the basis I. Show that for the basis I' we have $\bar{c}_i > 0$, and the ith basic feasible direction is a multiple of $-\mathbf{d}$.

(9) (Bertsimis-Tsitsiklis ex 3.6a) Show that if $\bar{c}_i > 0$ for all $i \notin I$ at some stage of the simplex algorithm then then the basic feasible solution $\mathbf{x}$ corresponding to I is the unique optimal solution (ie $\mathbf{c} \cdot \mathbf{x} < \mathbf{c} \cdot \mathbf{y}$ for all feasible $\mathbf{y}$ with $\mathbf{y} \neq \mathbf{x}$).

(10) (Bertsimis-Tsitsiklis ex 3.18) For each of the following statements, either give a proof or a counterexample.

   (a) An iteration of the simplex method may move the feasible solution by a positive distance while leaving the cost unchanged.
   (b) If at some stage in the algorithm we change I to $I \backslash \{i\} \cup \{l\}$, then at the next stage we cannot add i back again.
   (c) If there is a nondegenerate optimal solution to the linear program, then there is a unique solution to the linear program.

(11) Consider the linear program: Minimize $(0, 0, -1) \cdot (x, y, z)$ for $(x, y, z) \in \mathbb{R}^3$ satisfying $x \geq 0$, $y \geq 0$, $x + z \leq 1$, $y + z \leq 1$.

   (a) Sketch the feasible region of this linear program. How many vertices does it have?
   (b) Put this linear program into standard form.
   (c) Compute all basic solutions for this linear program. How many of these are feasible? How many are nondegenerate?