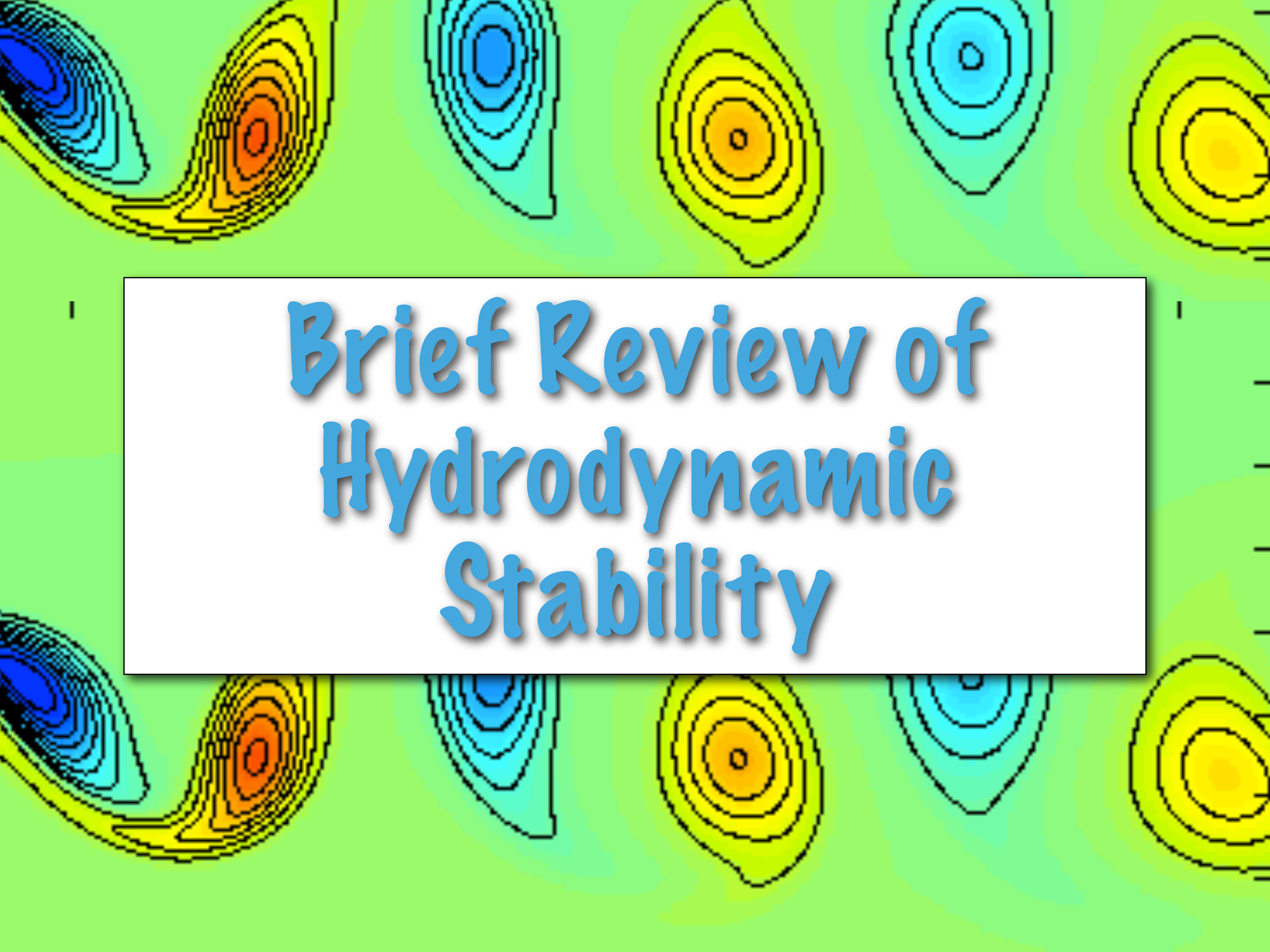
# Having Fun with Adjoints

$A^*A \neq AA^*$

$\langle y, Ax\rangle = \langle A^* y, x\rangle$

$A^*A$

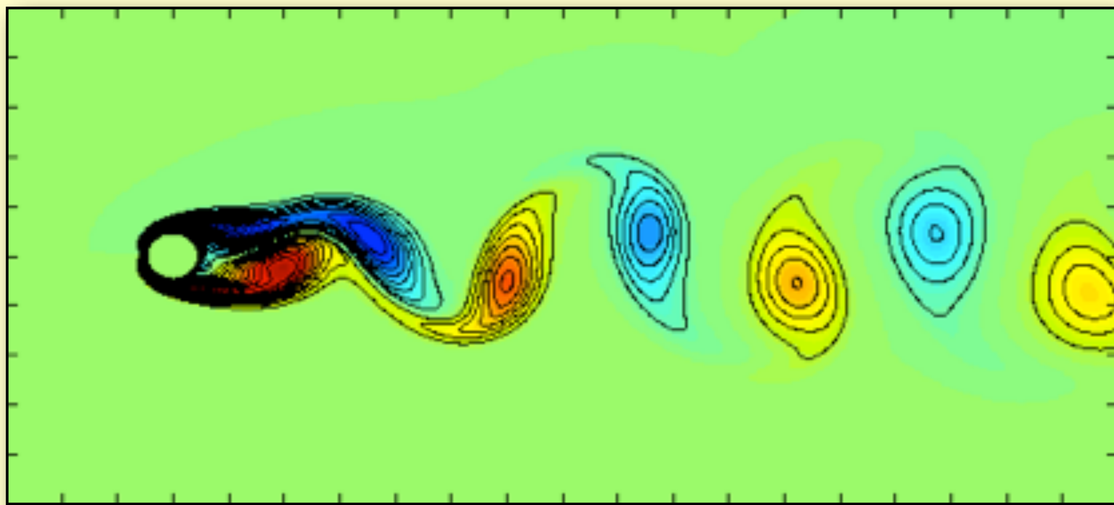
$(\partial_t + \partial_x + \partial_x^2)^* = (-\partial_t - \partial_x + \partial_x^2)$



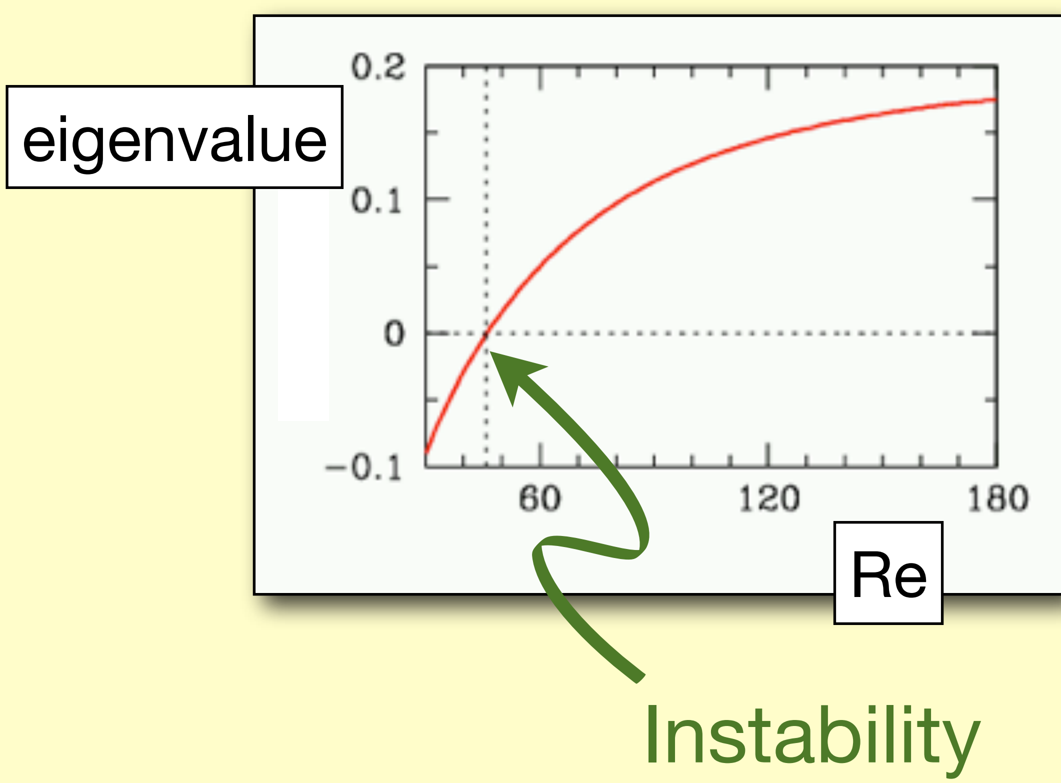
# Brief Review of Hydrodynamic Stability

# Two Examples:

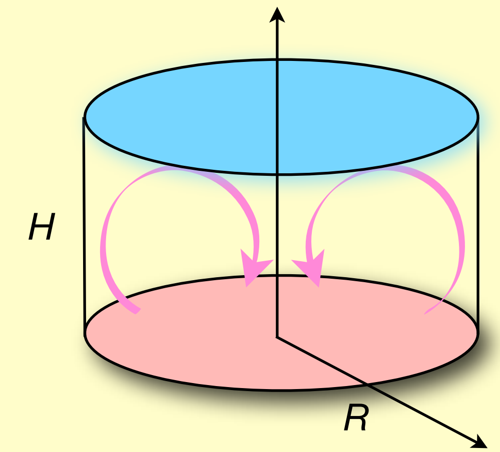
## Cylinder Wake

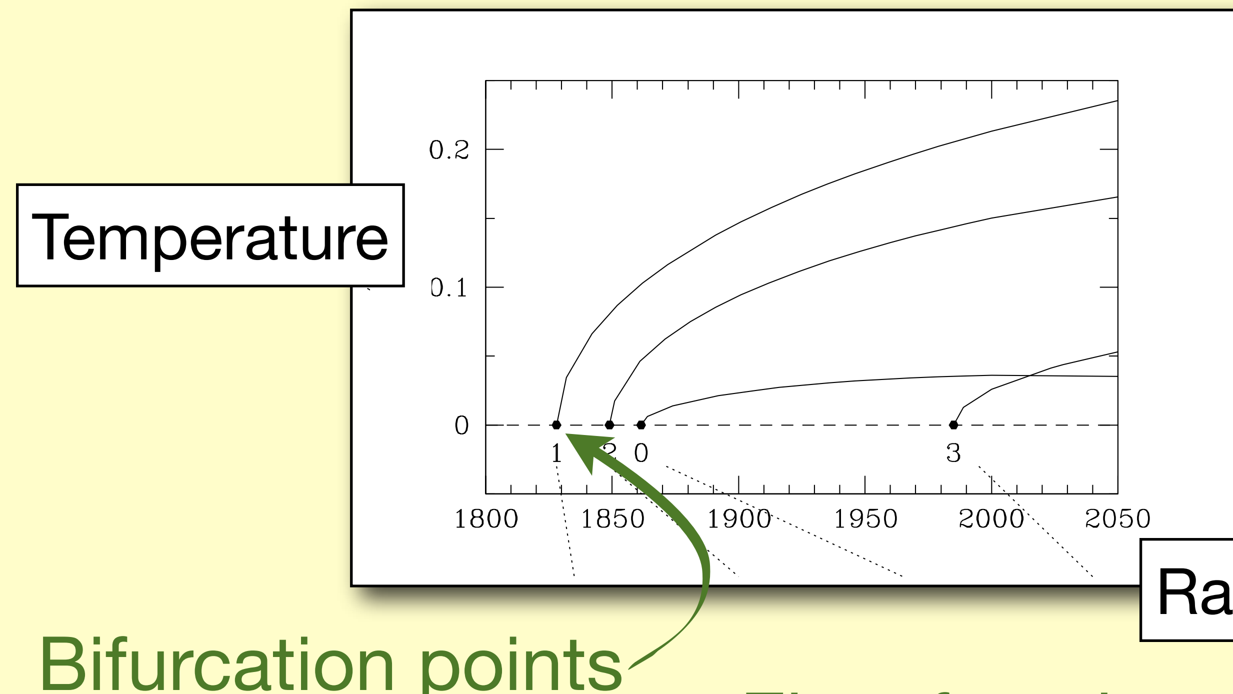


D Calhoun

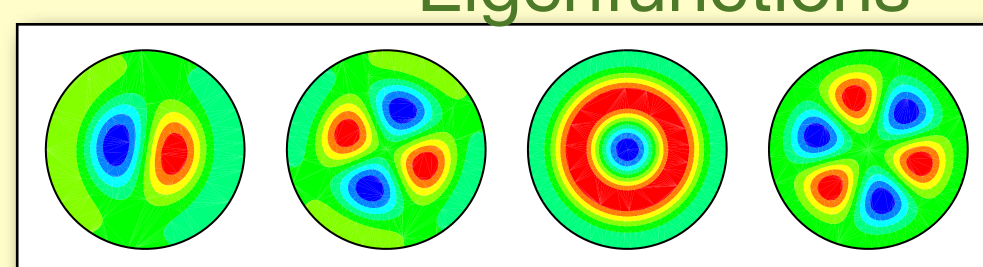



## Convection




Eigenfunctions



Boronska & Tuckerman

# Linear Stability Analysis

## Navier Stokes Equations

$$\partial_t \mathbf{u} = -(\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla p + \nu\nabla^2\mathbf{u}$$
$$\nabla\cdot\mathbf{u} = 0$$

### Base Solution

$$\mathbf{U}\ ,\quad P$$

### Infinitesimal Perturbation

$$\mathbf{U} + \epsilon\mathbf{u}'\ ,\quad P + \epsilon p'$$

## Linear Stability Equations

$$\partial_t \mathbf{u}' = -(\mathbf{U}\cdot\nabla)\mathbf{u}' - (\mathbf{u}'\cdot\nabla)\mathbf{U} - \nabla p' + \nu\nabla^2\mathbf{u}'$$
$$\nabla\cdot\mathbf{u}' = 0$$

### Linear Evolution

$$\partial_t \mathbf{u}' = \mathcal{L}\,\mathbf{u}'$$

### Modal Solution

$$\mathbf{u}'(\mathbf{x}, t) = \exp(\lambda t)\tilde{\mathbf{u}}(\mathbf{x})$$

## Eigenvalue Problem

$$\mathcal{L}\tilde{\mathbf{u}} = \lambda\tilde{\mathbf{u}} = (\sigma + i\omega)\tilde{\mathbf{u}}$$

## Instability

$$\sigma > 0$$

# Timestepper Approach

## Navier Stokes Equations

$$\partial_t \mathbf{u} = -(\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla p + \nu\nabla^2\mathbf{u}$$
$$\nabla\cdot\mathbf{u} = 0$$

## Linear Stability Equations

$$\partial_t \mathbf{u}' = -(\mathbf{U}\cdot\nabla)\mathbf{u}' - (\mathbf{u}'\cdot\nabla)\mathbf{U} - \nabla p' + \nu\nabla^2\mathbf{u}'$$
$$\nabla\cdot\mathbf{u}' = 0$$

DNS

Nonlinear Evolution

$$\mathbf{u}(t) = \mathrm{DNS}(\mathbf{u}(0))$$

Linear Evolution

$$\mathbf{u}'(t) = \mathcal{A}(t)\mathbf{u}'(0)$$

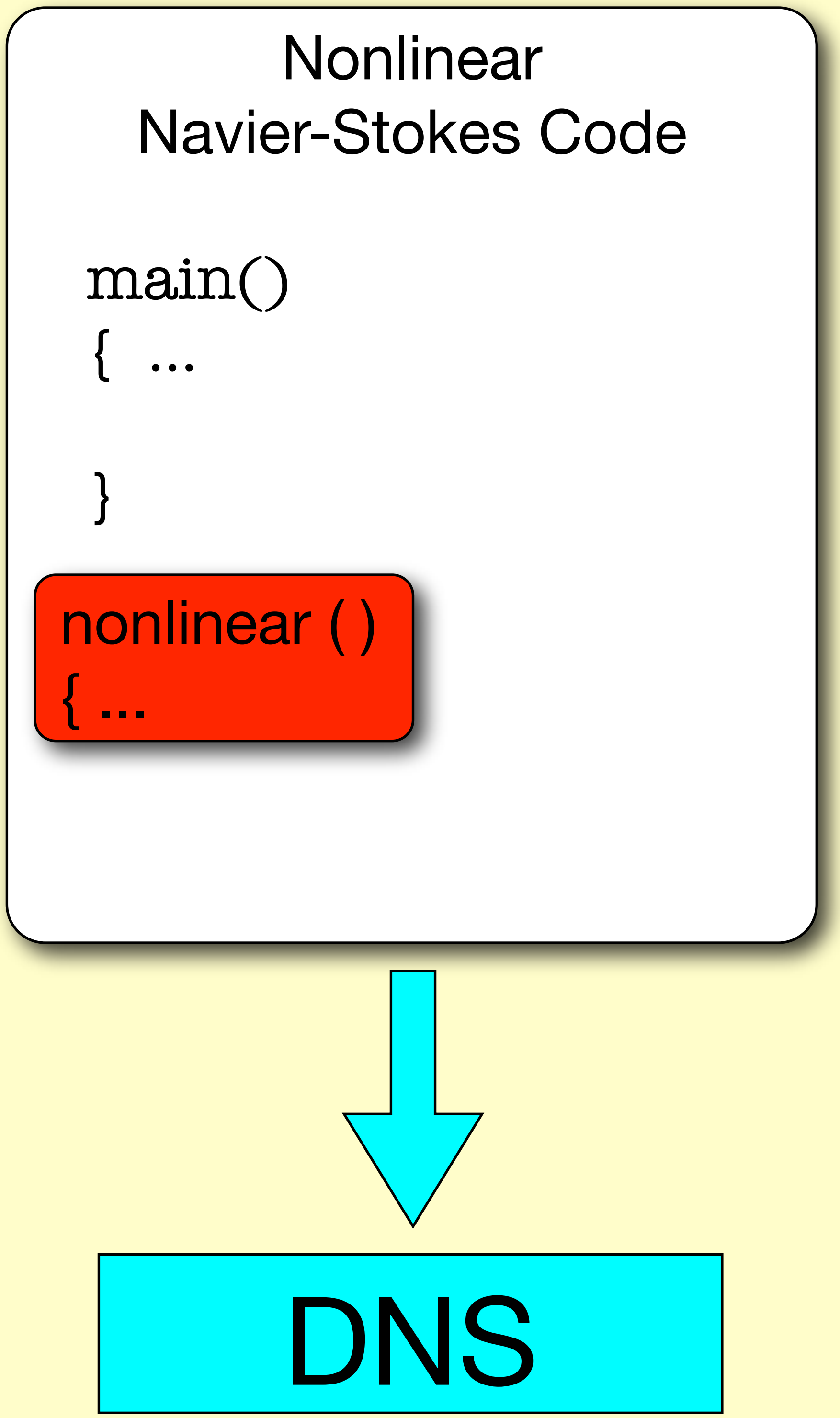
Fix a time interval $T$ and re-express eigenvalue problem $\mathcal{L}\tilde{\mathbf{u}} = \lambda\tilde{\mathbf{u}}$ in terms of $\mathcal{A}(T)$

## Eigenvalue Problem

$$\mathcal{A}(T)\tilde{\mathbf{u}} = \mu\tilde{\mathbf{u}} \qquad \mu = \exp(\lambda T)$$

Solve iteratively using matrix-free technique

# Timestepper Approach



# Timestepper Approach

Driver(s)

