Finite-Element Approximation of Elliptic Equations with a Neumann or Robin Condition on a Curved Boundary

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This paper considers a finite-element approximation of a second-order self-adjoint elliptic equation in a region $\Omega \subset \mathbb{R}^n$ (with $n = 2$ or 3) having a curved boundary $\partial \Omega$ on which a Neumann or Robin condition is prescribed. If the finite-element space defined over $D^h$, a union of elements, has approximation power $h^k$ in the $L^2$ norm, and if the region of integration is approximated by $\Omega^h$ with $\text{dist}(\Omega, \Omega^h) = Ch^k$, then it is shown that one retains optimal rates of convergence for the error in the $H^1$ and $L^2$ norms, whether $\Omega^h$ is fitted ($\Omega^h = D^h$) or unfitted ($\Omega^h \subset D^h$), provided that the numerical integration scheme has sufficient accuracy.

1. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ (with $n = 2$ or 3) having a smooth boundary $\partial \Omega$. Let $\sigma, c \in L^\infty(\Omega)$ and $\alpha \in L^\infty(\partial \Omega)$ be sufficiently smooth functions satisfying

\begin{align}
\sigma_1 &\equiv \sigma(x) \geq \sigma_0 > 0 \quad \text{and} \quad c_1 \equiv c(x) \geq c_0 \geq 0 \quad \text{a.e. in } \Omega, \\
\alpha_1 &\equiv \alpha(x) \geq \alpha_0 \geq 0 \quad \text{a.e. on } \partial \Omega.
\end{align}

Consider the numerical solution of the elliptic boundary-value problem

\begin{align}
Au &= -\nabla \cdot (\sigma \nabla u) + cu = f \quad \text{in } \Omega, \quad \sigma \frac{\partial u}{\partial \nu} + au = g \quad \text{on } \partial \Omega; 
\end{align}

where $\partial / \partial \nu$ denotes differentiation along the outward-pointing normal to $\partial \Omega$.

A standard practical finite-element approach would be to fit a mesh to $\Omega$ using isoparametric finite elements; that is, $\Omega$ is approximated by $\Omega^h$: a union of elements. However, for a Neumann condition on a curved boundary, it was shown in Barrett & Elliott (1984) that it is not necessary to fit the mesh to the boundary in order to retain the optimal rate of convergence. They consider the simplest trial spaces—piecewise linears on triangles and piecewise bilinears on rectangles—and replace the curved boundary $\partial \Omega$ by its chord in each element it intersects, thereby obtaining a polygonal approximation $\Omega^h$ (not a union of
elements) to $\Omega \subset \mathbb{R}^2$. The resulting finite-element approximation retains the optimal rate of convergence in the Dirichlet norm.

The effect of domain perturbation and numerical integration is well understood when using a fitted mesh. In the case of homogeneous Dirichlet boundary data, Ciarlet & Raviart (1972) and Nedoma (1979) have derived optimal $H^1$ and $L^2$ error bounds for isoparametric elements. This has been extended to non-homogeneous Dirichlet data in Barrett & Elliott (1987). For homogeneous Dirichlet data, an $L^\infty$ bound is given by Wahlbin (1978) for quadratic isoparametric elements in two dimensions. This has been extended to higher-order elements and higher dimensions in the absence of numerical integration by Schatz & Wahlbin (1982). We note in passing that, to explore the possibility of using an unfitted mesh for the Dirichlet problem, the boundary condition has to be imposed weakly; that is, a penalty formulation is required: see Barrett & Elliott (1986).

For the Neumann or Robin problem, the following results for an approximation on a fitted mesh have appeared. The effect of numerical integration without domain perturbation has been studied by Goldstein (1980). An optimal $H^1$ error bound in the presence of domain perturbation but with exact integration is given in Strang & Fix (1973). An optimal $H^1$ bound when using Zlamal's curved triangular elements and employing numerical integration has been obtained by Ženišek (1981a). Čermák (1983a) has derived optimal $H^1$ and $L^2$ error bounds for isoparametric elements in the presence of domain perturbation and numerical integration.

It is appropriate to mention here the recent work of Feistauer & Ženišek (1987) concerning variational crimes for a nonlinear elliptic problem and the work of Čermák & Zlamal (1986) and Čermák (1987) concerning the use of fitted and unfitted meshes for the finite-element approximation of moving-boundary problems for parabolic equations.

