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Thermodynamics of the Quasiperiodic Parameter Set at the Borderline of Chaos: Experimental Results

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A periodically driven relaxation-oscillator circuit is used to study experimentally the multifractal structure of the quasiperiodic parameter set on the critical line. The experimental data provide evidence of a phase transition in the thermodynamic free energy $q(\tau)$. The experimental thermodynamics compares well with that of the sine circle map and provides strong evidence for universality in the parameter scaling at the borderline of chaos.

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The recently developed thermodynamic formalism of multifractals[1-4] has been used to quantify fractal structures arising in a variety of physical contexts. Notable examples include fractals generated in chaotic dynamics,[1,2,4-7] fluid turbulence,[8] and fractal growth processes.[9,10] For many fractal sets,[10-15] nonanalyticities exist in the associated thermodynamic functions, e.g., the free energy $q(\tau)$. In light of the analogy to thermodynamics, these nonanalyticities have a natural, and in certain cases formally exact,[12] interpretation as phase transitions.

One fractal set of particular interest in the setting of chaotic dynamics is the quasiperiodic parameter set at the borderline of chaos, that is, the complement to the set of mode lockings on the critical line.[16] The quasiperiodic set provided the first physically important example of phase-transition phenomena in the scaling of multifractals,[11] and a detailed theoretical study of this set, with particular emphasis on the existence of first-order transitions, has recently appeared.[15]

Apart from the issue of phase transitions, the scaling of the (critical) quasiperiodic set is important because it represents a truly global manifestation of the transition from quasiperiodicity to chaos. For example, numerical[16] and experimental[17] studies indicate that at the transition to chaos, the fractal dimension, D_0, of the quasiperiodic set obtains a universal value. Because D_0 depends on the mode-locking structure *everywhere* along the critical line, the universality of D_0 implies a global universality in the quasiperiodic transition to chaos.[18] However, D_0 represents but one value in a continuum of dimensions, D_q, reflecting an infinity of scalings within the quasiperiodic set.[1,19,20] Until now there has been no experimental study to test the universality of the entire spectrum of scalings possessed by the quasiperiodic set.

We report here the experimental study of the quasiperiodic set for a periodically driven operational-amplifier oscillator which has previously been described.[21,22] We find strong *experimental* evidence for a phase transition in the thermodynamic functions for the quasiperiodic set. We also present the first experimental measurement of the scaling spectrum, $f(\alpha)$, for this set.

We begin by recalling the thermodynamic formalism.[1-4] Consider the covering of a fractal measure by intervals of length l_i. Each interval contains a corresponding weight or probability p_i. Then define the partition function by

$$\Gamma(q,\tau)=\sum_{i=1}^{N} p_i^q/l_i^\tau . \qquad (1)$$

In principle, one takes the limit $N\rightarrow\infty$; in practice, we consider finite N. Setting $\Gamma(q,\tau)=1$, we obtain $q(\tau)$, which in the thermodynamic analogy plays the role of free energy,[23] $F(\beta)$. In the study of phase transitions, it is useful to examine $\mu(\tau)=dq/d\tau$ and $c(\tau)=d^2q/d\tau^2$, which play the rule of internal energy and specific heat, respectively. One can alternatively consider τ as a function of q, and by Legendre transformation, obtain the spectrum of scaling indices $f(\alpha)=q\alpha-\tau(q)$, where $\alpha=d\tau/dq=1/\mu$.

The starting point for the thermodynamics of the (critical) quasiperiodic set is a collection of $N+1$ mode-locked intervals $\{P_i/Q_i\}$, labeled by their winding number. The mode lockings are thought of as a set of holes in the critical line on which the quasiperiodic set does not live. The covering, $\{l_i\}$, for the quasiperiodic set is the complement to the set of holes, i.e., interval l_i is the gap between neighboring mode lockings P_i/Q_i and P_{i+1}/Q_{i+1}.

Several different measures have been put forth for the quasiperiodic set,[1,15] as represented by different definitions of the probabilities p_i. Each measure gives rise to distinct thermodynamics. Here we follow Halsey *et al.*[1] and take the p_i to be the change in (dressed) winding number across covering interval l_i (normalized such that $\sum p_i=1$). We study this thermodynamics because it is experimentally the most accessible: We simply compute the probabilities from whatever mode lockings are found experimentally.