Linear stability analysis

$$\mathbf{u}_0, \mathbf{A}\mathbf{u}_0, \mathbf{A}^2\mathbf{u}_0, \ldots$$

$$\mathbf{A}\mathbf{Q}_k = \mathbf{Q}_k\mathbf{H}_k + h^*\mathbf{q}_k\hat{\mathbf{e}}_k^T$$

Nonlinear
Navier-Stokes Code

```
sub ()
{  ...

}
```

```
nonlinear ( )
{ ...
```

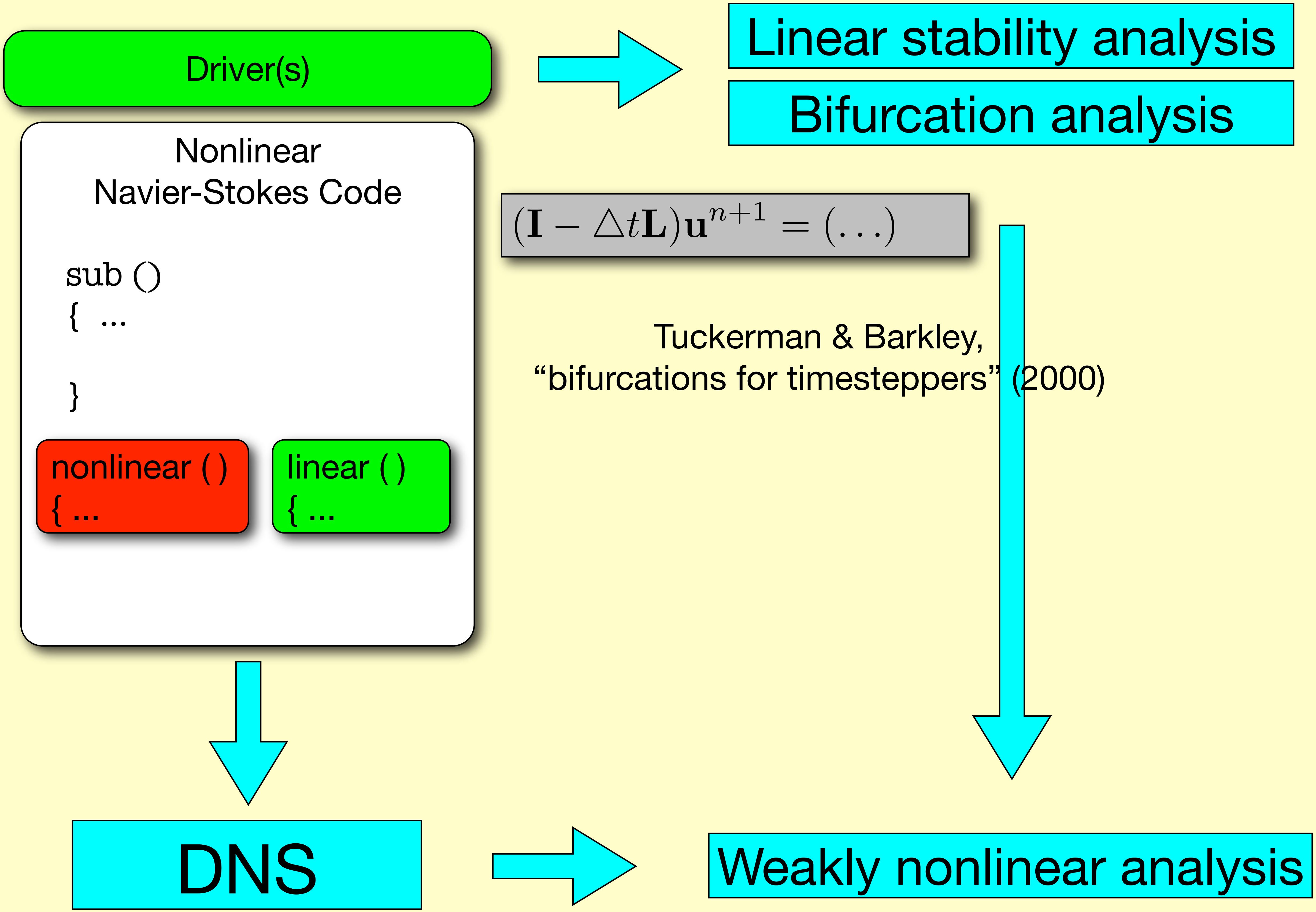
```
linear ( )
{ ...
```

$$(\mathrm{D}\mathbf{N}\mathbf{u})|_j = U_i \partial_i u_j + (\partial_i U_j) u_i$$

DNS

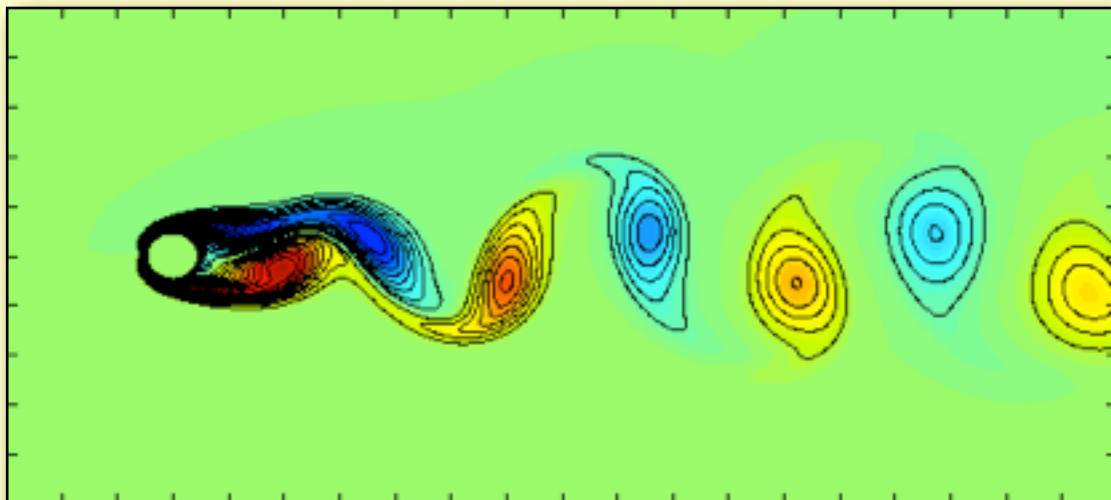
Tuckerman, PhD Thesis (1984), Marcus & Tuckerman (1987)
Goldhirsch, Orszag, and Maulik, J. Sci. Comp. 2, (1987)
Edwards, et al "Krylov methods for the incompressible Navier-Stokes equations" (1994)

# Timestepper Approach

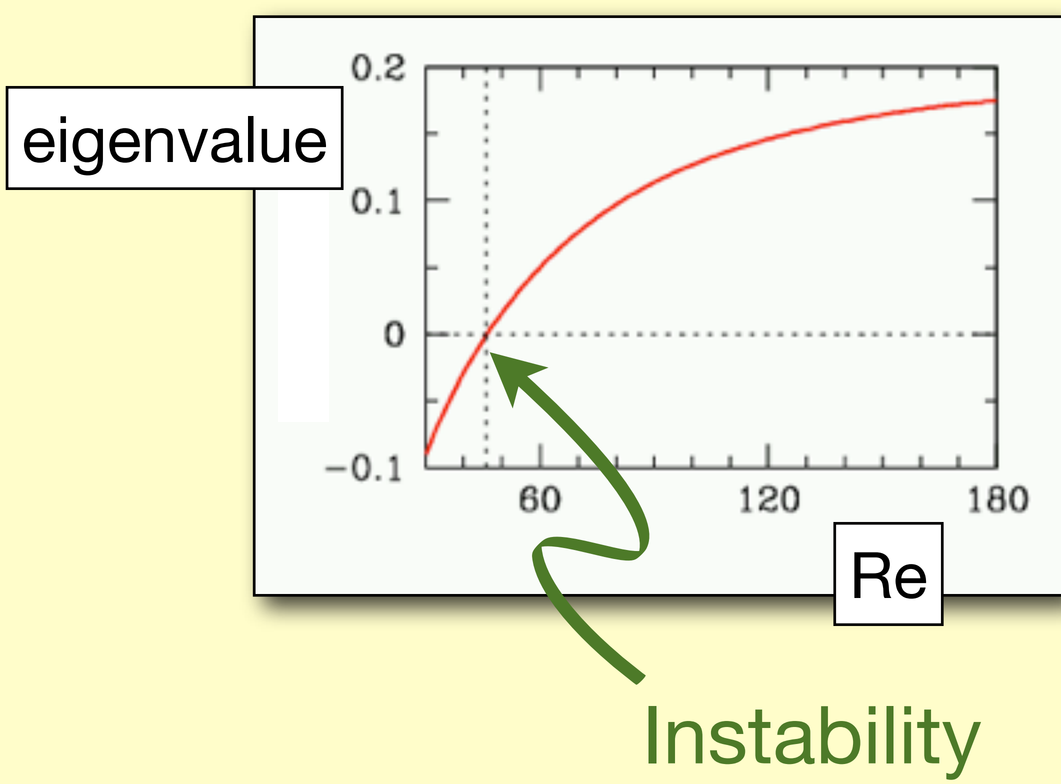


# Two Examples:

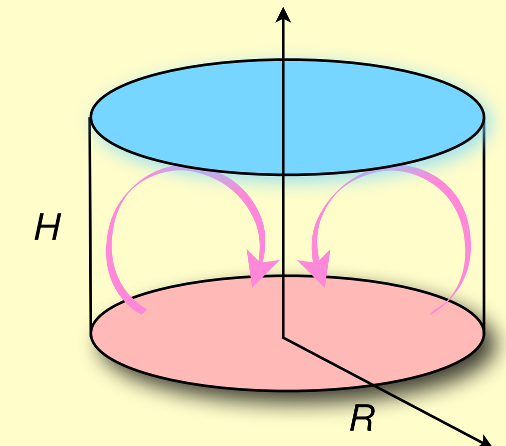
## Cylinder Wake

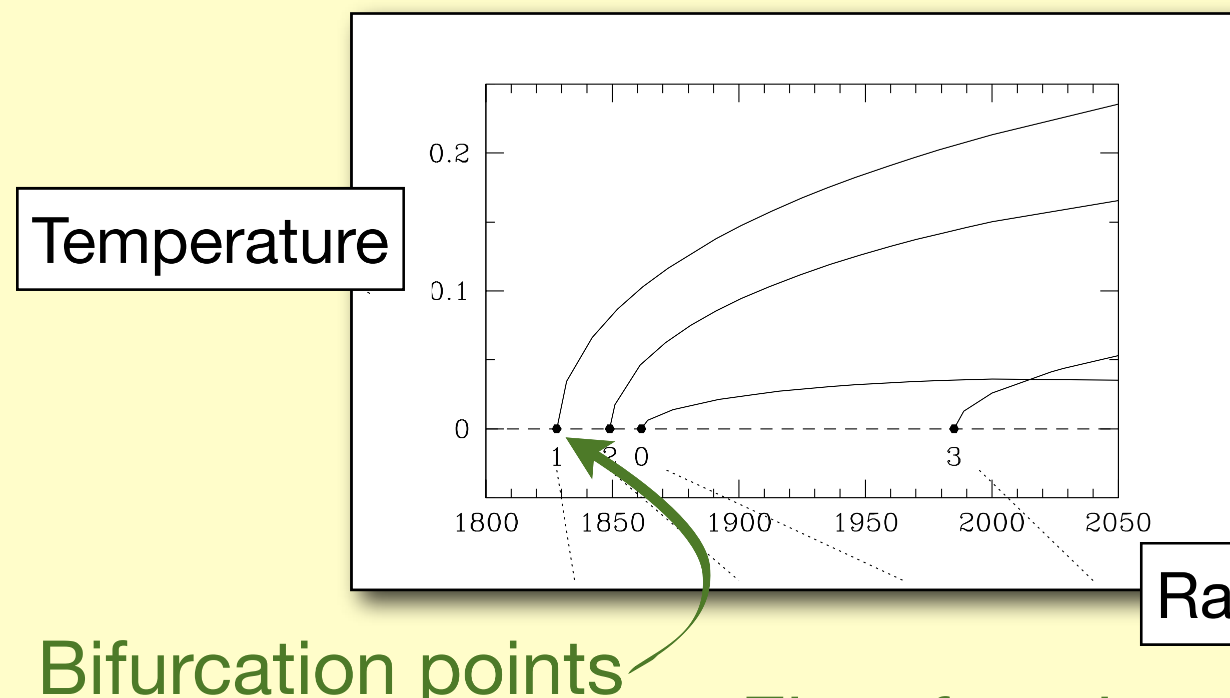


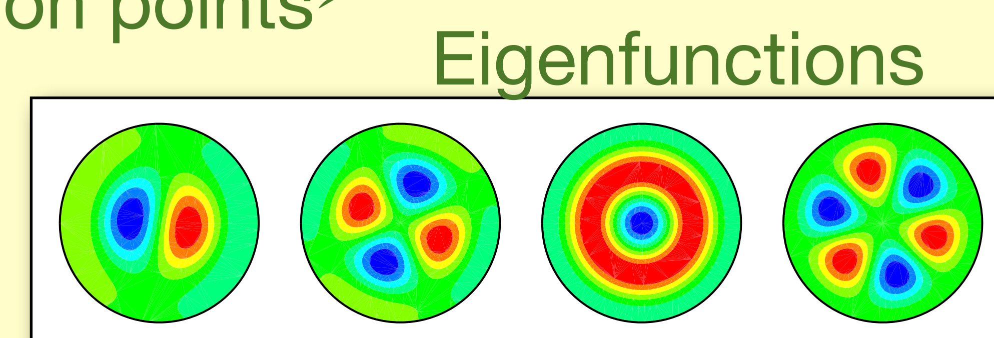
D Calhoun

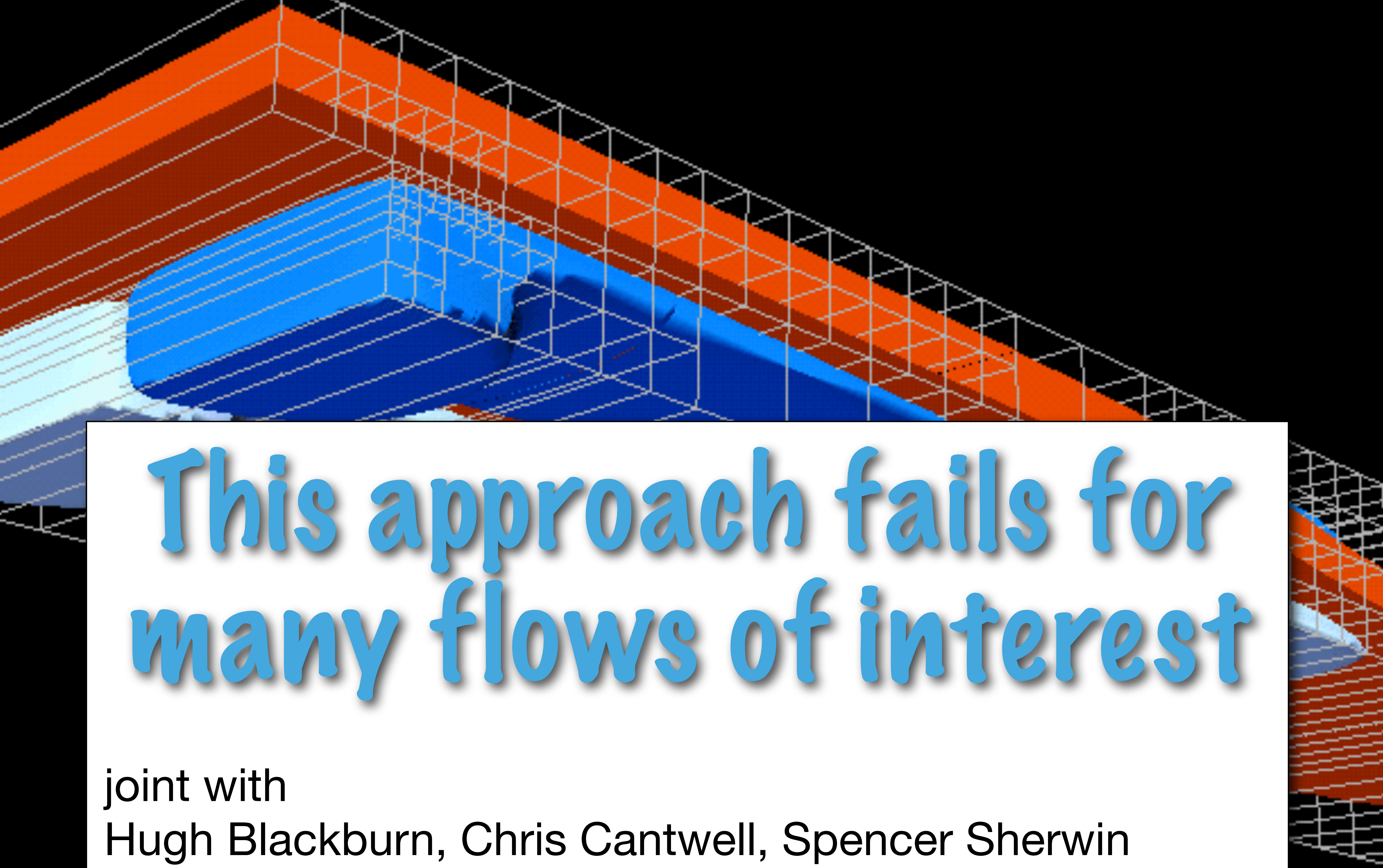



## Convection





# This approach fails for many flows of interest

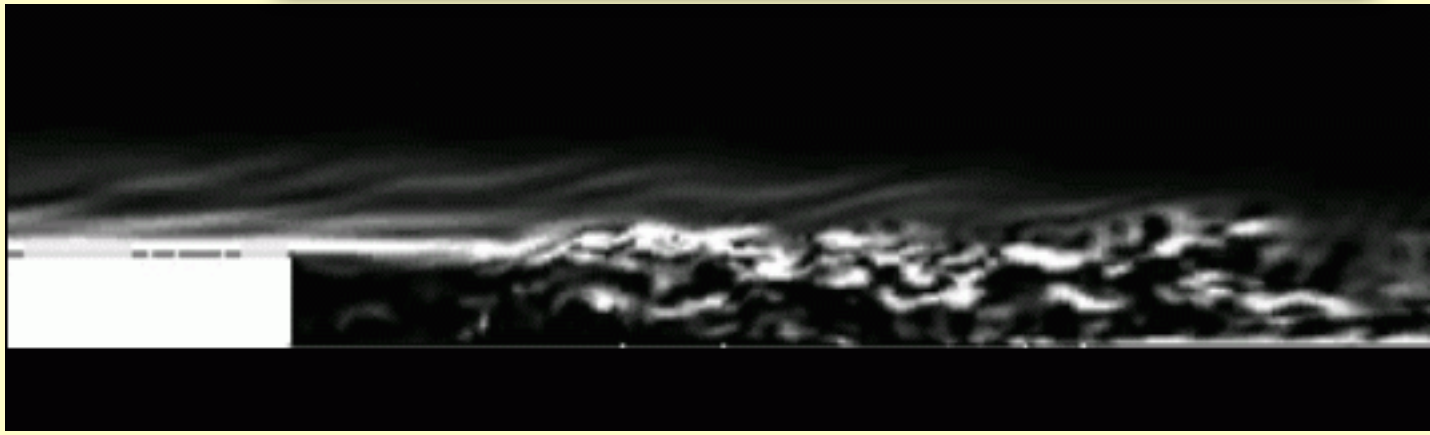
joint with
Hugh Blackburn, Chris Cantwell, Spencer Sherwin