In this paper, we give a simplified proof of Čermák's theorem; further, the proof is applicable to unfitted as well as fitted meshes. We show that, for a finite-element space defined over $\hat{D}^h$, a union of elements, with approximation power $h^k$ in the $L^2$ norm and with dist $(\Omega, \Omega^h) \leq Ch^k$, one retains optimal rates of convergence for the error in the $H^1$ and $L^2$ norms whether $\Omega^h$ is fitted ($\hat{\Omega}^h = \hat{D}^h$) or unfitted ($\hat{\Omega}^h \subset \hat{D}^h$), provided that the numerical integration scheme is of sufficient accuracy. Our proof is shorter than that of Čermák (1983a), avoiding the key technical lemma: Lemma 3.3, pp. 443–451. We note that unfitted meshes have useful practical applications to free-boundary and moving-boundary problems; see Barrett & Elliott (1982, 1985) for example.

The outline of the paper is as follows. In the next section we define a finite-element approximation to (1.2), stating our assumptions (A1)→(A5). (A1) is an approximation assumption on the finite-element space. (A2) is an assumption on the domain perturbation $\Omega$ to $\Omega^h$. (A3)→(A5) are assumptions on the numerical integration scheme. Under assumptions (A1)→(A5), optimal $H^1$ and $L^2$ error estimates are derived in Sections 3 and 4. Assumptions (A1) and (A2) are easily seen to be applicable to most practical finite-element schemes. The assumptions (A3)→(A5) are justified in Section 5. Finally, in Section 6, we report on a numerical example on an unfitted mesh.
Throughout this paper, we adopt the standard notation $W^{m,p}(G)$ for Sobolev spaces on a bounded domain $G$ with norm $\|\cdot\|_{m,p,G}$ and seminorm $|\cdot|_{m,p,G}$. For $p=2$, we adopt the convention $H^m(G) = W^{m,2}(G)$, with $\|\cdot\|_{m,G} = \|\cdot\|_{m,2,G}$ and $|\cdot|_{m,G} = |\cdot|_{m,2,G}$.

If $\partial G$, the boundary of $G$, is of class $C^{0,1}$ and piecewise $C^{m,1}$, i.e. $\partial G = \bigcup_{L \in L} \partial L G$ with $\partial L G$ of class $C^{m,1}$, we define

$$\|w\|_{m,p,\partial G} = \left(\sum_{t=1}^{N} \|w\|_{m,p,\partial L G}^p\right)^{1/p},$$

where $\|\cdot\|_{m,p,\partial G}$ is the standard norm on the space of traces $W^{m,p}(\partial G)$, see Kufner, John, & Fučík (1977: Ch. 6) for a full description.

The measure of a domain $G$ is denoted by $m(G)$. Throughout, $C$ denotes a positive constant independent of $h$ whose value may differ in different relations. We require also the trace inequalities: for $\partial G$ of class $C^{0,1}$, we have

$$\|D^n w\|_{0,\partial G} \leq C \|w\|_{n+1,G} \quad \forall w \in H^{n+1}(G). \quad (1.3a)$$

This implies that

$$\left\|\frac{\partial w}{\partial v}\right\|_{0,\partial G} \leq C \|w\|_{2,G} \quad \forall w \in H^2(G) \quad (1.3b)$$

and, for $\partial G$ of class $C^{0,1}$ and piecewise $C^{m,1}$,

$$\|w\|_{m,\partial G} \leq C \|w\|_{m+1,G} \quad \forall w \in H^{m+1}(G), \quad (1.3c)$$

where $v$ is the outward-pointing unit normal to $\partial G$ and $C$ is a constant independent of $w$; see Kufner, John, & Fučík (1977).

Finally, we require the Friedrichs inequality for $\partial G$ of class $C^{0,1}$:

$$\|w\|_{0,G}^2 \leq C (\|w\|^2_{1,G} + \|w\|^2_{0,\partial G}) \quad \forall w \in H^1(G), \quad (1.4)$$

where $C$ is a constant independent of $w$.