The experimental apparatus is an operational-amplifier relaxation oscillator which is driven by a sine wave whose frequency and amplitude can be varied. The drive frequency and voltage are computer controlled. The output voltage, sampled once every drive period, is filtered by a 500-kHz low-pass filter, digitized by a twelve-bit analog-to-digital converter, and passed to the computer for analysis. Complete details can be found in Refs. 21 and 22.

We first experimentally determined the critical line (drive voltage as a function of drive frequency) for the transition to chaos (see Fig. 1 of Ref. 21). We identified the transition to chaos by the appearance of a fold in the Poincaré sections (Fig. 2, Ref. 21). The critical line was then scanned by making 5-Hz steps in drive frequency and adjusting the drive amplitude so as to remain on the critical line. At each step the period and winding number of the driven oscillator were obtained by the computer. From the scan of the critical line, 249 of the possible 388 mode-locked intervals with periods $Q_i \leq 50$ were identified. Of the covering intervals l_i, obtained from the experimental mode lockings, four were narrower than 20 Hz and are not considered in our analysis because of the large relative error of these intervals.

Figure 1(a) shows the devil's staircase for the experimental data. For comparison, we plot the staircase for the sine circle map[16] with all mode lockings such that $Q_i \leq 50$. Figure 1(b) shows our experimental estimate of the fractal density of quasiperiodic states on the critical line: $\rho=p_i/l_i$ in covering interval i and zero elsewhere.

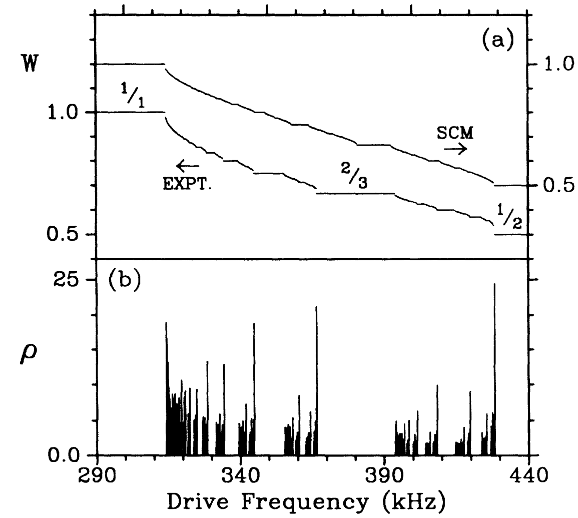

FIG. 1. (a) The devil's staircase obtained by plotting winding number, W, as a function of drive frequency for the 249 experimentally measured mode-locked intervals along the critical line. Three low-order lockings are labled. For comparison, the devil's staircase for the sine circle map (SCM) is plotted with the end points of the 1/1 and 1/2 mode lockings aligned with the experimental data. (b) Experimental estimate of the fractal density, ρ, for the quasiperiodic set. The holes due to low-order mode lockings are clearly seen.

Because of scale-dependent prefactors,[1,15,24] the thermodynamic functions computed directly from (1) often converge slowly as $N\rightarrow\infty$. This is particularly true for the experimental relaxation oscillator whose wide low-order (small-Q_i) mode lockings impair convergence as N becomes large. The ill effects of the low-order lockings can be greatly reduced by computing $\tau(q)$ [or alternatively, $q(\tau)$] from the ratio of partition functions:[1,15]

$$\Gamma(q,\tau)/\Gamma'(q,\tau)=1\,, \qquad (2)$$

where primed and unprimed functions refer to a coarse and fine covering of the quasiperiodic set, respectively. The coarse and fine coverings are determined from mode lockings whose periods Q_i do not exceed maximum periods Q' and Q, respectively.

In Fig. 2 we illustrate how, for the experimental data, results obtained from (2) depend on the choice of coverings. Shown is the fractal dimension, $D_0=-\tau(q=0)$, as a function of Q for several values of Q'. In the limit $Q\rightarrow\infty$, the results should be independent of the coarse covering given by Q', and we expect all curves in Fig. 2 to approach asymptotically the same value of D_0. However, at the resolution of the experimental data, $Q=50$, the finite-N estimates of D_0 depend considerably on the coarse covering. While in studies of circle maps the choice $Q'=[Q/\sqrt{2}]$ has been used[25] (square brackets denote integer part), it is impossible, *a priori,* to say what value of Q' is most appropriate for the finite-N experimental data. It is clear, however, that small values