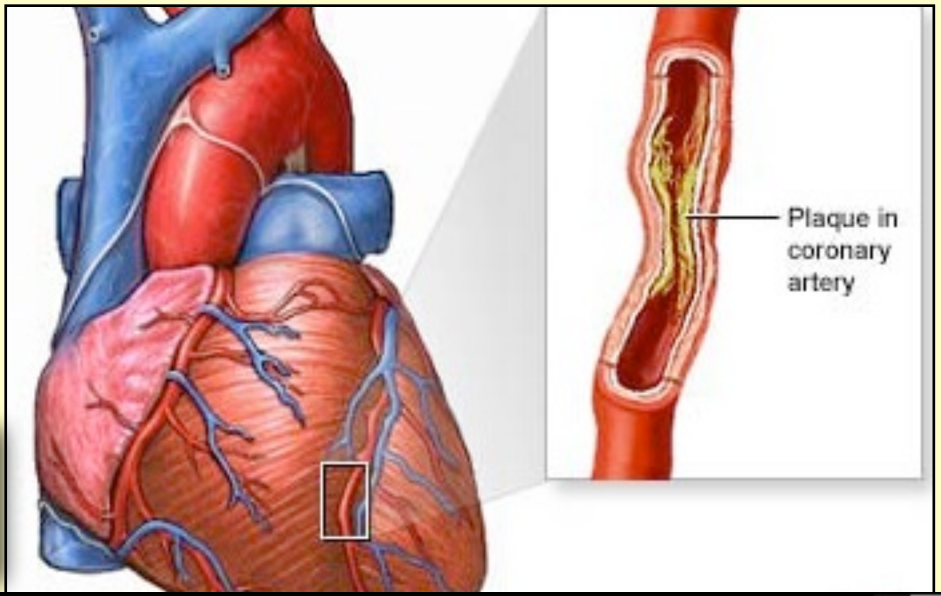
# Examples

Expanding Pipe

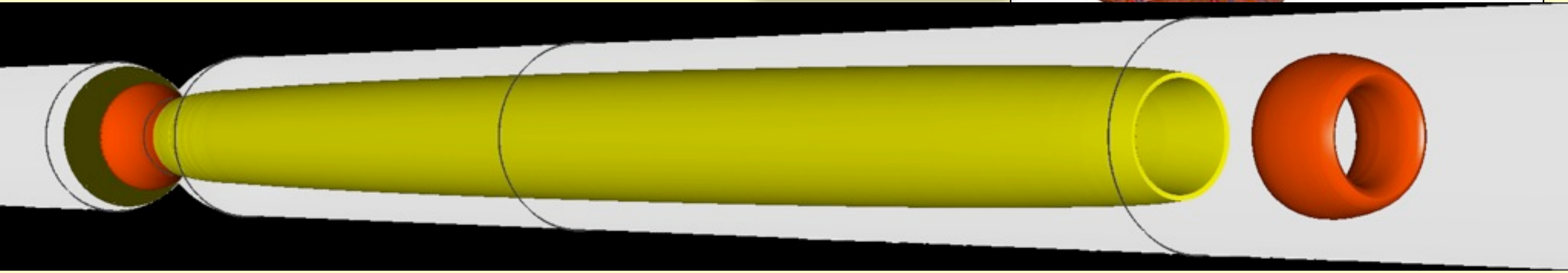
Backward-Facing Step

**Xiaohua Wu, George Homsy and Parviz Moin**



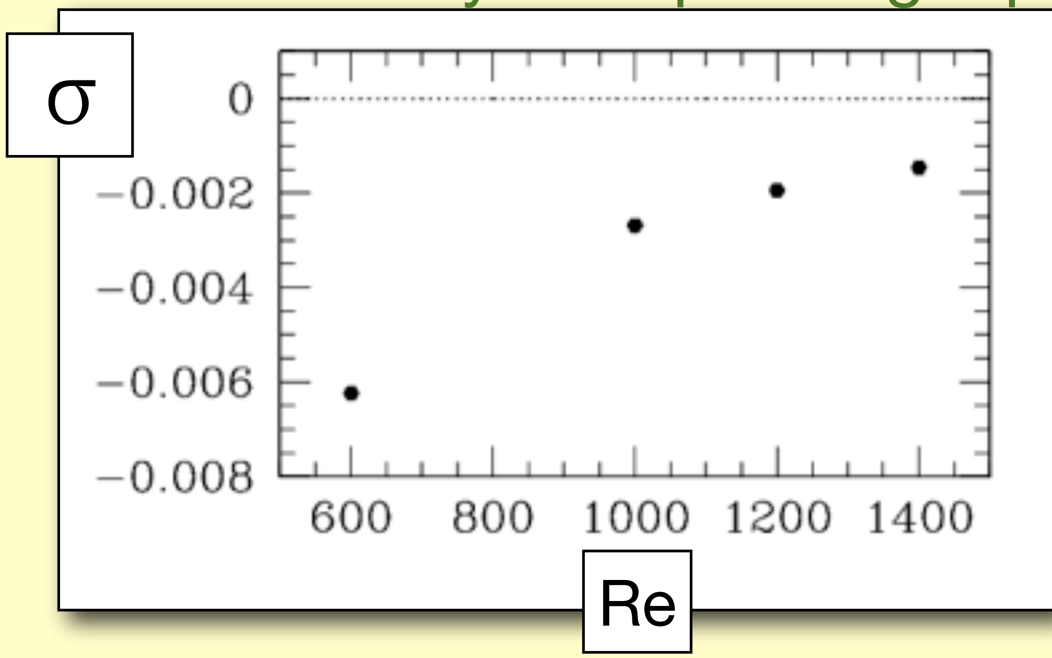



Stenosis



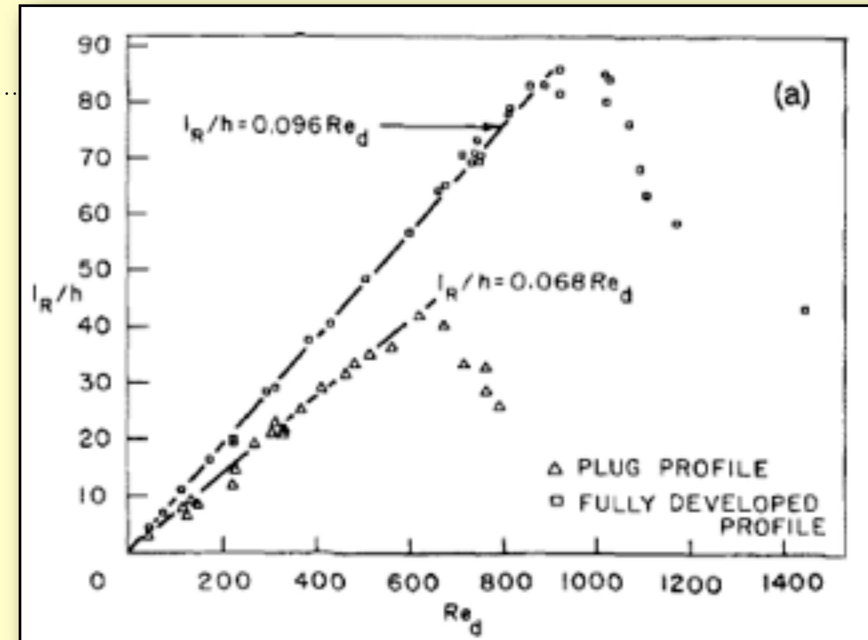
# Expanding Pipe

Numerical Computations of
Linear Stability of Expanding Pipe




Experiments

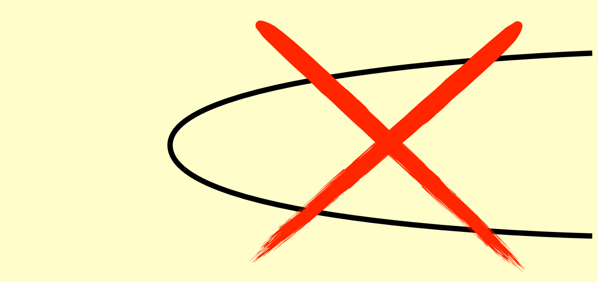
(Latornell and Pollard, Phys Fluids 1986)




Hall *et al*

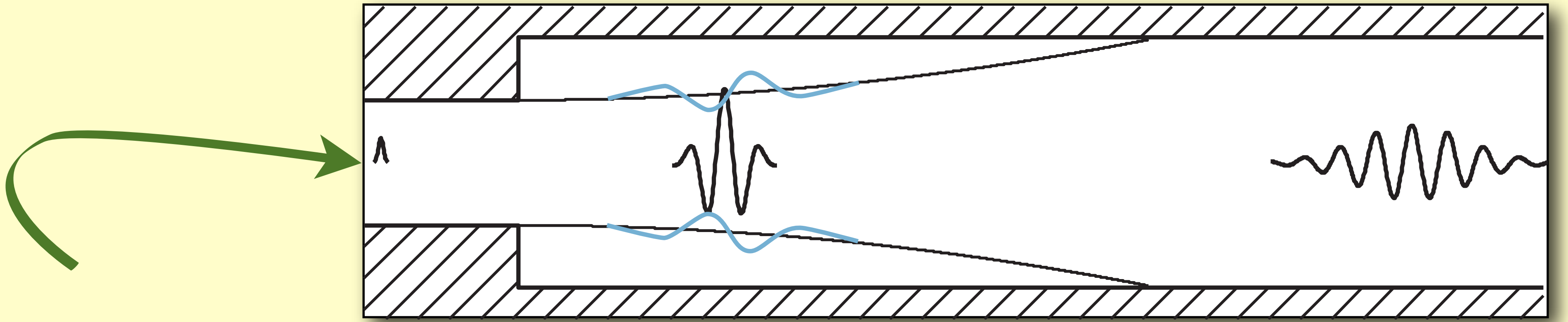
- Flow is linearly stable to large *Re*
- Flow undergoes oscillations beyond a poorly defined Re
- Nonlinearity is stabilizing and plays no significant role

(not subcritical instability)



# Fluid Dynamics

## Convectively unstable shear layer



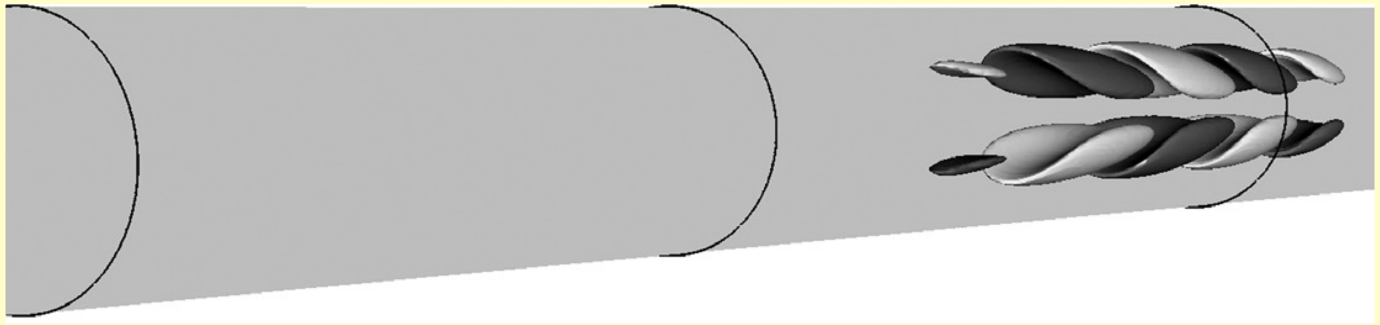
small perturbation
in upstream pipe

amplified by
highly unstable shear layer

advected downstream
where it decays

## How to really compute

- spatially developing flow
- non-trivial structures



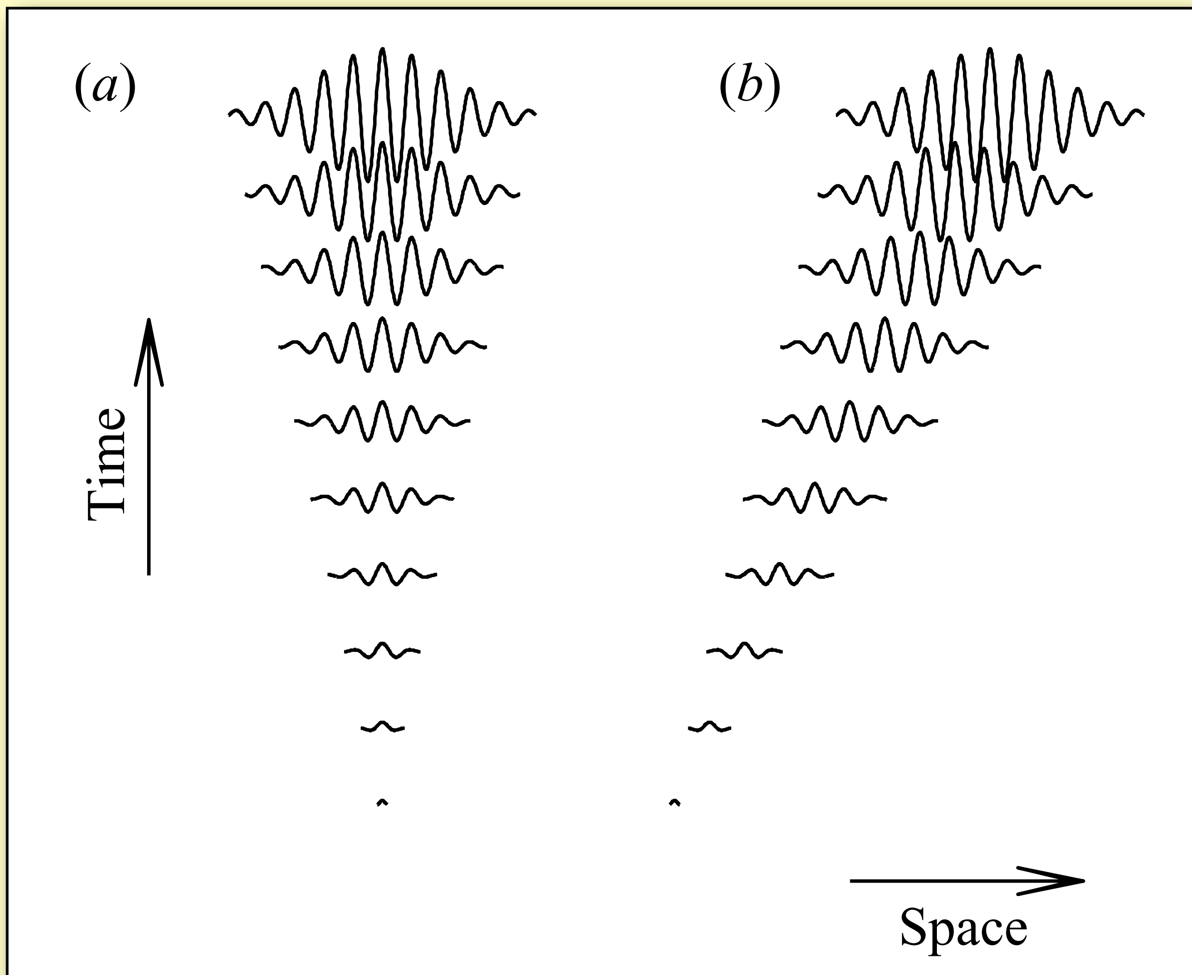
# Localized Convective Instability

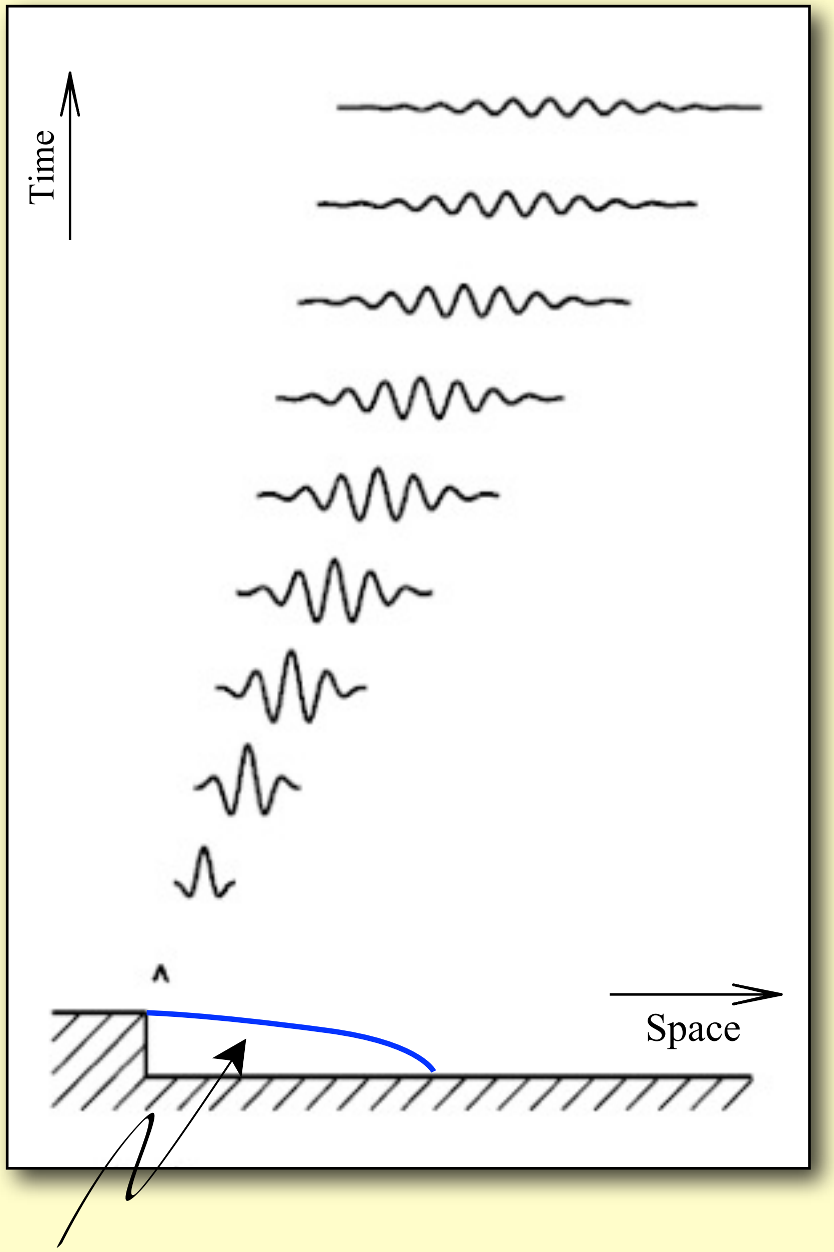
## homogeneous flow

## inhomogeneous flow

Absolute Instability

Convective Instability




The flows are linearly unstable and instability can be found by computing eigenvalues

$$||\mathbf{u}'(x,t)|| \sim e^{\lambda t + ikx}$$

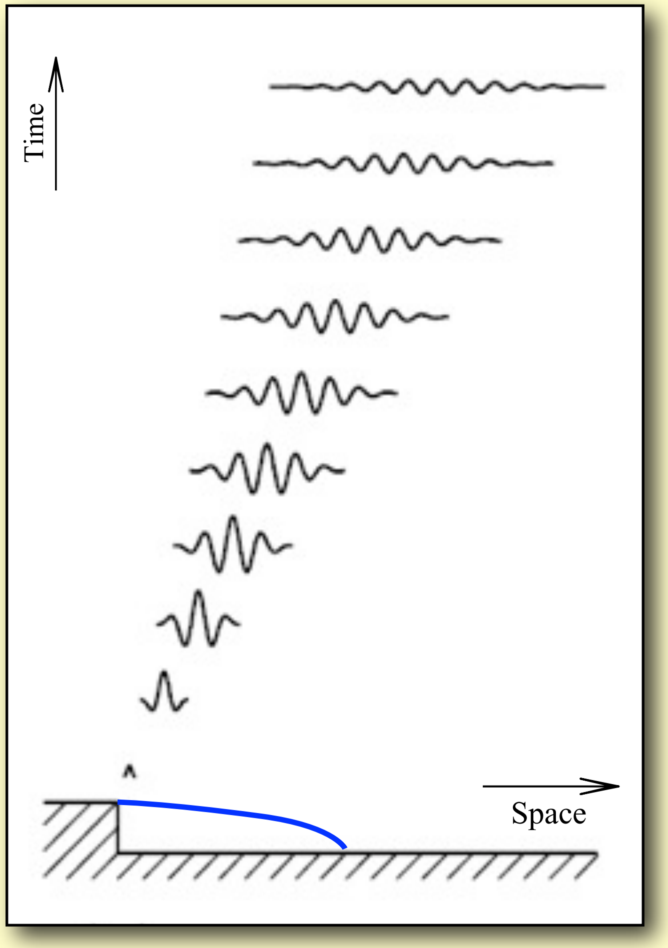
Localized region of convective instability.
The flow is linearly stable.
Dynamics can not be found by eigenvalues