2. Finite-element approximation

The variational form of (1.2) is: find $u \in H^1(\Omega)$ such that

$$a(u, v) = l(v) \quad \forall v \in H^1(\Omega), \quad (2.1)$$

where

$$a(w, v) = (\sigma \nabla w, \nabla v)_{\Omega} + (cw, v)_{\Omega} + (aw, v)_{\partial \Omega}, \quad (2.2a)$$

$$l(v) = (f, v)_{\Omega} + (g, v)_{\partial \Omega}; \quad (2.2b)$$

here we have adopted the notation $(w, v)_\Omega = \int_G uv \, dx$, and $(w, v)_{\partial \Omega} = \int_{\partial \Omega} uv \, ds$. It is assumed that either $c_0$ or $a_0$ is nonzero, so that $a(\cdot, \cdot)$ is coercive over $H^1(\Omega)$ and hence (2.1) is a well-posed variational problem for data $f \in L^2(\Omega)$ and $g \in L^2(\partial \Omega)$.

Consider the finite-element approximation of (2.1). The domain $\Omega$ and its boundary $\partial \Omega$ are approximated respectively by $\Omega^h$ and $\partial \Omega^h$. We wish our analysis to cover the case of unfitted as well as fitted meshes. We consider first the
case of a fitted mesh. By a fitted mesh we mean that

$$\Omega^h = \bigcup_{\tau \in T^h} \tilde{\tau},$$

where $T^h$ is a collection of disjoint open regular elements $\tau$, each of maximum diameter not exceeding $h$. It is assumed that each element $\tau$ has at most one face on $\partial \Omega^h$ and only faces on $\partial \Omega^h$ are allowed to be curved.

Associated with $T^h$ is a finite-dimensional subspace $S^h$ of $W^{1,m}(\Omega^h)$, depending upon an integer $k \geq 2$, such that $\chi|_{\tau} \in H^2(\tau)$ $\forall \chi \in S^h$ and $\forall \tau \in T^h$. Let $\pi_h : C^0(\tilde{\Omega}^h) \to S^h$ denote the interpolation operator. We assume that the following approximation property holds.

(A1) For integers $k'$ and $m$ satisfying $k' \geq m \geq 0$ and $k \geq k' \geq 2$,

$$\|w - \pi_h w|_{m, \tau}\leq Ch^{k'-m} \|w\|_{k', \tau} \quad \forall w \in H^{k'}(\tau) \quad \forall \tau \in T^h,$$  \hspace{1cm} (2.3)

where $C$ is a constant independent of $h$ and $w$. □

We make the following assumption on $\Omega^h$ in the case of a fitted mesh.

(A2) Setting

$$B^h = \{\tau \in T^h : m(\tilde{\tau} \cap \partial \Omega^h) \neq 0 \text{ in } \mathbb{R}^{n-1}\},$$  \hspace{1cm} (2.4)

then, for each element $\tau \in B^h$, there exists a local coordinate system $(X_\tau, Y_\tau)$ such that $X_\tau \in \Delta_\tau \subset \mathbb{R}^{n-1}$ and $Y_\tau \in \mathbb{R}$. The surface $\partial \Omega^h = \partial \Omega^h \cap \tilde{\tau}$ is locally described by $Y_\tau = \psi^h_\tau(X_\tau)$. The surface $\partial \Omega$ is locally described by $Y_\tau = \psi_\tau(X_\tau)$, and we denote this section of $\partial \Omega$ by $\partial \Omega_\tau$. It is assumed that $\psi_\tau, \psi^h_\tau \in C^{1,1}(\Delta_\tau)$ and that $\psi_\tau$ and $\psi^h_\tau$ vanish at the vertices of $\Delta_\tau$. This immediately implies that

$$\|\nabla \psi_\tau\|_{0, \infty, \Delta_\tau} \leq Ch \quad \text{and} \quad \|\nabla \psi^h_\tau\|_{0, \infty, \Delta_\tau} \leq Ch \quad \forall \tau \in B^h.$$  \hspace{1cm} (2.5a)

Further, for $\psi_\tau(\cdot) \in C^4(\Delta_\tau)$ it is assumed that

$$\|\psi_\tau - \psi^h_\tau\|_{m, \infty, \Delta_\tau} \leq Ch^{k'-m} \quad (m = 0, 1) \quad \forall \tau \in B^h.$$  \hspace{1cm} (2.5b)

