252 COMBINATORIAL OPTIMIZATION

HOMEWORK 4, FEBRUARY 2014

These exercises are at a mixture of levels. There may be typos or mistakes; please let me know as soon as you find one. I'm happy to discuss your solution attempts in my office hours (Monday 12pm).

Hand in Questions 1, 4, 5, 6, and 8 to the box outside the undergraduate office by Thursday, 27th February, at 2pm. You are strongly encouraged to do the other questions at this time as well, though you should not hand them in. You also are encouraged to work on these problems in groups, though your final write-up should be your own.

(1) Let $P = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1, x + y + z \leq 2\}$. Determine $\text{face}_{(2,-5,3)}(P)$, and $\text{face}_{(-1,-1,-1)}(P)$.

(2) The hypercube is the polytope $\{x \in \mathbb{R}^n : 0 \leq x_i \leq 1 \text{ for } 1 \leq i \leq n\}$. How many faces of each dimension does the n-dimensional hypercube have? How many faces in total are there? Check your answer for the square and the cube.

(3) The cross-polytope is the polytope $\{x \in \mathbb{R}^n : \sum \epsilon_i x_i \leq 1 \text{ for all } \epsilon \in \{-1, 1\}^n\}$. For example, when $n = 2$ it is $\{(x, y) \in \mathbb{R}^2 : x + y \leq 1, x - y \leq 1, -x + y \leq 1, -x - y \leq 1\}$, which is a square. The three-dimensional cross-polytope is the octahedron. How many faces of each dimension does the n-dimensional hypercube have? How many faces in total are there? Check your answer for the square and the octahedron.

(4) Put this LP in standard form: Maximize $x_1 + x_2$ for $\mathbf{x} \in \mathbb{R}^2$ subject to $x_1 \leq 0$, $x_1 + x_2 \geq -1$, and $x_1 + 2x_2 \leq 0$. Give a basic feasible solution $\mathbf{y} \in \mathbb{R}^5$ that achieve the optimum value for the standard form linear program.

(5) Draw the feasible region for the following linear programs. On your picture label clearly all the vertices $\mathbf{v}$, and indicate for each one a sample vector $\mathbf{c}$ for which $\{\mathbf{v}\} = \text{face}_{\mathbf{c}}(P)$, where P is the feasible region of the linear program.
   (a) Minimize $(0, -1) \cdot (x, y)$ for $(x, y) \in \mathbb{R}^2$ subject to $\{x \geq 0, x - 2y \leq 0, y - x \leq 2, y + 2x \leq 5\}$.
   (b) Minimize $(1, 0) \cdot (x, y)$ for $(x, y) \in \mathbb{R}^2$ subject to $\{x \geq 0, x - 2y \leq 0, y - x \leq 2, y + 2x \leq 5\}$.
   (c) Minimize $(1, 1) \cdot (x, y)$ for $(x, y) \in \mathbb{R}^2$ subject to $\{x \geq 0, y \geq 0, y \leq 1, x + y \leq 1\}$.

(d) Minimize $(0, 0, -1) \cdot (x_1, x_2, x_3)$ for $x \in \mathbb{R}^3$ subject to $\{x_1 \geq 0, x_2 \leq 0, x_3 \leq 0, x_1 + x_3 \leq 0, x_2 + x_3 \leq 0\}$.

(6) Recall that a basic solution to a standard form linear program is $\mathbf{x} \in \mathbb{R}^n$ with $A\mathbf{x} = \mathbf{b}$ with the property that $\{A_i : y_i \neq 0\}$ is linearly independent. In lecture we constructed basic solutions by starting with $I \subset \{1, \ldots, n\}$ such that $\{A_i : i \in I\}$ is a basis for $\mathbb{R}^d$. Explain why this construction finds all basic solutions for the linear program. For example, why does it find the basic solution $\mathbf{x} = (1, 0, 0)$ for the linear program: minimize $(2, 3, 4) \cdot \mathbf{x}$ for $\mathbf{x} \geq 0$ and

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}?$$

(7) Let $\mathbf{x}$ be a basic feasible solution corresponding to a set $I \subset \{1, \ldots, n\}$. Let $\mathbf{d}$ be the ith basic direction for some $i \notin I$. Show that if there is $\theta > 0$ with $\mathbf{x} + \theta\mathbf{d} \geq 0$ then $\{\mathbf{x} + \theta\mathbf{d} : \theta \geq 0, \mathbf{x} + \theta\mathbf{d} \geq 0\}$ is an edge of the feasible region. Do this by giving an explicit vector $\mathbf{c}$ for which this set is $\text{face}_{\mathbf{c}}(P)$, where P is the feasible region of the linear program.

(8) You know (at least) three ways to solve a matrix equation $B\mathbf{x} = \mathbf{b}$, where B is a $d \times d$ matrix and $\mathbf{b} \in \mathbb{R}^d$:

(a) Compute the matrix B^{-1}, and then do the multiplication $B^{-1}\mathbf{b}$.

(b) Use Cramer's rule: x_i equals the quotient $\det(B_i)/\det(B)$, where B_i is the matrix obtained by replacing the ith column of B by $\mathbf{b}$.

(c) Use Gaussian elimination: Row-reduce the $d \times (d + 1)$ augmented matrix whose last column in $\mathbf{b}$, and then read off the solution from the row-reduced form.

In this question you will see one reason why you should use the third option in practice. There are other reasons (see numerical analysis classes).

(a) Count the number of multiplications need to solve the equation

$$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

using each of the three methods. Show your working clearly.

(b) Multiplying a row by a scalar takes d multiplications. Calculate how many multiplications it takes to solve for $\mathbf{x}$ using each of the three methods. For the first one you will need to think about the best way to compute the inverse, and for the second you will need to think about the best way to compute the determinant.