# 2-Second History

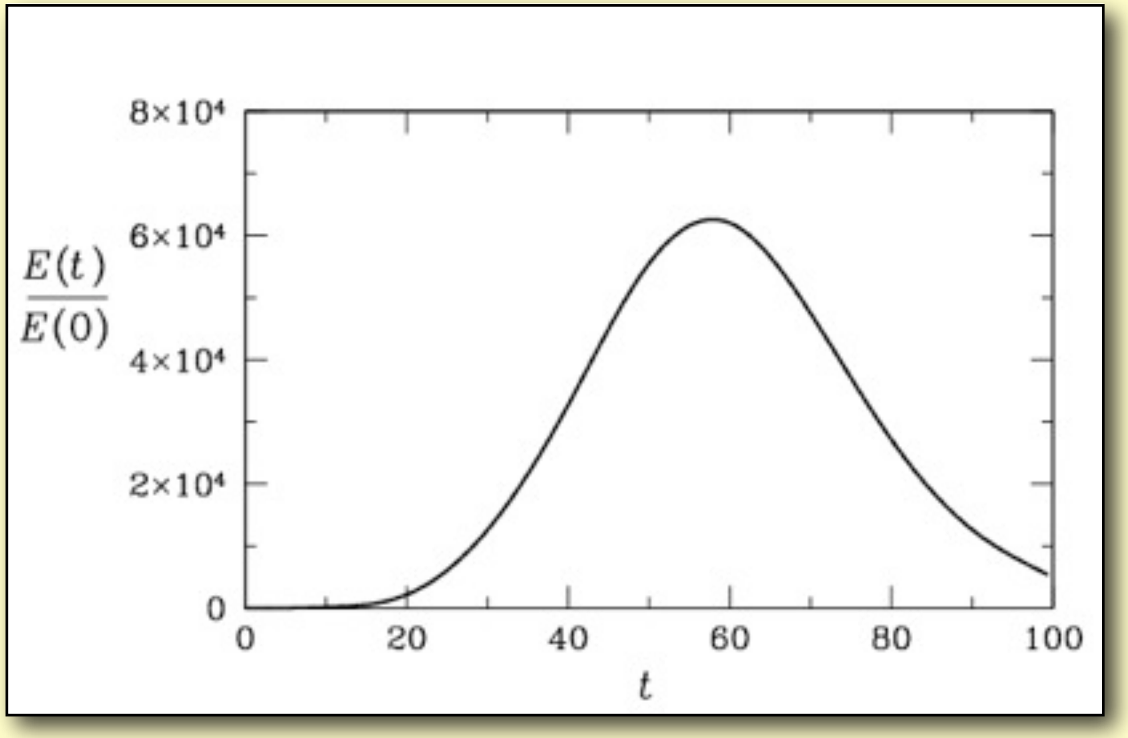
...
L. Gustavsson, J. Fluid Mech. 224, 241 (1991).
K. Butler and B. Farrell, Phys. Fluids A 4, 1637 (1992).
...
L. N. Trefethen, D. Henningson, P. Schmid et al (1993+)
...

Transient Growth. Subcritical Transition to Turbulence

C. Cossu and J. M. Chomaz, Phys. Rev. Let. 78, 4387 (1997).




Localized convective instability and transient growth



# Optimal Energy Growth

$$(\mathbf{u}, \mathbf{v}) \equiv \int_\Omega \mathbf{u} \cdot \mathbf{v} \, dv$$

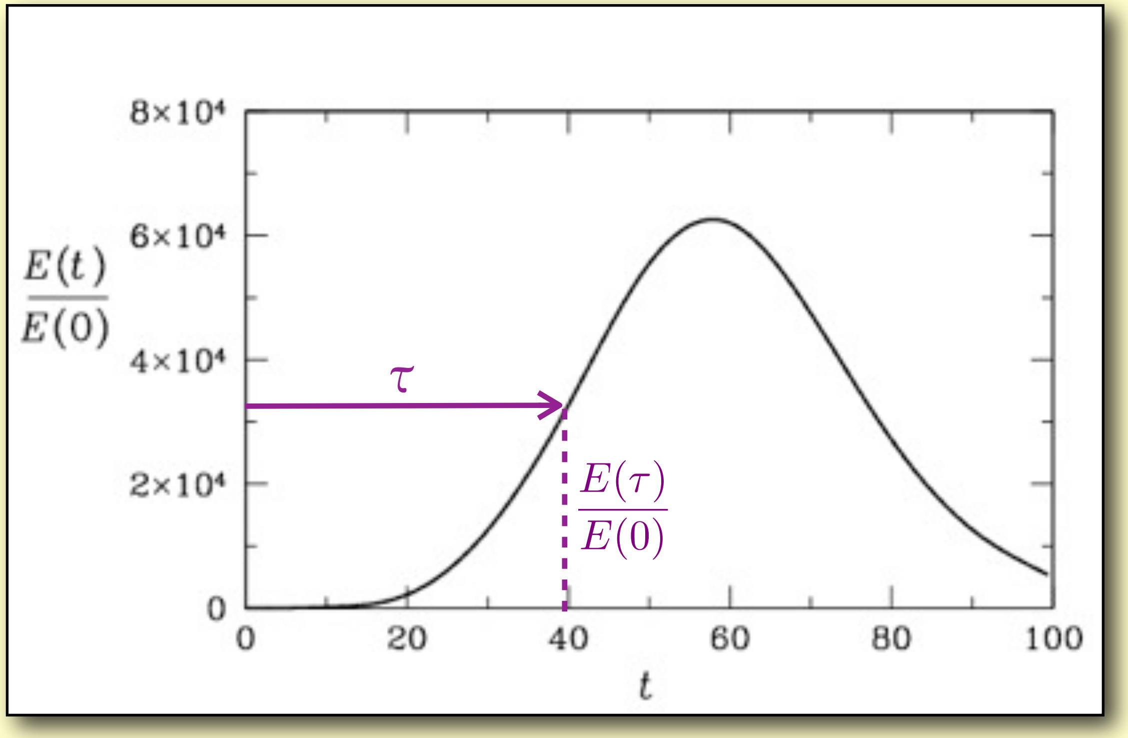
Start from normalized initial condition and look at evolved energy at t= $\tau$

$$||\mathbf{u}'(0)|| = 1$$

$$\frac{E(\tau)}{E(0)} = ||\mathbf{u}'(\tau)||^2 = (\ \mathbf{u}'(\tau)\ ,\ \mathbf{u}'(\tau)\ )$$

$$= (\ \mathcal{A}(\tau)\mathbf{u}'(0)\ ,\ \mathcal{A}(\tau)\mathbf{u}'(0)\ )$$

$$= (\ \mathbf{u}'(0),\ \mathcal{A}^*(\tau)\mathcal{A}(\tau)\mathbf{u}'(0)\ )$$




Consider eigenvalue problem

$$\mathcal{A}^*(\tau)\mathcal{A}(\tau)\mathbf{v}_j = \lambda_j \mathbf{v}_j \qquad ||\mathbf{v}_j|| = 1$$

Typically interested in largest (aka optimal) energy growth

$$G(\tau) = \max_j \lambda_j$$

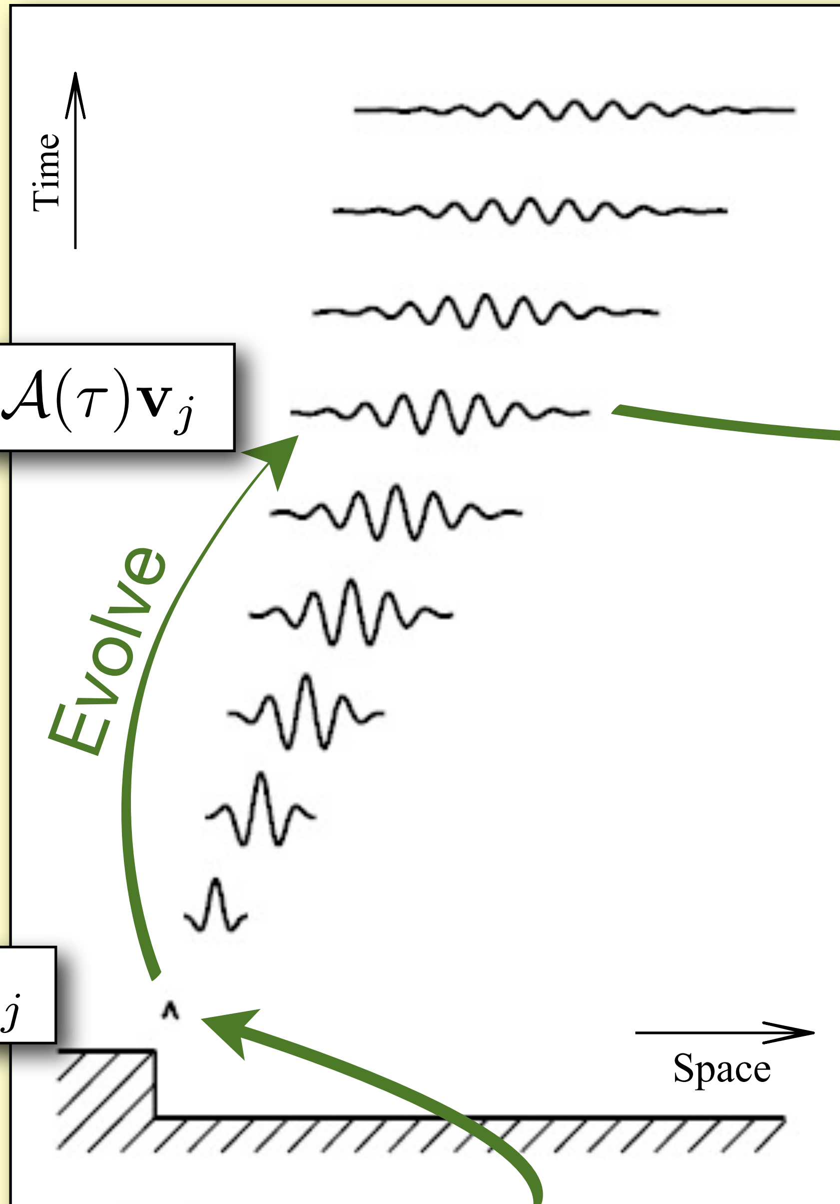
Starting from eigenfunction $v_j$ gives energy gain $\lambda_j$

$$\mathbf{u}'(0) = \mathbf{v}_j \qquad \frac{E(\tau)}{E(0)} = \lambda_j$$

# Equivalently in terms of SVD

$$\mathbf{u}'(\tau) = \mathcal{A}(\tau)\mathbf{v}_j$$

Evolve




$$\mathbf{u}'(0) = \mathbf{v}_j$$

Initialize with normalized eigenvector

$$\mathcal{A}^*(\tau)\mathcal{A}(\tau)\mathbf{v}_j = \lambda_j \mathbf{v}_j$$

normalize result

$$\mathbf{u}_j = \frac{\mathbf{u}'(\tau)}{\sigma_j} \qquad \sigma_j = ||\mathbf{u}'(\tau)||$$

to obtain

SVD $$\mathcal{A}(\tau)\mathbf{v}_j = \sigma_j \mathbf{u}_j$$

right singular vector (initial conditions) $||\mathbf{v}_j|| = 1$

singular value (amplification)

left singular vector (final conditions) $||\mathbf{u}_j|| = 1$

$$G(\tau) = \max_j \lambda_j = \max_j \sigma_j^2$$

# A little more formalism

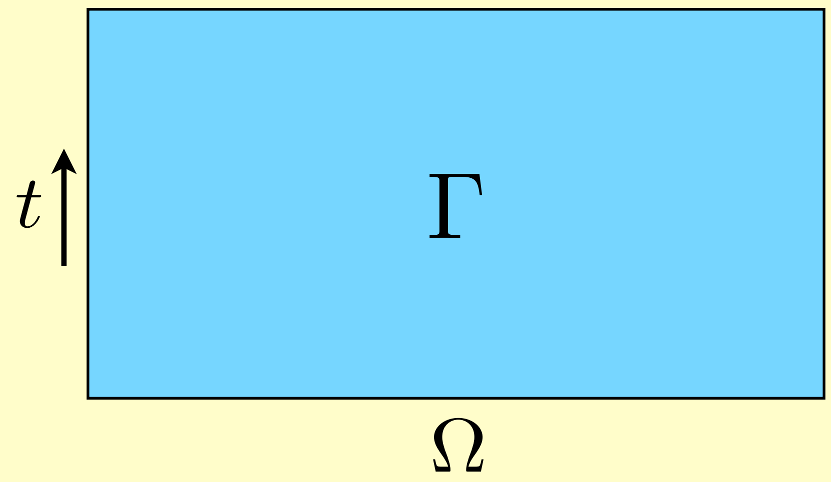
$$\mathbf{q} = \begin{pmatrix} \mathbf{u}' \\ p' \end{pmatrix} \qquad \langle \mathbf{q}, \mathbf{q}^* \rangle = \int_0^\tau \int_\Omega \mathbf{q}\cdot\mathbf{q}^* \, \mathrm{d}v \, \mathrm{d}t \qquad \mathbf{q}^* = \begin{pmatrix} \mathbf{u}^* \\ p^* \end{pmatrix}$$

**Linearized Navier Stokes Eqs**

$$\mathscr{H}\mathbf{q} = 0 \quad (\mathbf{x}, t) \in \Gamma$$

$$\mathbf{u}(t=0) = \mathbf{u}_0$$

$$\mathbf{u}(\partial\Omega) = 0$$




**Adjoint Linearized Navier Stokes Eqs**

$$\mathscr{H}^*\mathbf{q}^* = 0 \quad (\mathbf{x}, t) \in \Gamma$$

$$\mathbf{u}^*(t=\tau) = \mathbf{u}^*_\tau$$

$$\mathbf{u}^*(\partial\Omega) = 0$$

where

$$\mathscr{H} = \left[ \begin{array}{c|c} -\partial_t - \mathrm{DN} + Re^{-1}\nabla^2 & -\boldsymbol{\nabla} \\ \hline \boldsymbol{\nabla}\cdot & 0 \end{array} \right] \qquad \mathscr{H}^* = \left[ \begin{array}{c|c} \partial_t - \mathrm{DN}^* + Re^{-1}\nabla^2 & -\boldsymbol{\nabla} \\ \hline \boldsymbol{\nabla}\cdot & 0 \end{array} \right]$$

$$\mathrm{DN}\mathbf{u}' = (\mathbf{U}\cdot\nabla)\mathbf{u}' + (\mathbf{u}'\cdot\nabla)\mathbf{U} \qquad \mathrm{DN}^*\mathbf{u}^* = -(\mathbf{U}\cdot\nabla)\mathbf{u}^* + (\boldsymbol{\nabla}\mathbf{U})^{\mathrm{T}}\cdot\mathbf{u}^*$$

$$\boxed{\mathbf{u}(t+s) = \mathscr{A}(s)\mathbf{u}(t)} \qquad \boxed{\mathbf{u}^*(t-s) = \mathscr{A}^*(s)\mathbf{u}^*(t)}$$