Remark 2.1. The assumptions (A1) and (A2) are satisfied for a $(k - 1)$-regular family of simplicial Lagrangian isoparametric elements as introduced by Ciarlet & Raviart (1972). In this case, $\psi^h_\tau$ agrees with $\psi_\tau$ at all nodes lying on $\partial \Omega^h$; see Čermák (1983a) for a full description. □

We now consider the case of an unfitted mesh as introduced by Barrett & Elliott (1984). Let $D^*$ be a bounded domain in $\mathbb{R}^n$ containing $\Omega$ such that $D^* = \bigcup_{\tau \in T^*} \tilde{\tau}$, where $T^*$ is a collection of disjoint regular open elements $\tau$, each of maximum diameter not exceeding $h$ and having no curved faces. We stress that this partition of $D^*$ is totally independent of the domain $\Omega$, and would be a uniform partition in most cases. We assume that

$$\tilde{\Omega} \subseteq \tilde{D} \subseteq D^*,$$

where $\tilde{D} = \bigcup_{\tau \in T^*} \tilde{\tau}$ and $T = \{\tau \in T^* : \tau \cap \Omega \neq \emptyset\}$.

One could then define a finite-element approximation to (2.1): find $u_*^h \in S^h$
such that

$$a(u^h, \chi) = l(\chi) \quad \forall \chi \in S^h,$$

(2.6)

where $S^h$ is a finite-dimensional subspace of $W^{1,\infty}(D)$ with interpolation operator $\pi_h : C^0(\bar{D}) \to S^h$ satisfying (A1) for all $\tau \in T$. The approximation $u^h$ of (2.6) is an example of what we call 'an unfitted mesh approximation', since $u^h$ is defined over $\bar{D}$, which is a union of regular elements, but $\bar{Q}$ is not a union of regular elements. We note that approximations of this type were mentioned by Babuška (1971). However, the approximation (2.6) is not practical, since it requires integrals to be computed over the curved regions $\Omega \cap \tau$ and $\partial \Omega \cap \bar{\tau}$ ($\tau \in T$).

A practical unfitted-mesh approximation is obtained by replacing $\bar{Q}$ by $\bar{Q}^h$ so that (A2) is satisfied and the integrals of the type $\Omega^h \cap \tau$ and $\partial \Omega^h \cap \bar{\tau}$ can be evaluated to sufficient accuracy using standard quadrature rules. We now seek an approximation in $S^h$, a finite-dimensional subspace of $W^{1,\infty}(D^h)$, with interpolation operator $\pi_h : C^0(\bar{D}^h) \to S^h$ satisfying (A1) for all $\tau \in T^h$, where

$$\Omega^h \subseteq \bar{D}^h \subseteq \bar{D}^* \subseteq \mathbb{R}^n, \quad \bar{D}^h = \bigcup_{\tau \in T^h} \bar{\tau}, \quad T^h = \{ \tau \in T^* : \tau \cap \Omega^h \neq \emptyset \}. \quad (2.7a,b,c)$$

We now give an explicit construction for $\Omega^h$ so that (A2) holds in the case of continuous piecewise linears on regular simplices $\tau$ ((A1) holds with $k = 2$). Consider first the case $n = 2$. We assume that $\partial \Omega$ is sufficiently smooth and $h$ sufficiently small that $\partial \Omega$ crosses any one triangle side at most twice. A polygonal domain $\Omega^h$ approximating $\Omega$ is constructed in the following way. For a triangle $\tau \in T^*$ such that $m(\bar{\tau} \cap \partial \Omega) \neq 0$ in $\mathbb{R}$, and with at least one vertex in $\Omega$, then the arc $\partial \Omega$ in $\tau$ is approximated by its chord joining the points where it intersects the boundary of the triangle. If $\partial \Omega$ crosses the boundary of such a triangle more than twice, then the approximating chord is taken to be the one that joins the first point of entry to the last point of exit. Note that $\tau \in T$ does not necessarily imply that $\tau \in T^h$. See Fig. 2.1 for examples. Choosing the local coordinate system such

![Diagram](https://via.placeholder.com/150)

**Fig. 2.1.** Examples of the construction of $\partial \Omega^h$ for $n = 2$. 
that $Y_\tau = 0$ corresponds to $\partial \Omega^h$, we obtain from standard interpolation theory that (A2) holds. The above construction generalizes in a natural way to the case $n = 3$. In each simplex $\tau \in T^*$ such that $m(\bar{\tau} \cap \partial \Omega) \neq 0$ in $\mathbb{R}^2$ and with at least one vertex in $\Omega$, then the surface $\partial \Omega$ in $\tau$ is approximated by a plane. Again with the choice of the local coordinate system such that $Y_\tau = 0$ corresponds to $\partial \Omega^h$, it follows from standard interpolation theory that (2.5b) holds for $m = 0$ and (2.5a) holds, which immediately implies that (2.5b) holds for $m = 1$ as well.