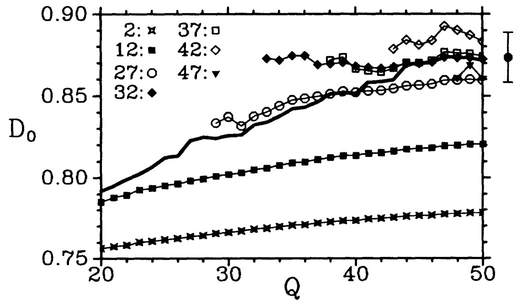

FIG. 2. Fractal dimension, $D_0 = -\tau(0)$, for the experimental data as a function of Q for values of Q' indicated in the legend. Q and Q' determine the fine and coarse coverings used in (2). The bold curve, without symbols, is for $Q' = [Q/\sqrt{2}]$, where brackets denote integer part. The experimental error bar for D_0 is shown to the right.

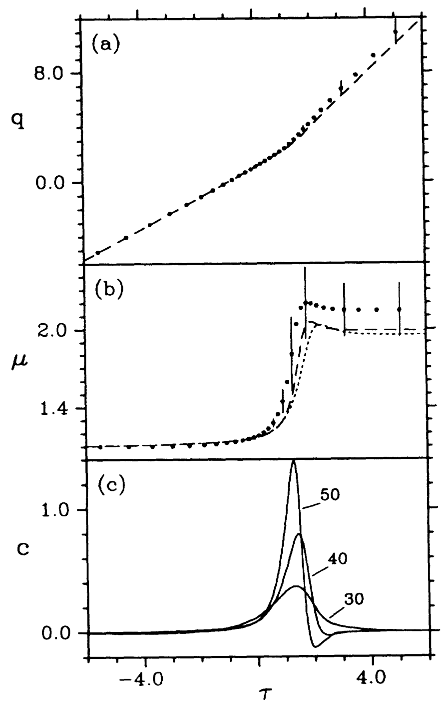

FIG. 3. The free energy $q(\tau)$, internal energy $\mu(\tau) = dq/d\tau$, and specific heat $c(\tau) = dq^2/d\tau^2$, as functions of inverse temperature, τ. Points are from experiment. Sine-circle-map results are shown for two values of Q: $Q = 180$ (long-dashed curve) and $Q = 50$ (short-dashed curve). The short-dashed curve is not shown in (a) as it is not clearly distinguishable from the long-dashed curve. In (c) the experimental data are plotted with solid lines, without error bars, at three values of Q (with $Q' = [Q/\sqrt{2}]$): $Q = 30$, 40, and 50, corresponding to $N = 121$, 186, and 244. For clarity, circle-map results are not shown in (c), but they are comparable to experiment. The abrupt change in the slope of the free energy and the peaking of the specific heat at $\tau \simeq 1.3$ indicates a first-order phase transition. The experimental deviation from the circle-map curve at large τ is presumably due to finite-N effects (see text).

of Q' are inappropriate for estimating the asymptotic value of D_0.

Based on Fig. 2, the experimental error bar for the fractal dimension is chosen to include *all* finite-N values of D_0 for Q near 50 and Q' greater than 27. We have estimated uncertainty in D_0 due to experimental error in determining the end points of the mode-locked intervals and have found this to be negligible in comparison with the uncertainty due to the arbitrariness of Q'. While it is not possible to extrapolate unequivocally the asymptotic value of D_0, we find it significant that the curves $Q' = 32$ and $Q' = 37$ are quite flat as a function of Q (as are all curves for Q' in the range $30 \le Q' < 40$). These curves indicate that the asymptotic value of D_0 for the experimental data is near 0.87, the (conjectured) universal value for the fractal dimension of the quasiperiodic set.[16]

The free energy $q(\tau)$ and its derivatives are shown in Fig. 3. The scaling spectrum $f(\alpha)$ is shown in Fig. 4. As in the case of the fractal dimension (Fig. 2), all error bars in Figs. 3 and 4 are based on variations in the thermodynamic variables for $Q \simeq 50$ and $Q' > 27$. We believe that for $\tau \lesssim 1$ the error bars bound the asymptotic thermodynamics; for $\tau \gtrsim 1$ we cannot be certain of the asymptotic limit and the error bars should be interpreted as the uncertainty in the finite-N experimental results. For comparison, we show results obtain by numerically computing mode lockings for the sine circle map and using (2) with $Q = 180$ and $Q' = [Q/\sqrt{2}] = 127$. This is our best numerical estimate of the asymptotic thermodynamics for the circle map. The choice $Q = 180$ has no significance other that it is the largest Q for which we could feasibly obtain all mode lockings. In Fig. 3(b) we also show circle-map results for $Q = 50$ and $Q' = 35$, cor-

