252 COMBINATORIAL OPTIMIZATION

HOMEWORK 2, 2014

These exercises are at a mixture of levels. There may be typos or mistakes; please let me know as soon as you find one. I'm happy to discuss solution attempts in my office hour (Monday at 12pm).

Hand in Questions 1, 2, 3 and 6 to the box outside the undergraduate office by **Thursday, 30th January, at 2pm**.

You are *strongly* encouraged to do the other questions at this time as well, though you should not hand them in. You also are encouraged to work on these problems in groups, though your final write-up should be your own.

(1) For each of the graphs shown in Figure 1, compute the shortest path from a to z (list it explicitly), and also compute the distance from a to all other vertices.

(2) Given a weighted digraph G, and collections of vertices $R, S \subset V$, consider the problem of finding a shortest length path joining a vertex $r \in R$ to a vertex $s \in S$. Reduce this to the shortest path problem considered in class.

(3) Give an example to show that Dijkstra's algorithm can give incorrect results if negative costs are allowed.

(4) Show, by giving an example, that a subpath of a shortest simple path need not be a shortest simple path, if a negative-cost circuit exists.

(5) A digraph G is acyclic if it has no circuits. Show that if G is acyclic then we can find the shortest path from a vertex v to all other vertices in time $O(|E|)$. Hint: Choose a careful ordering on the vertices, and thus on the edges.

(6) This question is a (compulsory!) review of some linear algebra that we will use later.

  (a) Compute the row reduced form of the following matrix:

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 & -1 \\ 1 & 1 & 0 & 2 & -1 \\ 1 & 0 & 3 & 2 & 0 \end{pmatrix}.$$

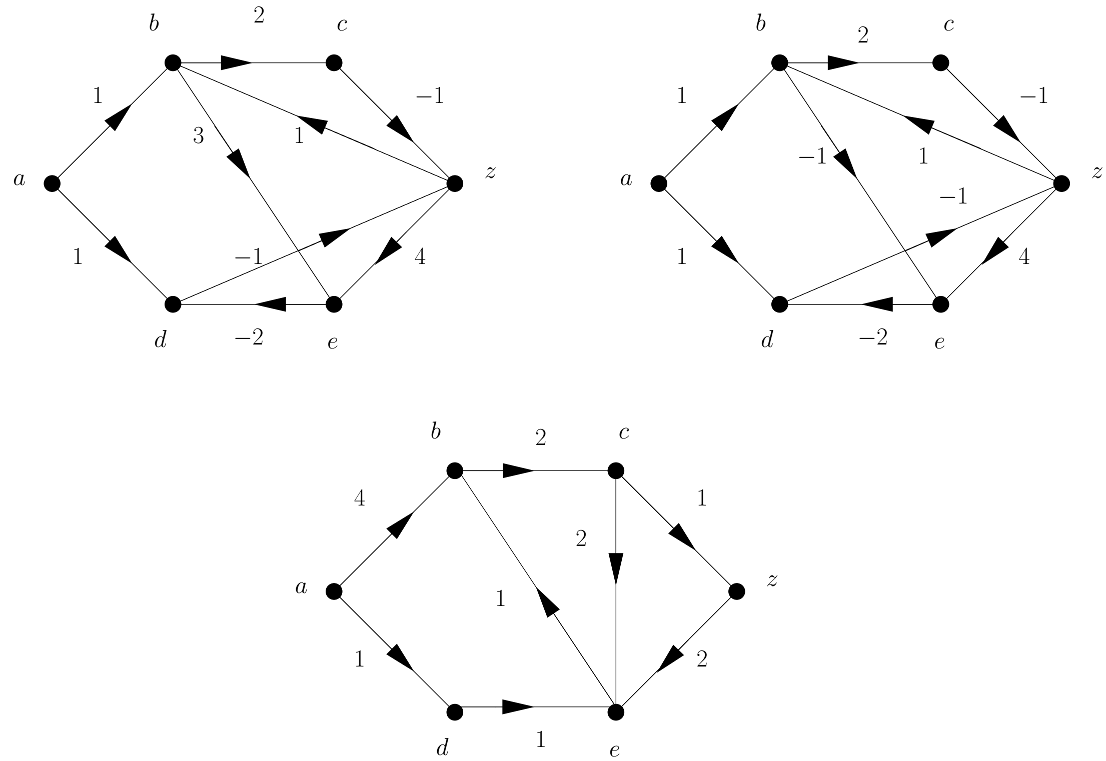

FIGURE 1.

(b) Let B be the 3×3 matrix consisting of the first three columns of A:

$$B = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 0 & 3 \end{pmatrix}.$$

Compute the inverse B^{-1}.

(c) Compute $B^{-1}A$ by doing the multiplication.

(d) Write a few sentences explaining the connection between your answers to the first and third part.

(e) Let C be the matrix consisting of the second, third, and fourth columns of A:

$$C = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 0 & 3 & 2 \end{pmatrix}.$$

Compute $C^{-1}A$. Explain how this could be computed using row operations.