The generalization of the above to higher-order simplicial Lagrangian elements, with $k > 2$, requires approximating $\partial \Omega$ in an element $\tau$ by an interpolating polynomial of degree $k - 1$ such that $n$ of the interpolation points occur where $\partial \Omega$ crosses either the element sides ($n = 2$) or edges ($n = 3$). In the case $n = 2$, it follows from standard interpolation theory that (A2) is satisfied. However, for $n = 3$, it is possible that condition (2.5b) is satisfied only for $m = 0$, since $\Delta_\epsilon$ could be degenerate. In addition, for a practical scheme, one requires quadrature formulae to evaluate the integrals $\Omega^h \cap \tau$ and $\partial \Omega^h \cap \bar{\tau}$, and there are problems in the case $k > 2$ since these regions are curved. Therefore an unfitted-mesh approximation is at present only practical for continuous piecewise linears on simplices or continuous piecewise bilinears on quadrilaterals; see Barrett & Elliott (1984).

Throughout this paper we prove our results under the assumptions (A1) and (A2). In order to present a unified treatment of fitted and unfitted meshes, we extend the notation of (2.7) to fitted meshes by setting $D^h = \bar{\Omega}^h$. Next we note the following result.

**Lemma 2.1** The trace inequalities (1.3) for $0 \leq m \leq k$ and the Friedrichs inequality (1.4) hold for $G = \Omega^h$ with the constants $C$ independent of $h$.

**Proof.** Inequalities (1.3a,b,c) follow from the proof of the trace theorem in Nečas (1967: p. 15). (1.4) follows from the proof of Friedrichs inequality in Rektorys (1977: Chs 18, 30). $\square$

**Remark 2.2.** We note that, in order to extend the results of this paper to the mixed boundary-value problem, i.e. (1.2a) with (1.2b) replaced by

$$u = g_1 \quad \text{on} \quad \partial_1 \Omega, \quad \sigma \frac{\partial u}{\partial \nu} + au = g_2 \quad \text{on} \quad \partial_2 \Omega,$$

where $\partial \Omega = \partial_1 \Omega \cup \partial_2 \Omega$, the mesh would have to be fitted to the Dirichlet boundary, and one would have to generalize the discrete Friedrichs inequality results of Ženěšek (1981b) and Čermák (1983b) to an unfitted mesh. $\square$.

Because (in general) $\Omega^h \notin \Omega$, it is necessary to extend the data. It is convenient to introduce a domain $\tilde{\Omega} \subset \mathbb{R}^n$ with a smooth boundary such that

$$\Omega^h \subseteq D^h \subseteq \tilde{\Omega} \quad \forall \ h < h_0, \quad \Omega \subseteq \tilde{\Omega}. \quad (2.8)$$

For all integers $s \geq 0$, there exists an extension operator $E : H^s(\Omega) \rightarrow H^s(\tilde{\Omega})$ such that

$$Ew = w \quad \text{on} \quad \Omega, \quad \|Ew\|_{s, \tilde{\Omega}} \leq C \|w\|_{s, \Omega}, \quad (2.9a,b)$$

where $C$ is independent of $w$. (See Kufner, John, & Fučík (1977).)
We make the following regularity assumptions on the data.

\( f \in H^k(\Omega) \);  

(2.10a)

g and \( \alpha \) are the restrictions to \( \partial \Omega \) of functions \( \tilde{g} \) and \( \tilde{\alpha} \) such that

\( \tilde{g} \in H^{k+1}(\tilde{\Omega}), \quad \tilde{\alpha} \in C^{k-1}(\tilde{\Omega}) \);  

(2.10b)

and \( \sigma \) and \( c \) are the restrictions to \( \Omega \) of \( \tilde{\sigma} \) and \( \tilde{c} \) such that

\( \tilde{\sigma} \in C^{k+1}(\tilde{\Omega}), \quad \tilde{c} \in C^k(\tilde{\Omega}) \);  