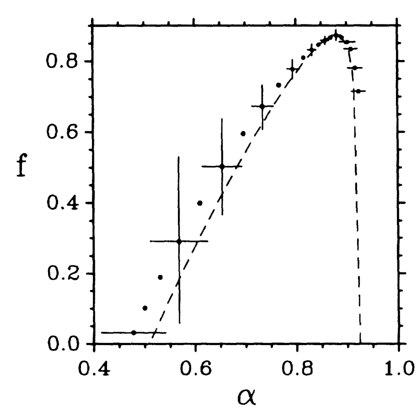

FIG. 4. The scaling spectrum $f(\alpha)$ for the experimental data (points) and the sine circle map with $Q = 180$ (dashed curve). The experimental uncertainties in f are very large on the right branch and vertical error bars are not shown. The phase transition is contained in the left branch and extends to the α axis. The experimental deviation from the circle-map curve along the left branch is presumably due to slow convergence at the phase transition.

responding approximately to the resolution of experiment.

The abrupt change in the slope of the free energy $q(\tau)$ at $\tau \simeq 1.3$ clearly suggests a phase transition. From finite-N experimental or numerical data it is impossible to conclude with certainty the existence of a true first-order phase transition. Nevertheless, the experimental specific heat [Fig. 3(c)] has not reached a limiting value as a function of N, and the data suggest that, in the "thermodynamic limit" $N \rightarrow \infty$, the specific heat tends to a δ function. This is a strong indication that the experimental data are a finite-N reflection of a first-order phase transition. We cannot, however, rule out the possibility that the phase transition is infinite order.[4,13] To our knowledge, Fig. 3 provides the most direct and convincing experimental evidence to date of a true phase transition in the thermodynamics of a multifractal set.

The experimental points systematically deviate from the circle-map curve in the high-τ phase. We attribute this to finite-N effects, and note that similar deviations have been observed in numerical studies of phase transitions.[12,15] For the quasiperiodic set, the behavior at large τ is governed by the scaling of the harmonic series within the quasiperiodic set, and these series have extremely poor convergence properties.[20] For example, we know[15,16,20] that in the thermodynamic limit μ should equal 2 at large τ. However, the convergence to this limit is very slow as the circle-map curves in Fig. 3(b) illustrate. Because we do not expect universality in finite-N results, we cannot expect the finite-N experimental and circle-map results to agree. Given that for the circle map with $Q=50$, μ at large τ differs by 2% from the thermodynamic limit, we regard the corresponding 7% difference in experiment as reasonable.

The agreement between the experimental and circle-map thermodynamics is very significant. The low-order mode lockings in the two staircases of Fig. 1 are quite different and would appear to reflect different parameter scalings along the critical line, and yet, a comparison of either the free energies $q(\tau)$ or the scaling spectra $f(\alpha)$ reveals that this is not the case. Except for differences which can be expected when computing thermodynamic functions from a finite number of mode lockings, the scaling of the quasiperiodic set for the driven relaxation-oscillator circuit is the same as that of the one-dimensional sine circle map.

A transfer-matrix method has been proposed for obtaining the free energy from low-order approximations to the scaling function.[2-4] We are not able to apply this technique directly to the problem considered here, however, because neither the probabilities p_i nor covering intervals l_i are uniform. We hope that the transfer-matrix method can be extended to the case of nonuniform probabilities as this approach should lend further insight into the phase transition. We are presently working along these lines.

In conclusion, we have presented the first experimental study of the thermodynamics of the quasiperiodic set, and have provided strong experimental evidence for a phase transition in this multifractal set. We have obtained, in experiment, thermodynamic functions, $q(\tau)$ and $f(\alpha)$, which agree well with those of the sine circle map. This provides the most complete confirmation to date of universality in the mode-locking structure at the transition from quasiperiodicity to chaos.

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