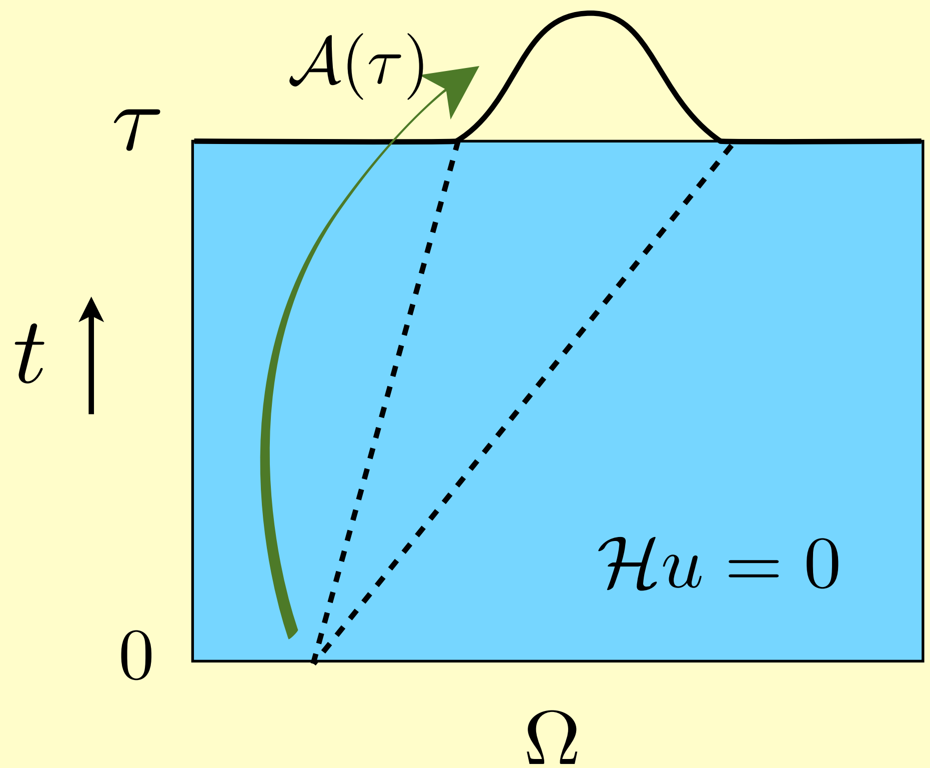
# A little intuition

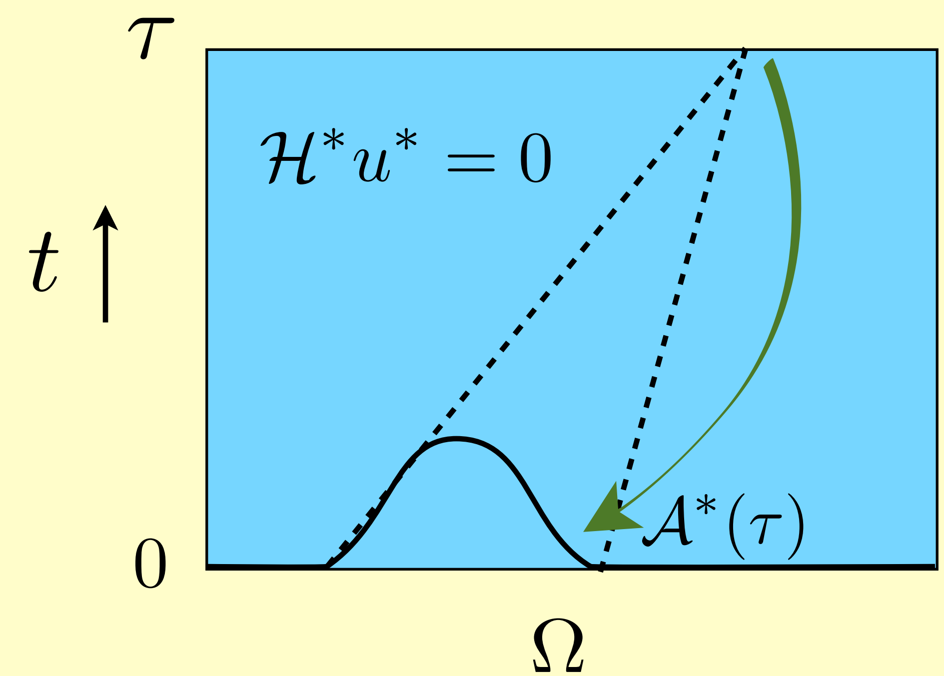
Advection-diffusion equation

$$\left(-\partial_t + \mu - c\partial_x + \partial^2_{xx}\right) u = 0$$

$$\left(\partial_t + \mu^* + c\partial_x + \partial^2_{xx}\right) u^* = 0$$

Green's functions



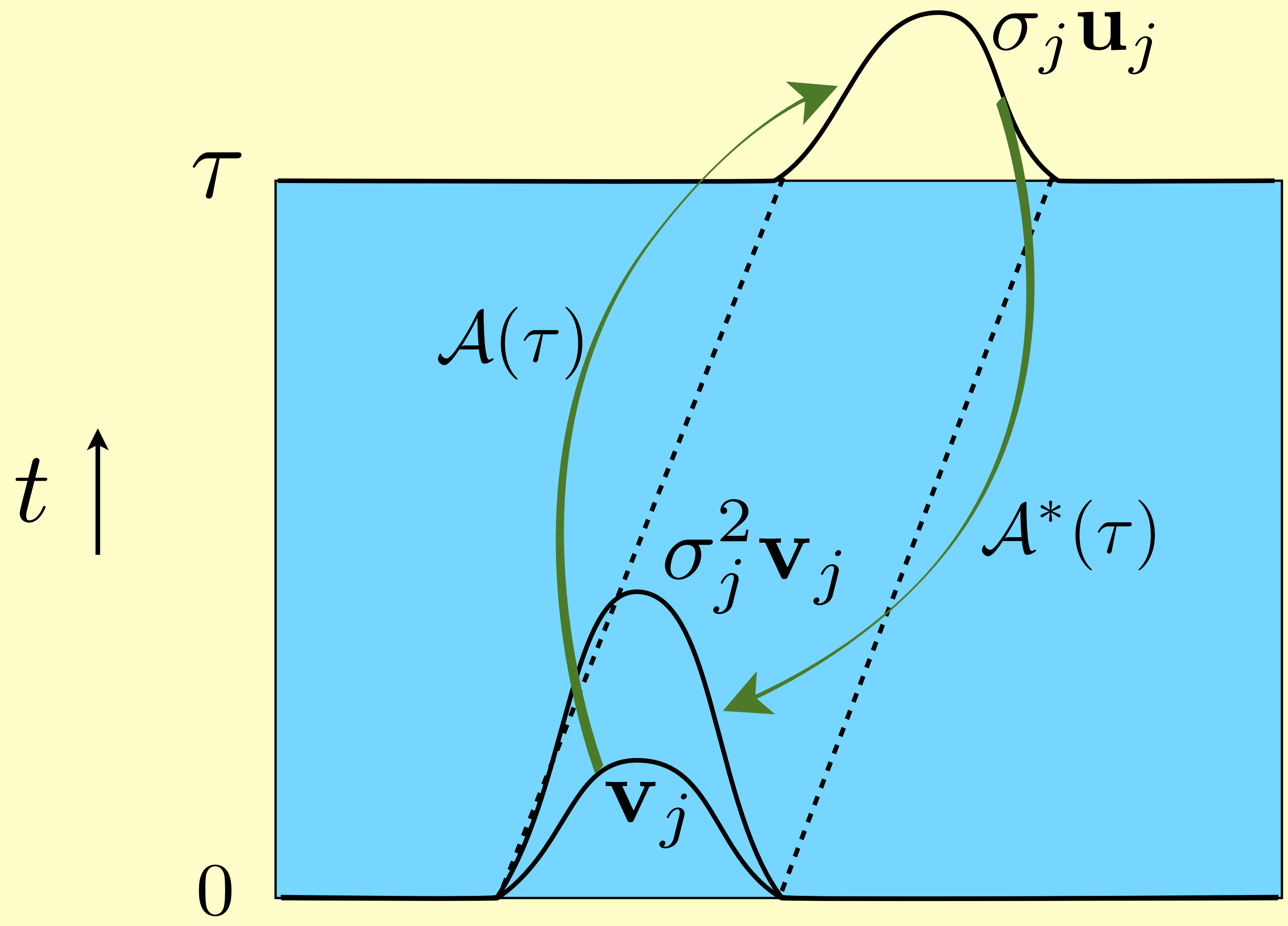
# A little more intuition

$$G(\tau) = \max_j \lambda_j = \max_j \sigma_j^2$$

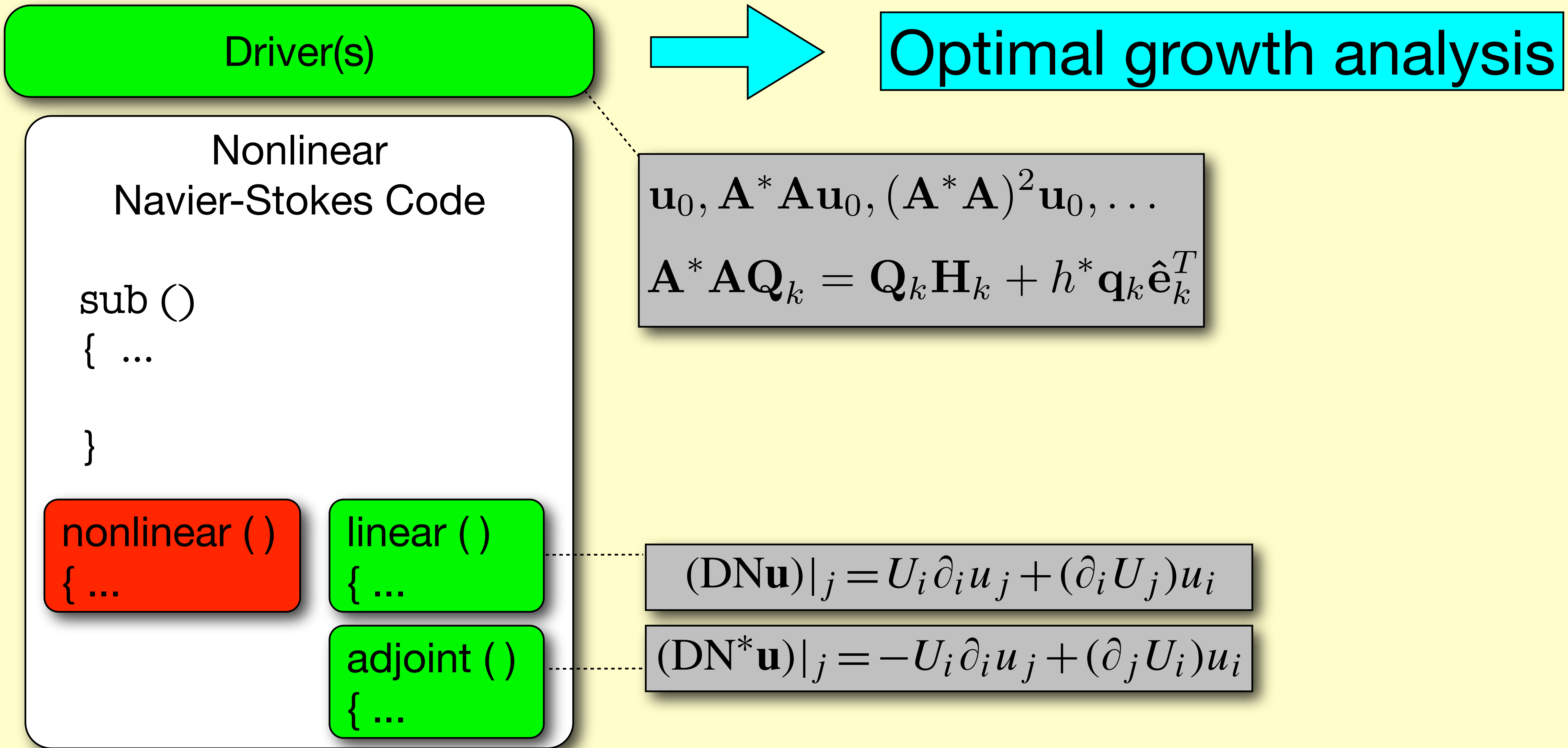
$$\mathcal{A}^*(\tau)\mathcal{A}(\tau)\mathbf{v}_j = \lambda_j \mathbf{v}_j$$

$$\mathcal{A}(\tau)\mathbf{v}_j = \sigma_j \mathbf{u}_j$$

$$\mathcal{A}^*(\tau)\mathbf{u}_j = \sigma_j \mathbf{v}_j$$



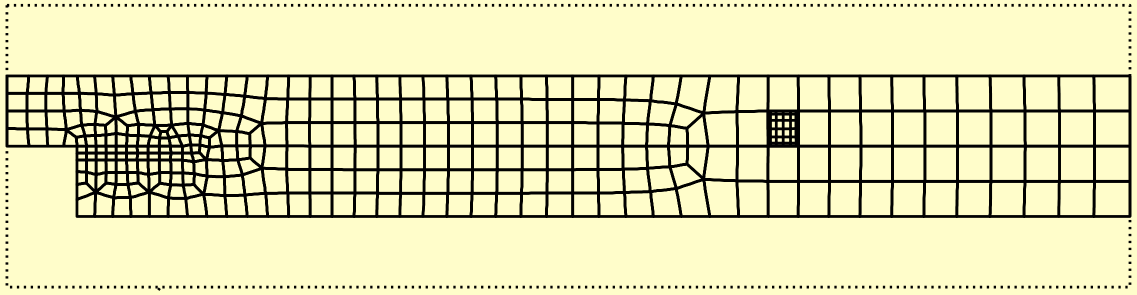
# Timestepper Approach



# Highlights of General Interest

Implemented in 3 independent spectral-element codes:

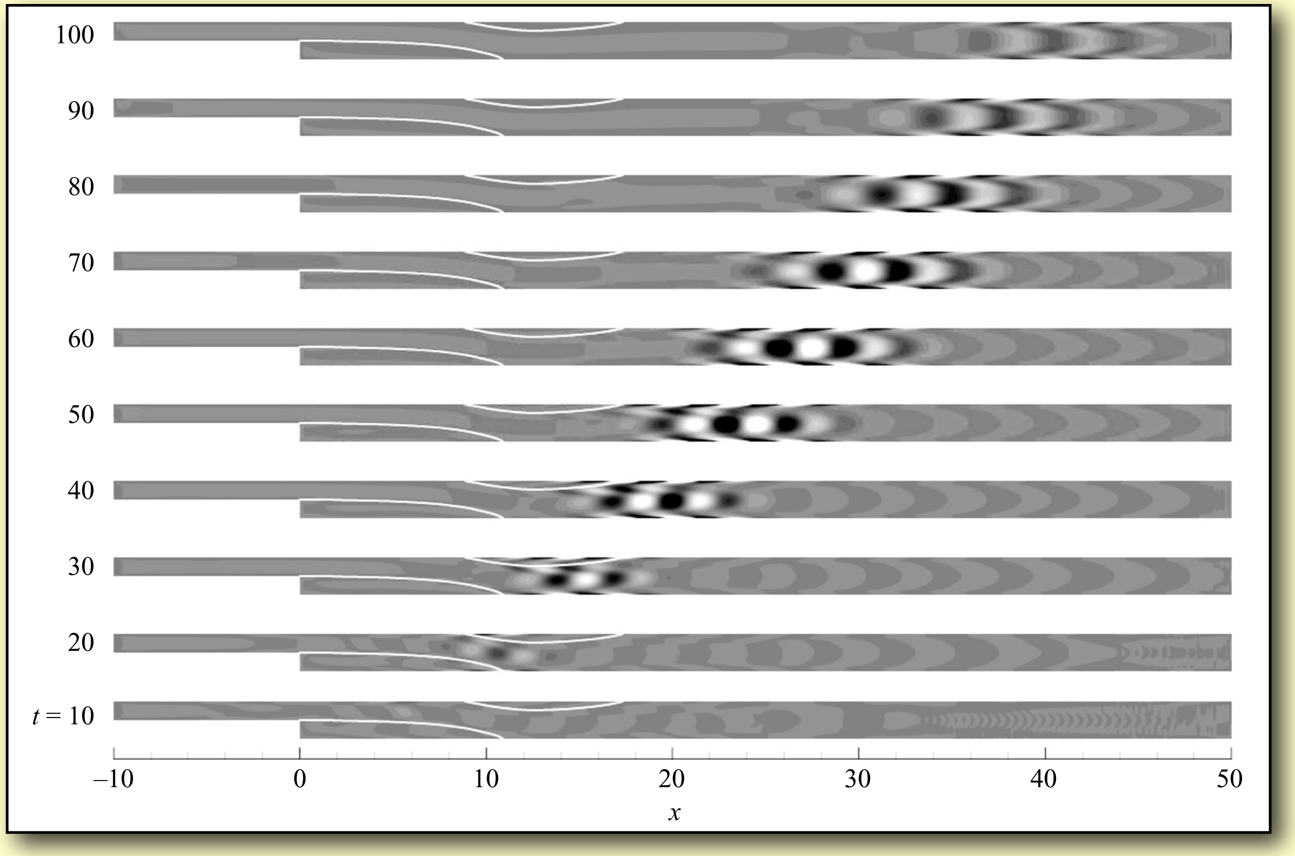
*Prism, Semtex, Nektar*



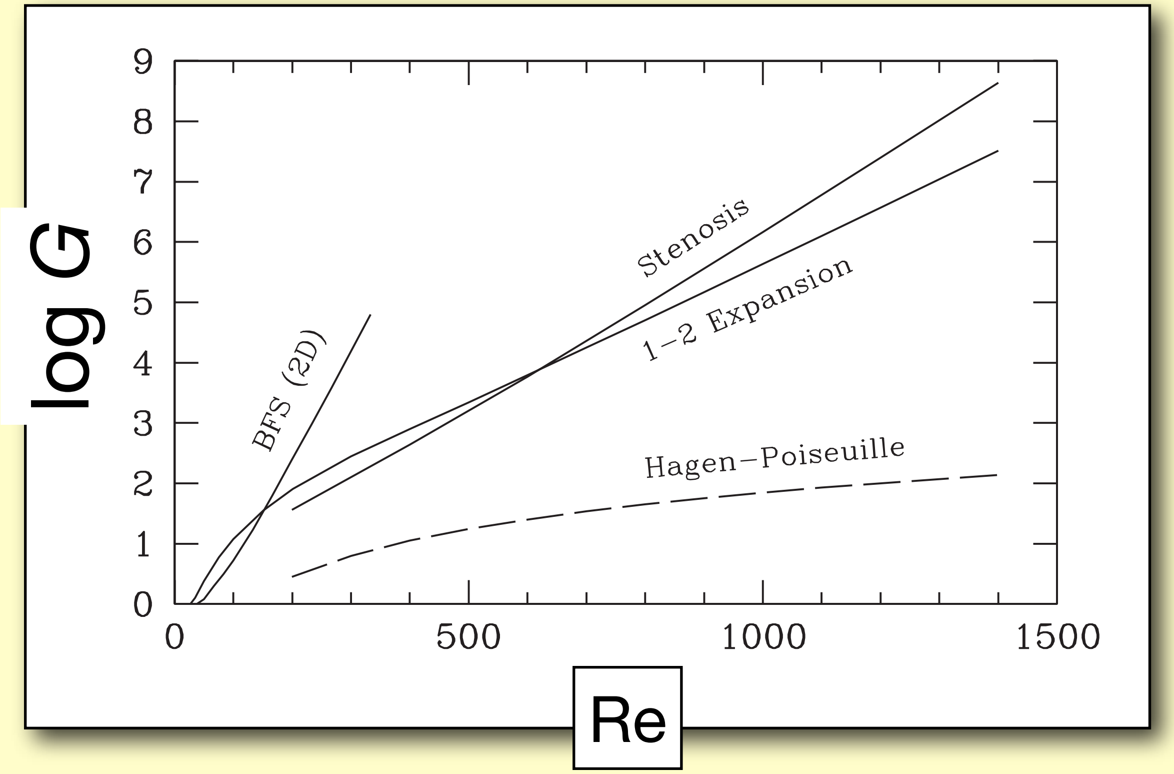
Several prototype geometries:

backward-facing step, stenosis, expanding pipe, cylinder wake

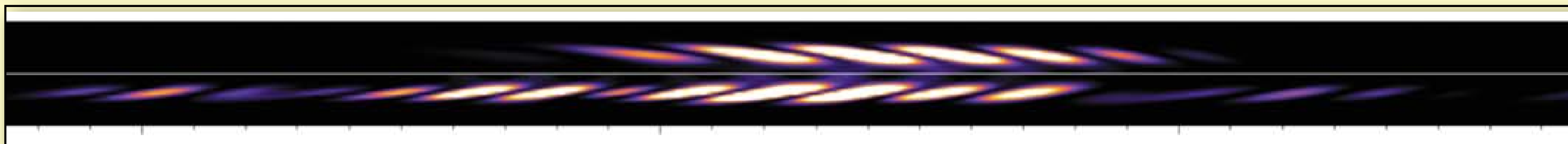
## Convective Instability




## growth ~ $e^{\alpha RE}$

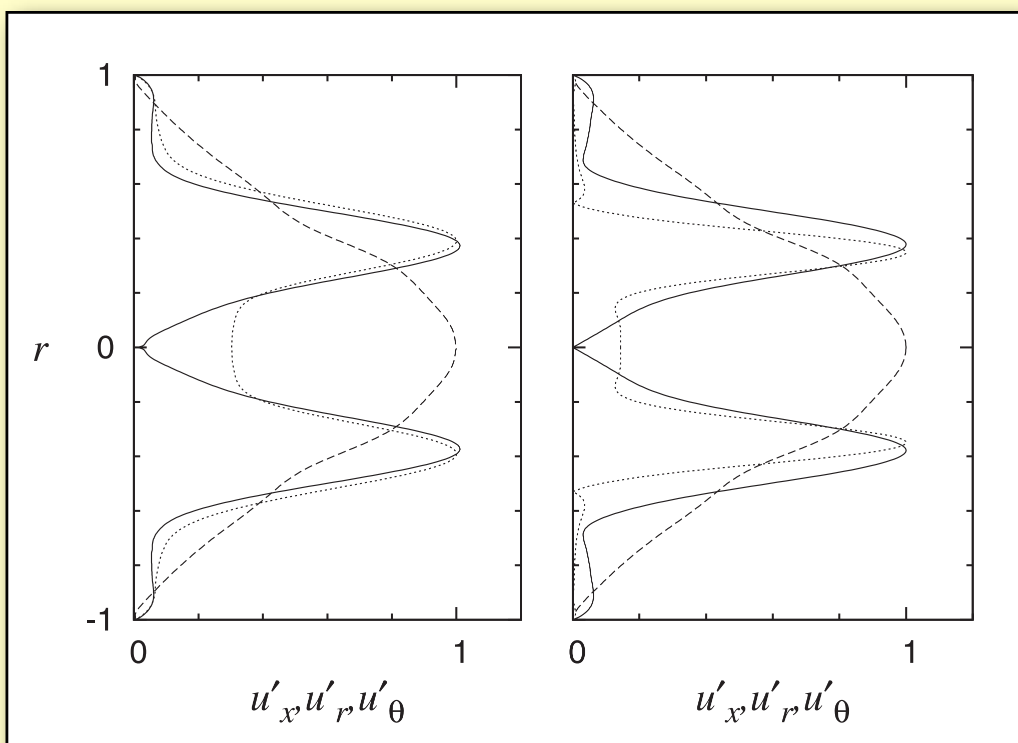


# Highlights of General Interest

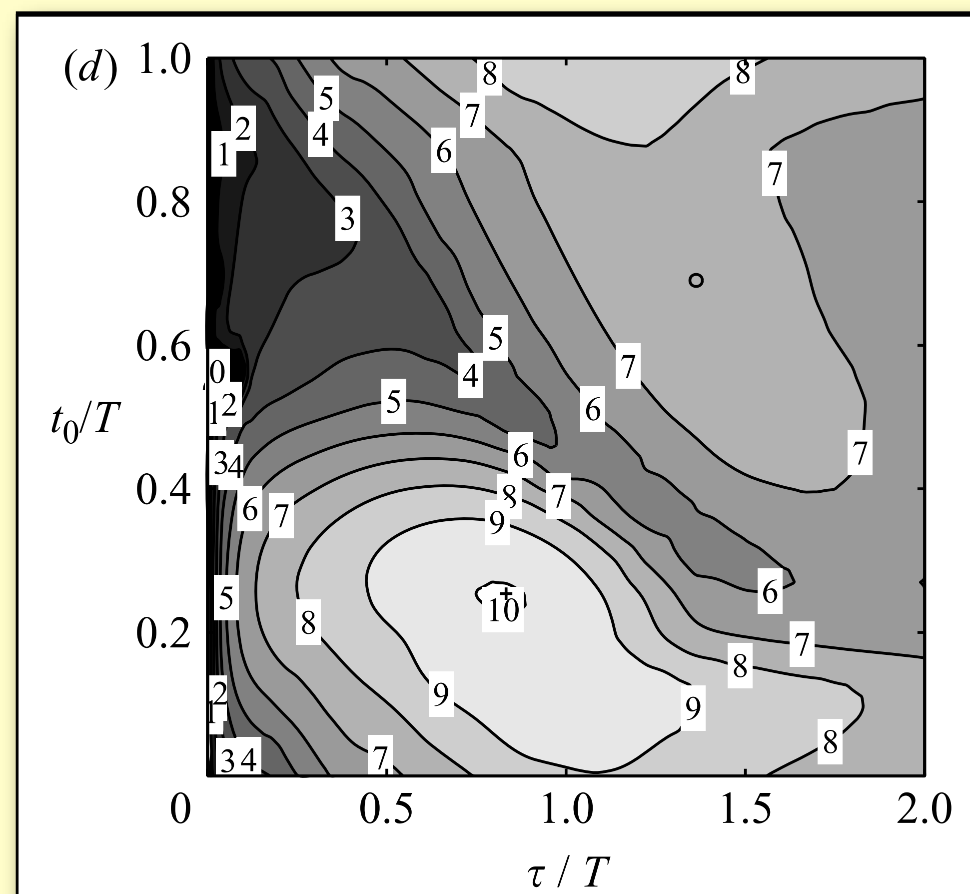


## Physical Structures

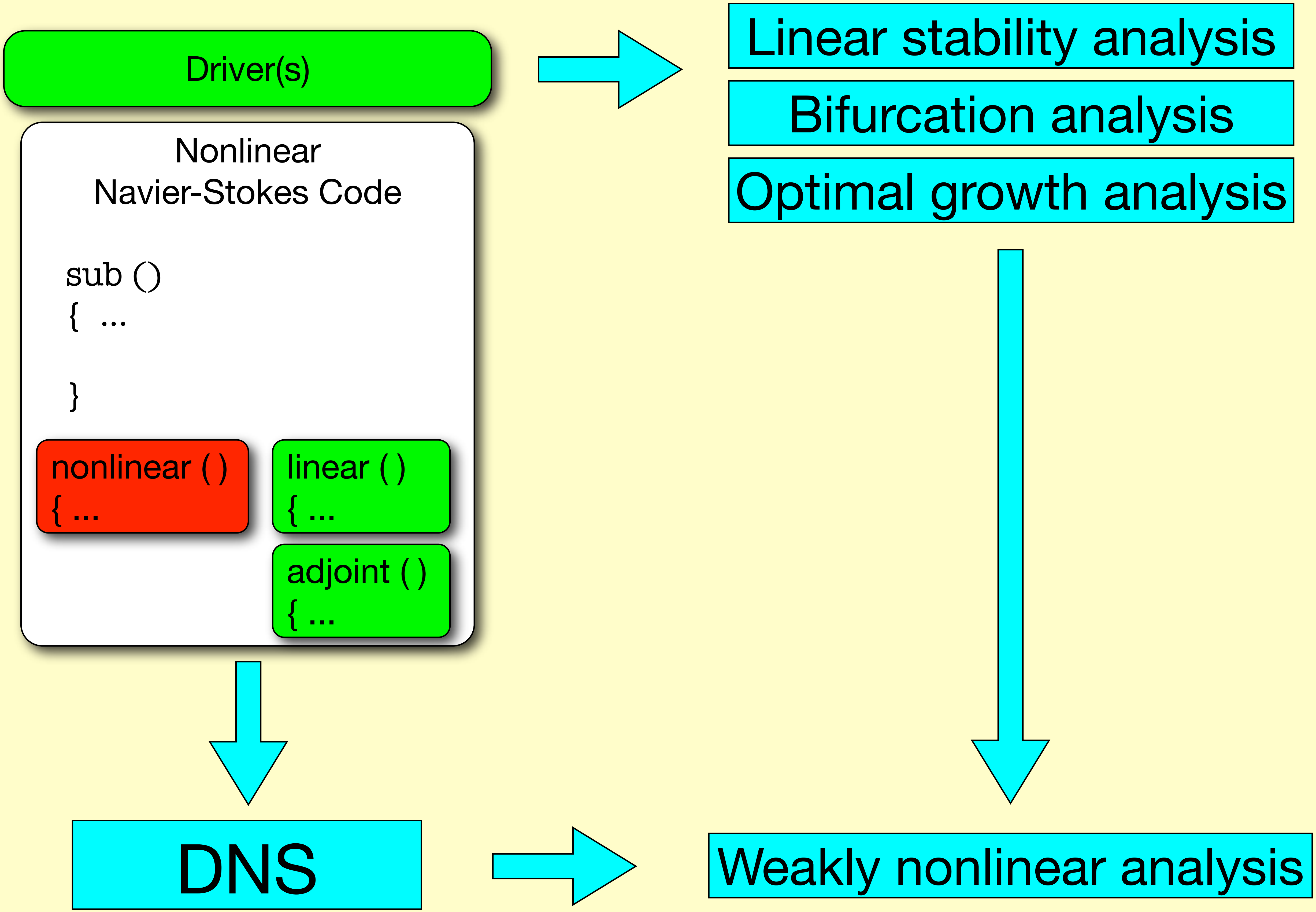
## Compare with full DNS




## Complex Cases



# Timestepper Approach



# Excitable Media

joint with Irina Biktasheva
Vadim Biktashev
Andy Foulkes

# Reaction-Diffusion Models

$$\partial_t \mathbf{u} = \mathbf{f}(\mathbf{u}) + \mathbf{D}\nabla^2 \mathbf{u}$$

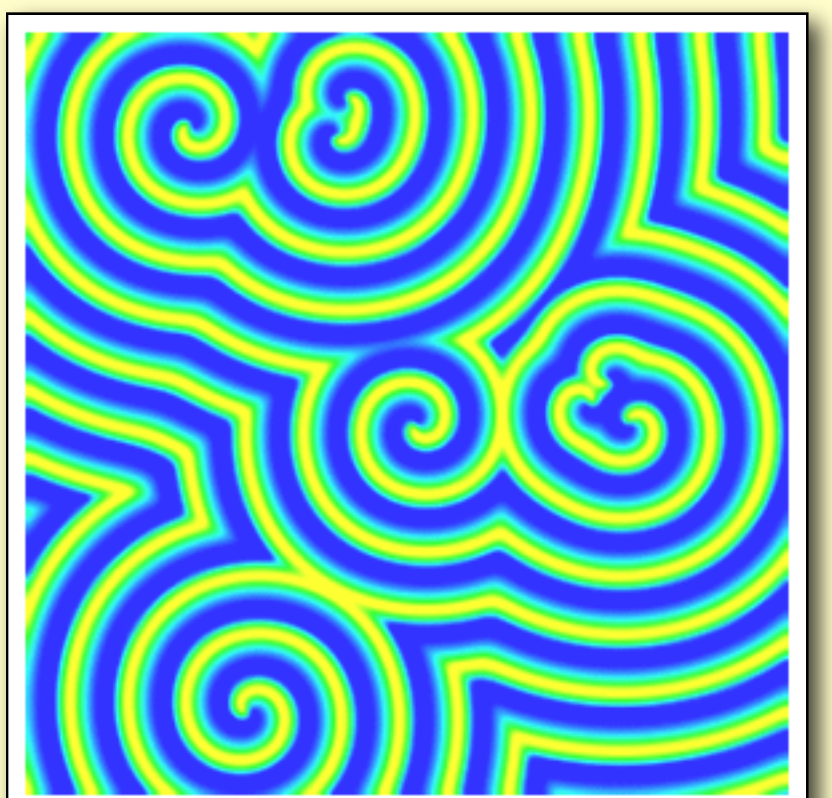
$\mathbf{u},\ \mathbf{f} \in \mathbb{R}^{\ell},\ \mathbf{D} \in \mathbb{R}^{\ell\times\ell}$

Consider
two-component examples,
but methods are general

$$\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix}$$

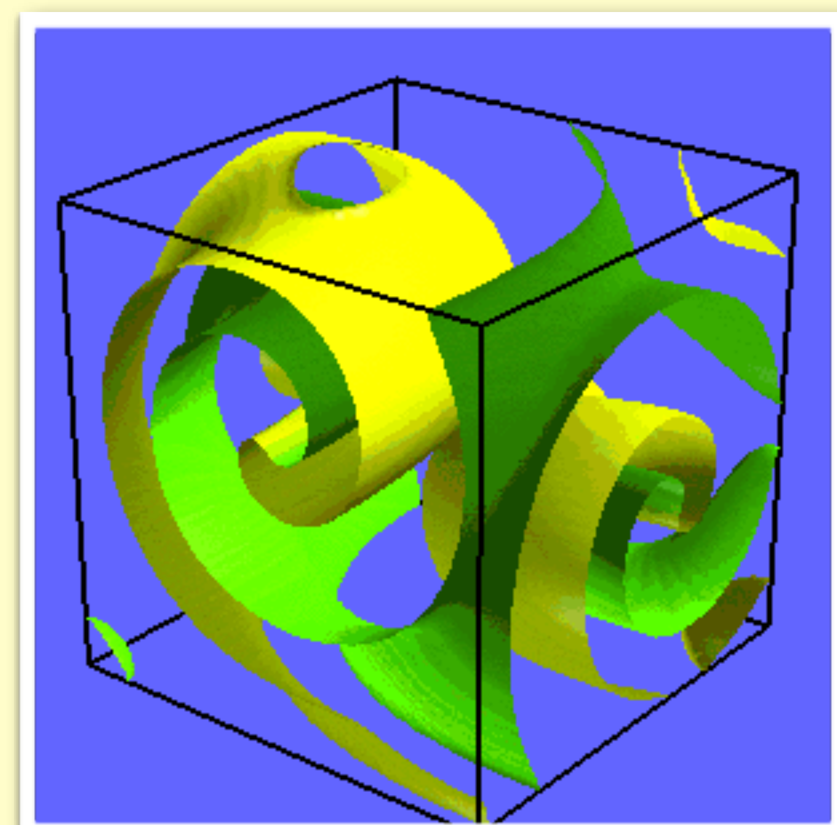
$u$: Fast

$v$: Slow

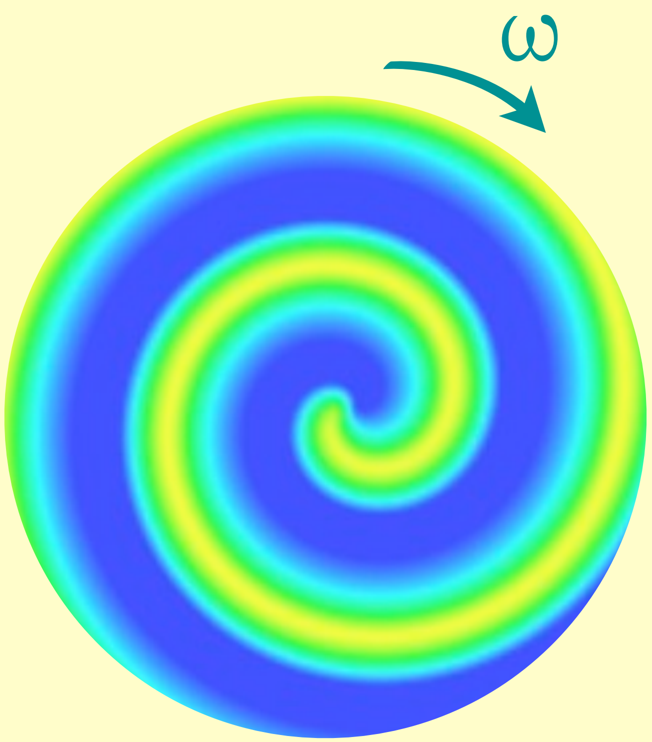


Spiral waves

Scroll waves



# Linear Stability and Symmetry




Base solution: $\mathbf{U}$ rotating wave
steady in rotating frame

$$0 = \mathbf{f}(\mathbf{U}) - \omega \partial_\theta \mathbf{U} + \mathbf{D}\nabla^2 \mathbf{U}$$

Stability Spectrum:

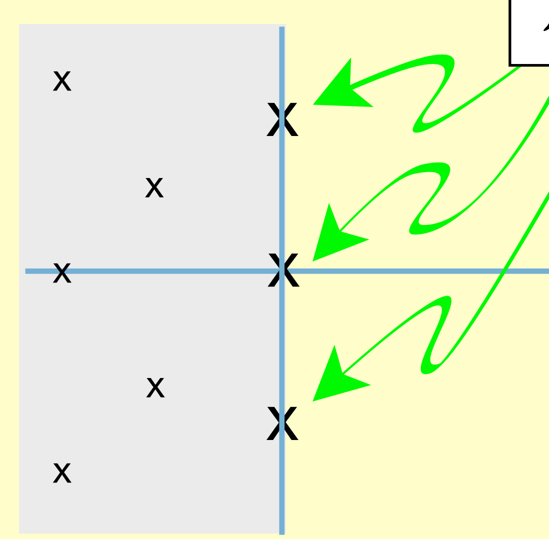
$$\mathcal{L}\mathbf{V} = \lambda \mathbf{V}$$ where $$\mathcal{L} = \mathbf{Df} - \omega\partial_\theta + \mathbf{D}\nabla^2$$

Consider linearly stable spirals on the plane

Three neutral eigenvalues
due to symmetry

$0$ rotational symmetry

$\pm i\omega$ translational symmetry
(in rotating frame)



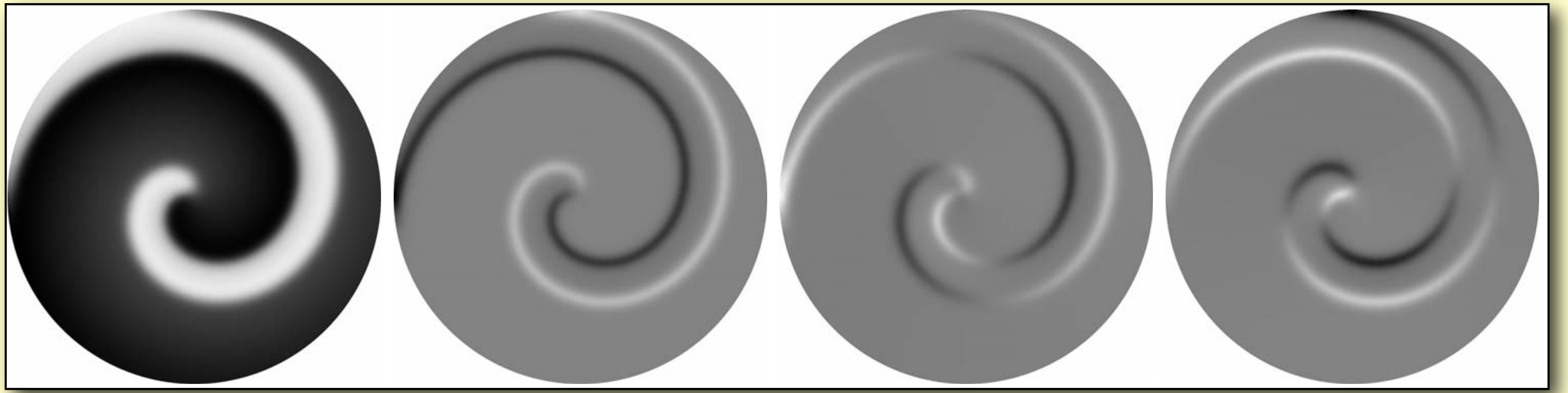
$$\mathcal{L}\mathbf{V}^{(n)} = in\omega\mathbf{V}^{(n)},$$

$$n = -1, 0, 1$$

# Neutral Eigenfunctions

Spiral Wave    Neutral Eigenfunctions

$$\mathcal{L}\mathbf{V}^{(n)} = \lambda_n \mathbf{V}^{(n)}$$



$\mathbf{V}^{(0)}$    $\mathrm{Re}(\mathbf{V}^{(1)})$    $\mathrm{Im}(\mathbf{V}^{(1)})$

Numerics:
accurate, high-order polar grid
efficient via Cayley transform

# Adjoint Neutral Eigenfunctions aka Response Functions

$$\mathcal{L}^\dagger \mathbf{W}^{(n)} = -in\omega \mathbf{W}^{(n)}, \quad n = -1, 0, 1$$