(2.10c)

here,

\( \tilde{\sigma}(x) \geq \sigma_0 > 0, \quad \tilde{c}(x) \geq c_0 > 0, \quad \tilde{\alpha}(x) \geq \alpha_0 > 0, \) (2.11a,b,c)

for all \( x \in \tilde{\Omega} \). We assume that either \( \sigma_0 \) or \( \alpha_0 \) is nonzero. In addition we assume that \( \partial \Omega \) is of class \( C^{k+1,1} \), and then it follows from elliptic regularity theory that the solution \( u \) of (2.1) is such that

\( u \in H^{k+2}(\Omega) \).  

(2.12)

We define

\[ \tilde{A} w = - \nabla \cdot (\tilde{\sigma} \nabla w) + \tilde{c} w, \]  

(2.13)

\[ \tilde{a}^h(w, v) = (\tilde{\sigma} \nabla w, \nabla v)_\Omega + \langle \tilde{c} w, v \rangle_\Omega + \langle \tilde{\alpha} w, v \rangle_{\partial \Omega^h}, \]  

(2.14a)

\[ = (\tilde{A} w, v)_\Omega + \langle \frac{\partial w}{\partial v^h} + \tilde{\alpha} w, v \rangle_{\partial \Omega^h}, \]  

(2.14b)

where \( \partial \Omega^h \) denotes differentiation along the outward-pointing normal to \( \partial \Omega^h \).

We define

\[ \tilde{l}^h(v) = \langle \tilde{f}, v \rangle_\Omega + \langle \tilde{g}, v \rangle_{\partial \Omega^h}, \]  

(2.15)

where \( \tilde{f} \) is an extension of \( f \). It follows from (2.8) and the trace inequality that there exists a constant \( C \) such that, for all \( w, v \in H^1(\Omega^h) \),

\[ |\tilde{a}^h(w, v)| \leq C \| w \|_{1, \Omega^h} \| v \|_{1, \Omega^h}. \]  

(2.16)

A possible finite-element approximation to (2.1) which takes into account the change in domain is: find \( u^h \in S^h \) such that

\[ \tilde{a}^h(u^h, \chi) = \tilde{l}^h(\chi) \quad \forall \chi \in S^h. \]  

(2.17)

Note that \( \tilde{u}^h \) depends on the extensions \( \tilde{\sigma}, \tilde{c}, \tilde{\alpha}, \tilde{f}, \) and \( \tilde{g} \). Since it is usually impossible or computationally inconvenient to evaluate exactly the integrals in (2.17), a fully practical scheme is: find \( u^h \in S^h \) such that

\[ a^h(u^h, \chi) = l^h(\chi) \quad \forall \chi \in S^h, \]  

(2.18)

where \( a^h(\cdot, \cdot) \) and \( l^h(\cdot) \) are approximations to \( \tilde{a}^h(\cdot, \cdot) \) and \( \tilde{l}^h(\cdot) \).

In order to guarantee the well-posedness of (2.18), we make the following assumption.

(A3) There exists a constant \( C \), independent of \( h \) and \( \chi \), such that

\[ a^h(\chi, \chi) \geq C \| \chi \|_{1, \Omega^h}^2 \quad \forall \chi \in S^h. \]  

(2.19)
To obtain asymptotic orders of convergence for the error \( u - u^h \), it is necessary to make an accuracy assumption concerning the numerical integration method. Setting

\[
\tilde{a}^h(w, v) = \tilde{a}_1^h(w, v) + \tilde{a}_2^h(w, v), \quad \tilde{l}^h(v) = \tilde{l}_1^h(v) + \tilde{l}_2^h(v),
\]

where

\[
\tilde{a}_1^h(w, v) = (\tilde{\sigma} \nabla w, \nabla v)_{Q^*} + (\tilde{c} w, v)_{Q^*}, \quad \tilde{a}_2^h(w, v) = \langle \tilde{a} w, v \rangle_{\partial Q^*},
\]

\[
\tilde{l}_1^h(v) = \langle \tilde{f}, v \rangle_{\partial Q^*}, \quad \tilde{l}_2^h(v) = \langle \tilde{g}, v \rangle_{\partial Q^*},
\]

and \( a_1^h(\cdot, \cdot), \; a_2^h(\cdot, \cdot), \; l_1^h(\cdot), \) and \( l_2^h(\cdot) \) are their respective approximations, we make the following assumption.