Adjoint linearization

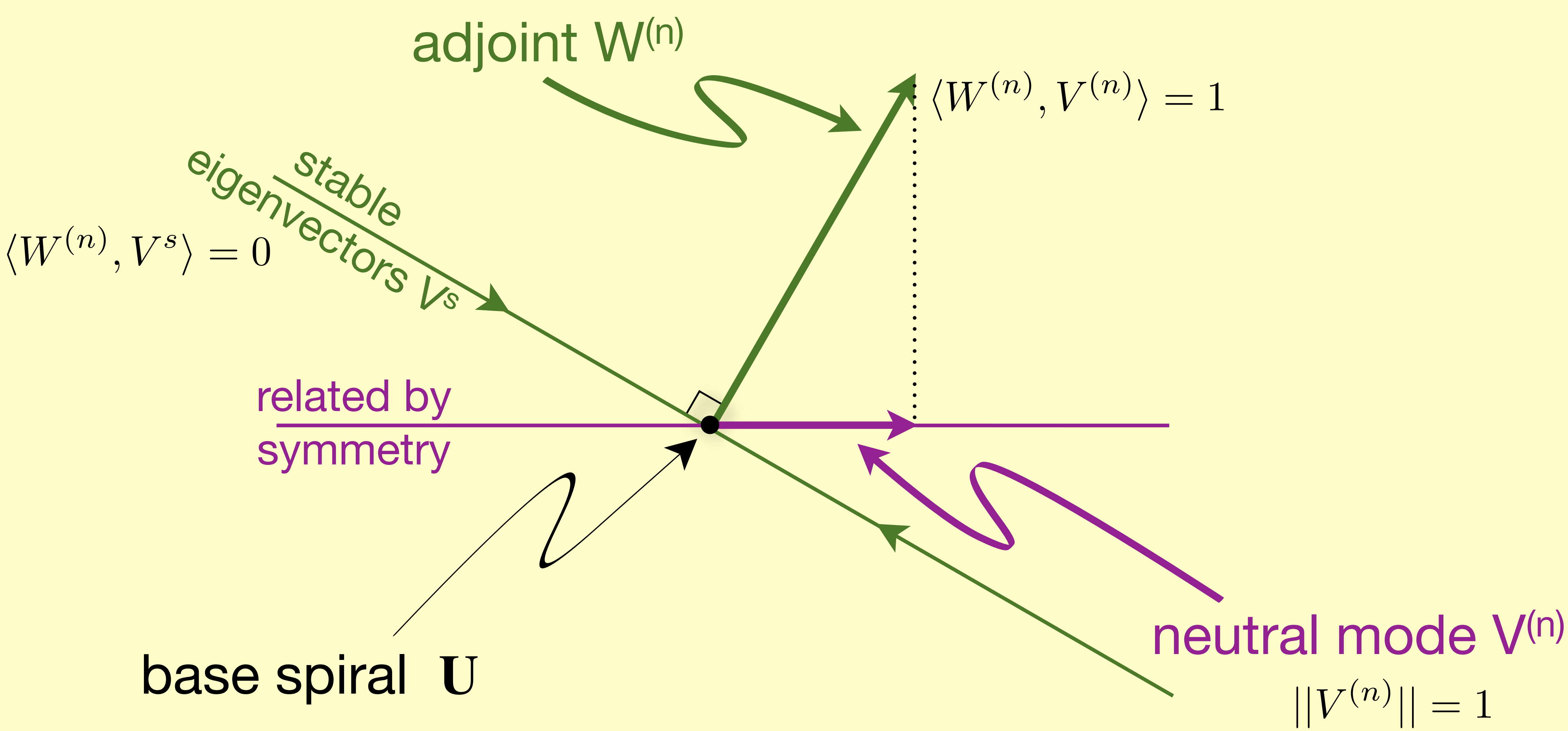
$$\mathcal{L}^\dagger = \mathbf{Df}^T + \omega\partial_\theta + \mathbf{D}\nabla^2$$


**Keener JP,** *Physica D,* 31(2), pp 269-276, **1988**

**Biktashev VN and Holden AV**, *Chaos Solitons & Fractals,* vol. 5, Issue: 3-4, pp 575-622, **1995**

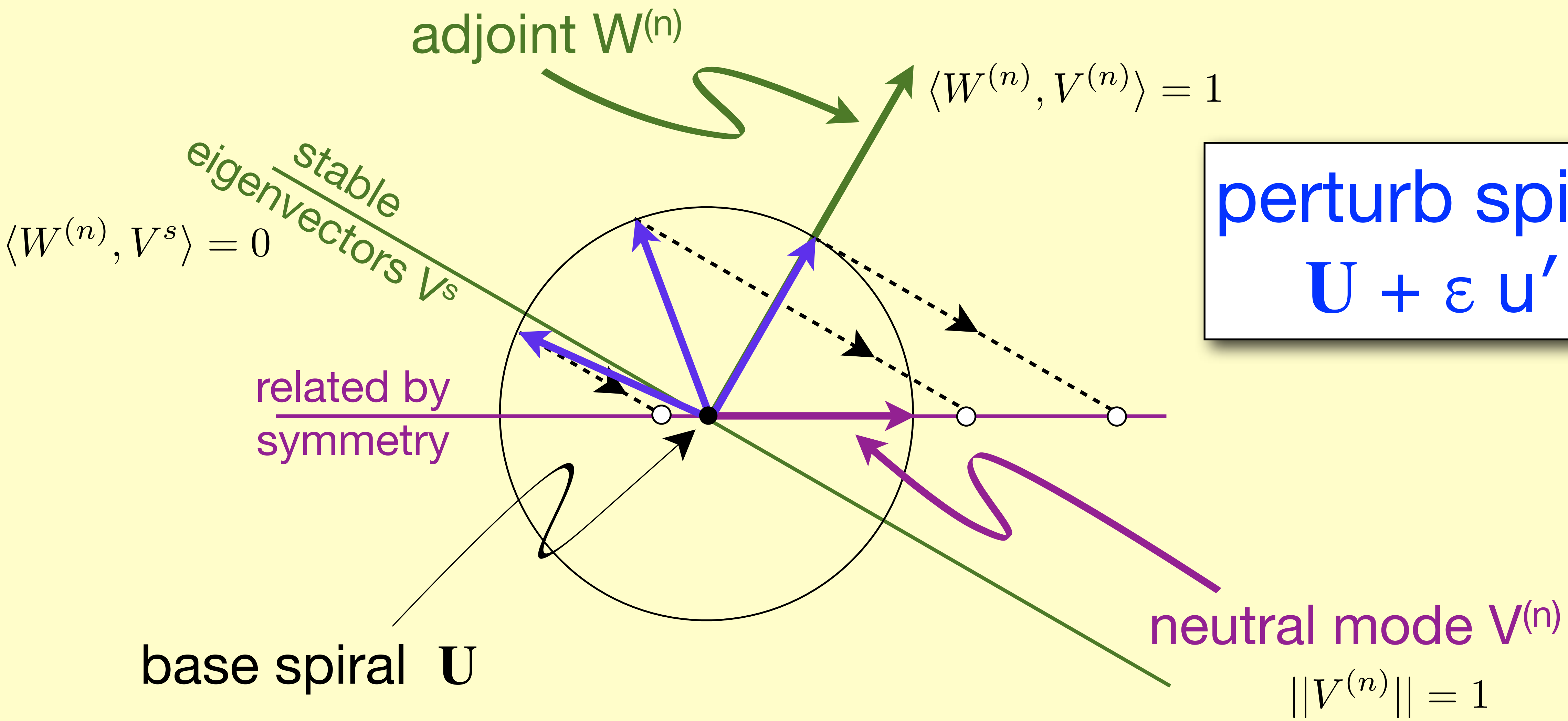
# Adjoint Neutral Eigenfunctions aka Response Functions

$$\mathcal{L}^\dagger \mathbf{W}^{(n)} = -in\omega \mathbf{W}^{(n)}, \quad n = -1, 0, 1$$



# Adjoint Neutral Eigenfunctions aka Response Functions

$$\mathcal{L}^{\dagger}\mathbf{W}^{(n)} = -in\omega\mathbf{W}^{(n)}, \quad n = -1, 0, 1$$

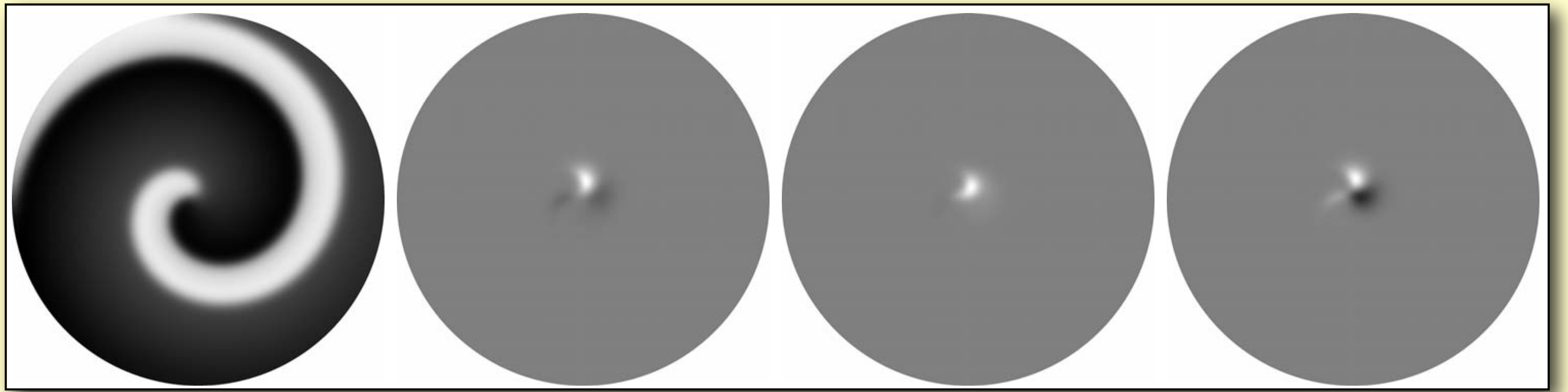



perturb spiral
$\mathbf{U} + \varepsilon\, u'$

# Response Functions in Excitable Media

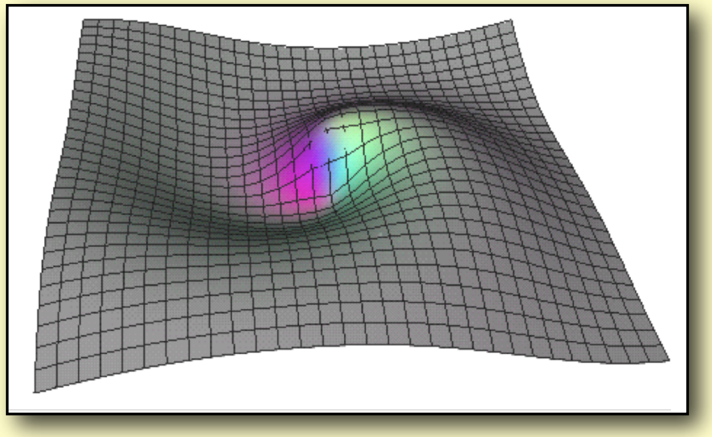
Spiral Wave        Response Functions

$$\mathcal{L}^{+}\mathbf{W}^{(n)} = \mu_n \mathbf{W}^{(n)}$$



$\mathbf{W}^{(0)}$    $\mathrm{Re}(\mathbf{W}^{(1)})$    $\mathrm{Im}(\mathbf{W}^{(1)})$

**Localization in CGLE** - **I.V. Biktasheva, Yu.E. Elkin, and V.N. Biktashev**, *Phys. Rev. E*, **57**(3):2656-2659, **1998**



## Wave-particle dualism

**I.V. Biktasheva, V.N. Biktashev**, *Phys. Rev. E*, **67**: 026221, **2003**

**H. Henry, V. Hakim**, *Phys. Rev. E*, **65 (4): 046235, 2002**

# Equations of Motion

Perturb Equation

$$\partial_t \mathbf{u} = \mathbf{f}(\mathbf{u}) + \mathbf{D}\nabla^2\mathbf{u} + \epsilon\mathbf{h}, \quad \mathbf{h} \in \mathbb{R}^\ell, \quad |\epsilon| \ll 1$$

perturbation

Use solvability condition
to obtain equations for (slow) motion for spiral core

Frequency Shift

$$\dot{\Phi} = \epsilon \int_0^{2\pi} \left\langle \mathbf{W}^{(0)}, \tilde{\mathbf{h}}(\mathbf{U}, \rho, \theta, \phi) \right\rangle \frac{\mathrm{d}\phi}{2\pi} + \mathcal{O}(\epsilon^2),$$

Motion

$$\dot{R} = \epsilon \int_0^{2\pi} e^{-i\phi} \left\langle \mathbf{W}^{(1)}, \tilde{\mathbf{h}}(\mathbf{U}, \rho, \theta, \phi) \right\rangle \frac{\mathrm{d}\phi}{2\pi} + \mathcal{O}(\epsilon^2)$$

adjoint translation eigenfunction

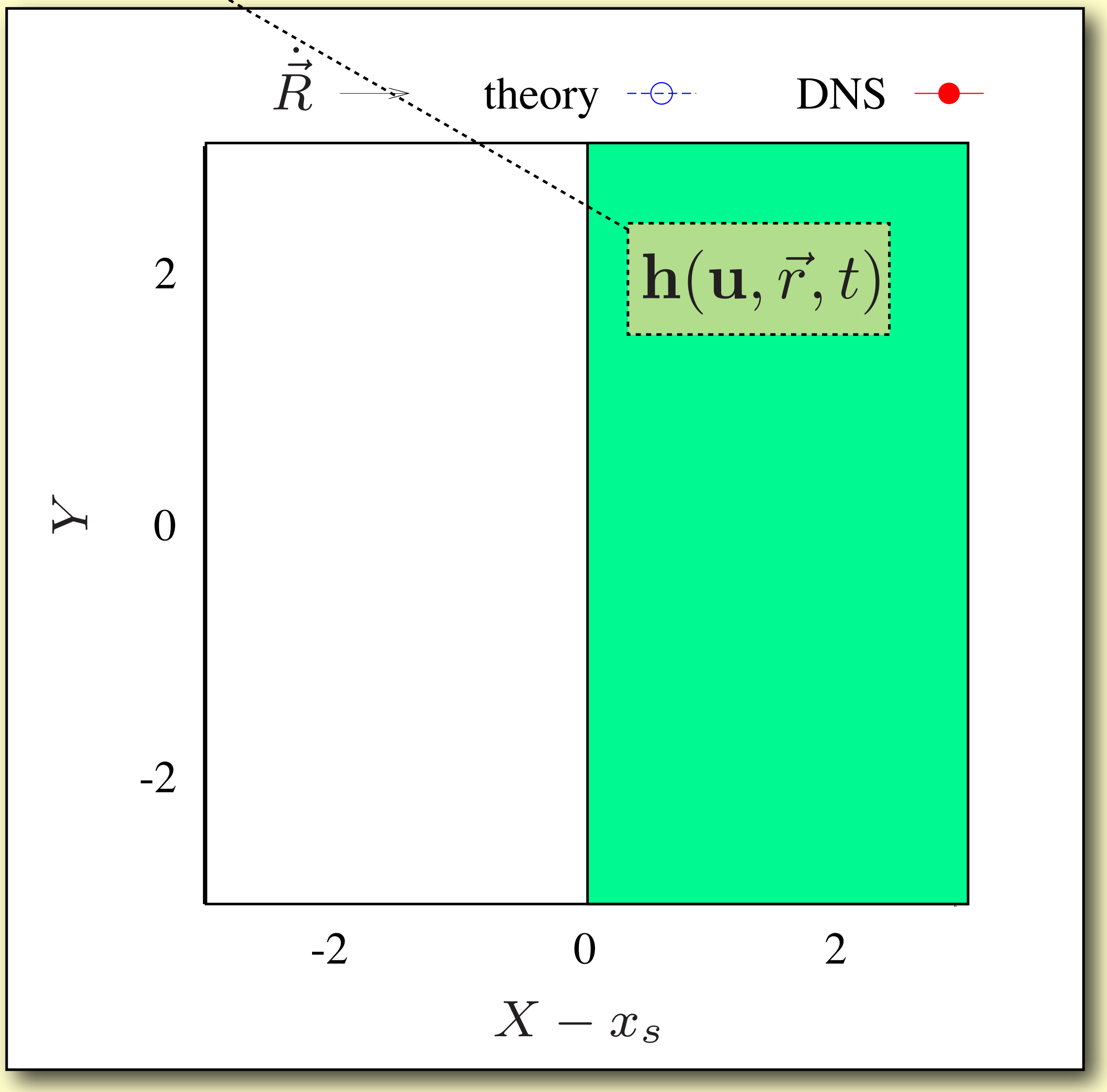
perturbation

# Example: Step Heterogeneity

$$\mathbf{f} = \mathbf{f}(\mathbf{u}, p), \qquad p = p(\vec{r}) = p_0 + \epsilon p_1(\vec{r}) \qquad p_1(x) = \mathrm{H}(x - x_s)$$

Heaviside function

$$\partial_t \mathbf{u} = \mathbf{D}\nabla^2\mathbf{u} + \mathbf{f}(\mathbf{u}, p_0) + \epsilon p_1(\vec{r})\partial_p \mathbf{f}(\mathbf{u}, p_0)$$



# Example: Step Heterogeneity

$$\mathbf{f} = \mathbf{f}(\mathbf{u}, p), \qquad p = p(\vec{r}) = p_0 + \epsilon p_1(\vec{r}) \qquad p_1(x) = \mathrm{H}(x - x_s)$$

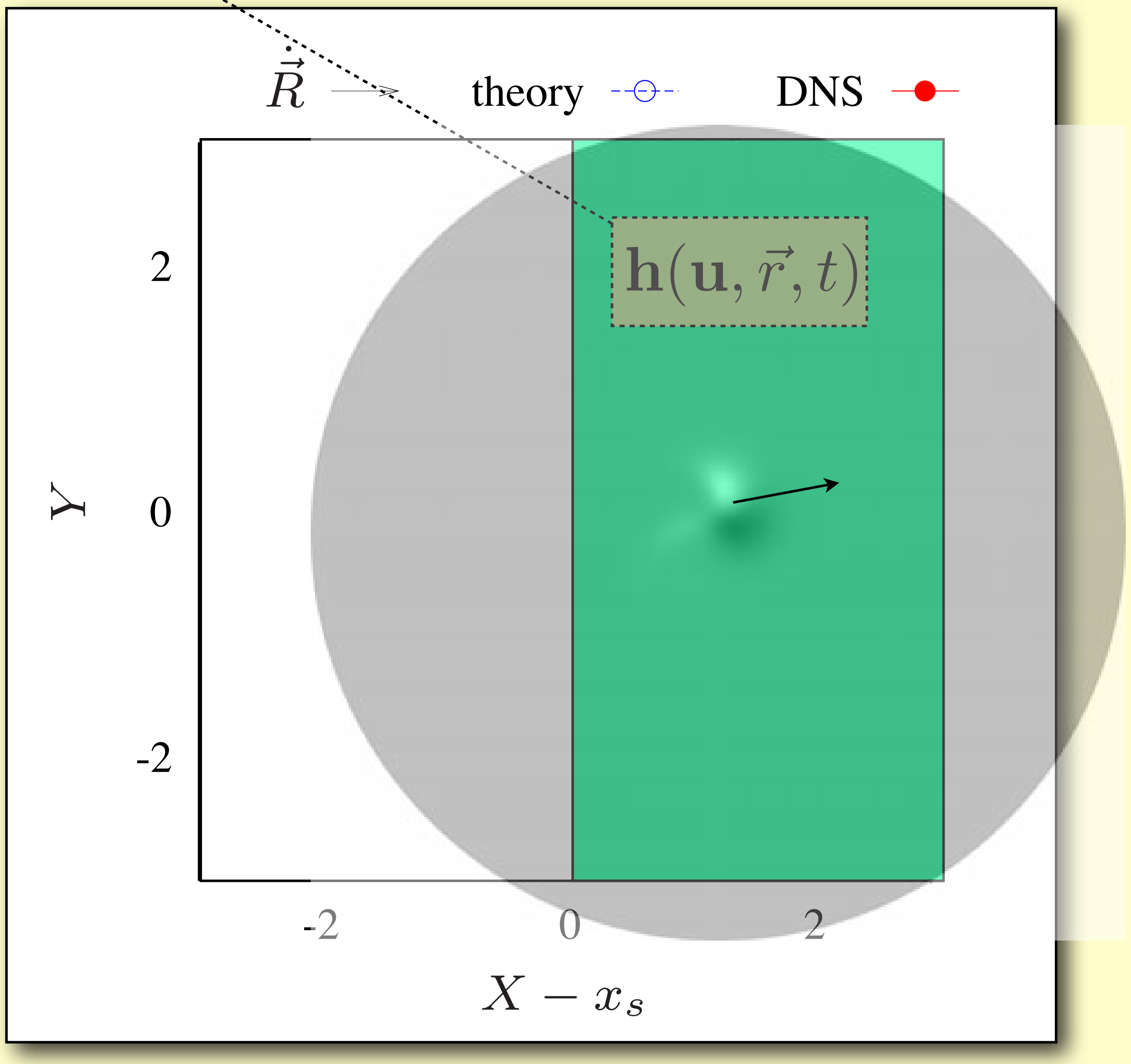
Heaviside function

$$\partial_t \mathbf{u} = \mathbf{D}\nabla^2 \mathbf{u} + \mathbf{f}(\mathbf{u}, p_0) + \epsilon p_1(\vec{r})\partial_p \mathbf{f}(\mathbf{u}, p_0)$$

## Motion

$$\dot{R} = \frac{\epsilon}{\pi} \int_0^{2\pi} \int_{|x_s - X|}^{\infty} w^{(1)}(\rho,\theta) e^{-i\theta} \sqrt{1 - \left(\frac{x_s - X}{\rho}\right)^2} \rho \, d\rho \, d\theta$$

where, $w^{(n)}(\rho,\theta) = \left[\mathbf{W}^{(n)}(\rho,\theta)\right]^{+} \partial_p \mathbf{f}(\rho,\theta; p_0)$



# Example: Step Heterogeneity

$$\mathbf{f} = \mathbf{f}(\mathbf{u}, p), \qquad p = p(\vec{r}) = p_0 + \epsilon p_1(\vec{r}) \qquad p_1(x) = \mathrm{H}(x - x_s)$$

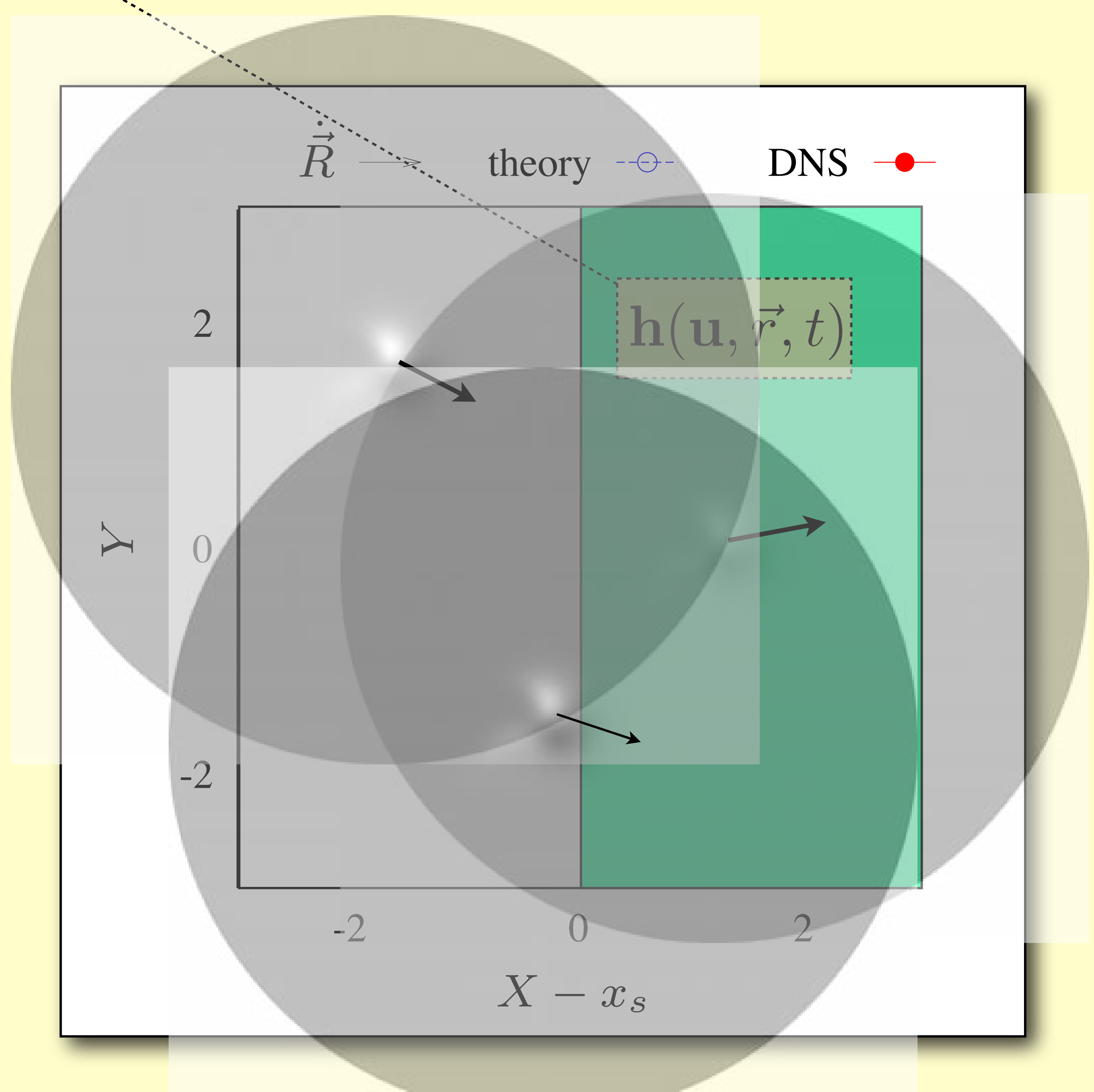
Heaviside function

$$\partial_t \mathbf{u} = \mathbf{D}\nabla^2 \mathbf{u} + \mathbf{f}(\mathbf{u}, p_0) + \epsilon p_1(\vec{r})\partial_p \mathbf{f}(\mathbf{u}, p_0)$$

## Motion

$$\dot{R} = \frac{\epsilon}{\pi} \int_0^{2\pi} \int_{|x_s - X|}^{\infty} w^{(1)}(\rho,\theta) \mathrm{e}^{-\mathrm{i}\theta} \sqrt{1 - \left(\frac{x_s - X}{\rho}\right)^2} \rho \, \mathrm{d}\rho \, \mathrm{d}\theta$$

$$\text{where,} \quad w^{(n)}(\rho,\theta) = \left[\mathbf{W}^{(n)}(\rho,\theta)\right]^+ \partial_p \mathbf{f}(\rho,\theta; p_0)$$



# Example: Step Heterogeneity

$$\mathbf{f} = \mathbf{f}(\mathbf{u}, p), \qquad p = p(\vec{r}) = p_0 + \epsilon p_1(\vec{r}) \qquad p_1(x) = \mathrm{H}(x - x_s)$$

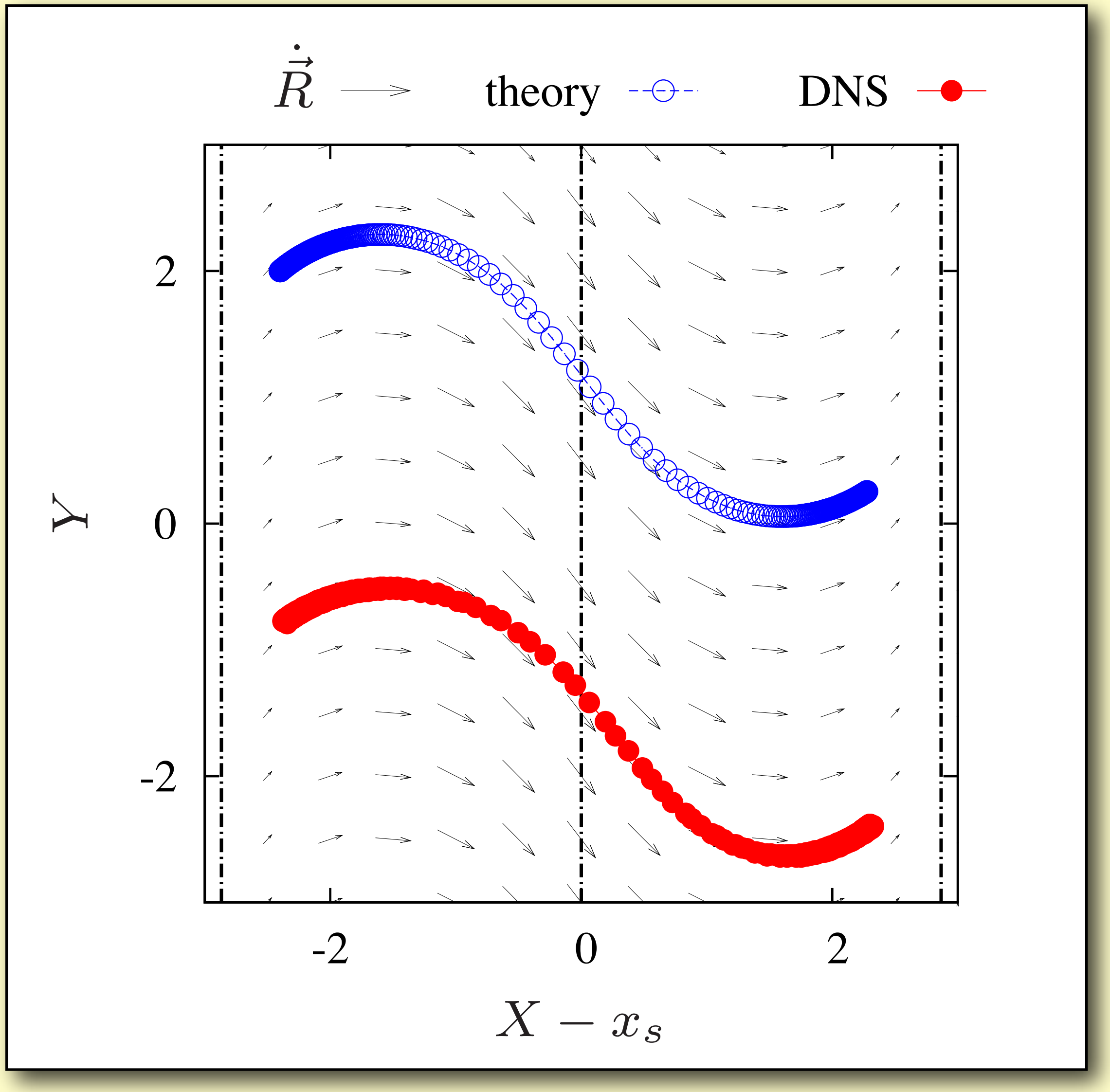
Heaviside function

$$\partial_t \mathbf{u} = \mathbf{D}\nabla^2 \mathbf{u} + \mathbf{f}(\mathbf{u}, p_0) + \epsilon p_1(\vec{r})\partial_p \mathbf{f}(\mathbf{u}, p_0)$$

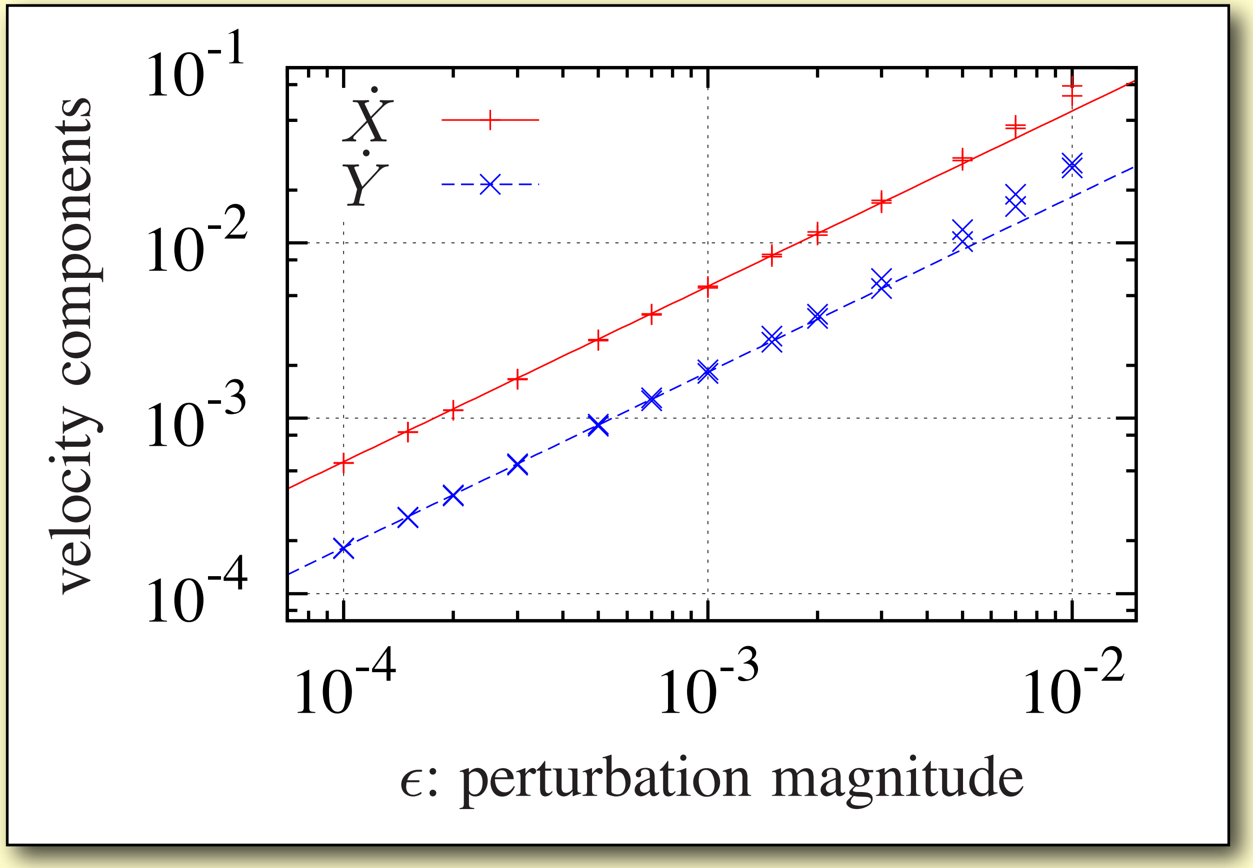
## Motion

$$\dot{R} = \frac{\epsilon}{\pi} \int_0^{2\pi} \int_{|x_s - X|}^{\infty} w^{(1)}(\rho, \theta) \mathrm{e}^{-\mathrm{i}\theta} \sqrt{1 - \left(\frac{x_s - X}{\rho}\right)^2} \rho \, \mathrm{d}\rho \, \mathrm{d}\theta$$

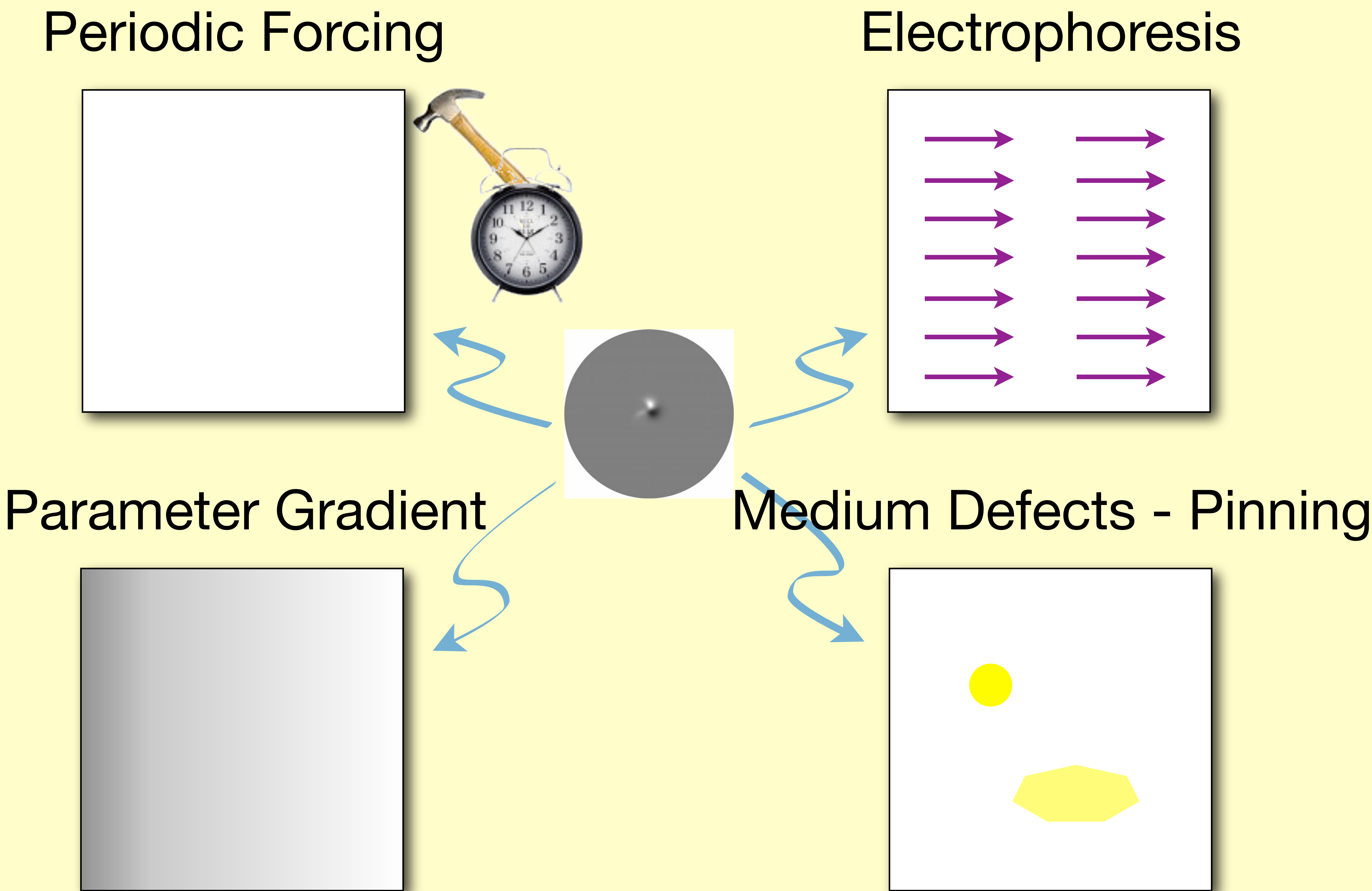
where, $w^{(n)}(\rho, \theta) = \left[\mathbf{W}^{(n)}(\rho, \theta)\right]^+ \partial_p \mathbf{f}(\rho, \theta; p_0)$



# Range of Validity, Scaling



# Applicable to other cases




Exciting Details at 3pm Today

# Acknowledgments

Laurette Tuckerman

Chris Cantwell
Hugh Blackburn
Spencer Sherwin

Irina Biktasheva
Vadim Biktashev
Andy Foulkes

Bjorn Sandstede

EPSRC
Leverhulme Trust/Royal Society