**(A4)** For \( h < h_0 \), it is assumed, for all \( w \in H^k(\tilde{Q}) \) and for all \( w^h \) and \( \chi \in S^h \), that

\[
|\tilde{a}^h(w^h, \chi) - a^h(w^h, \chi)| \leq C(h^{-1} \|w\|_{k, Q^*} + \|w - w^h\|_{1, Q^*}) \|\chi\|_{1, Q^*},
\]

\[
|M\chi - \tilde{M}\chi| \leq C h^{-1} (\|\tilde{M}\chi\|_{k, Q^*} + \|\tilde{g}\|_{k+1, Q^*}) \|\chi\|_{1, Q^*},
\]

\[
|M\chi - \tilde{M}\chi| \leq C h^k (\|\tilde{M}\chi\|_{k, Q^*} + \|\tilde{g}\|_{k+1, Q^*}) \left( \sum_{te T^h} \|\chi\|_{2, t}^2 \right)^{1/2}.
\]

In addition, we make the following assumption.

**(A5)** (i) \( a_1^h(\cdot, \cdot) \) and \( l_1^h(\cdot) \) are assumed to depend on the evaluation of \( \tilde{\sigma}, \tilde{c}, \) and \( \tilde{f} \) at points in \( \tilde{Q} \). (ii) \( a_2^h(\cdot, \cdot) \) and \( l_2^h(\cdot) \) are assumed to depend on the evaluations of \( \tilde{a} \) and \( \tilde{g} \) at points on \( \partial Q \).

**Remark 2.3.** It follows from assumption (A5) that the approximation \( u^h \) is independent of all the extensions. However, (ii) is not necessary for the error analysis, and practical schemes dependent on computable extensions of \( \alpha \) and \( g \) can also be derived.

Under the assumptions (A1)→(A5), it is shown in Sections 3 and 4 that \( u^h \) retains the optimal rate of convergence in the \( H^1 \) and \( L^2 \) norms. Thus the optimal rate is achieved, whether the mesh is fitted or unfitted, provided that (A3)→(A5) hold. Finally, in Section 5, we justify these numerical-integration assumptions for a \( (k - 1) \)-regular family of simplicial Lagrangian isoparametric fitted elements and for an unfitted-mesh approximation using piecewise linears on simplices.

### 3. \( H^1 \) error bound

Since the approximation \( u^h \) is independent of the extension \( f, \tilde{f} \) may be chosen for the convenience of the analysis. A suitable choice is

\[
\tilde{f} = \tilde{A} \tilde{u},
\]

where \( \tilde{u} = E u \). Clearly

\[
\tilde{f} = Au = f \quad \text{in } \Omega
\]
and (3.1) together with (2.12) and (2.9) imply that
\[
\|\tilde{f}\|_{k,\Omega^h} = \|\tilde{A}u\|_{k,\Omega^h} \leq C \|\bar{u}\|_{k+2,\Omega^h} \\
\leq C \|u\|_{k+2,\Omega^h}
\] (3.2)

The following estimates are useful in deriving the error bounds.

**Lemma 3.1** Assume that (A2) holds. Let \(w, \beta, \) and \(q\) satisfy
\[
\frac{\partial w}{\partial \nu} + \beta w = q \quad \text{on} \quad \partial \Omega,
\] (3.3)

where \(\beta \in W^{1,\infty}(\tilde{\Omega})\).

If \(w \in H^2(\tilde{\Omega})\) and \(q \in H^1(\tilde{\Omega})\), then
\[
\left\| \frac{\partial w}{\partial \nu^h} + \beta w - q \right\|_{0, \partial \Omega^h} \leq C h^k (\|w\|_{2,\tilde{\Omega}} + \|q\|_{1,\tilde{\Omega}}).
\] (3.4a)

If \(w \in W^{2,\infty}(\tilde{\Omega})\) and \(q \in W^{1,\infty}(\tilde{\Omega})\), then
\[
\left\| \frac{\partial w}{\partial \nu^h} + \beta w - q \right\|_{0, \partial \Omega^h} \leq C h^{k-1} (\|w\|_{2,\tilde{\Omega}} + \|q\|_{1,\tilde{\Omega}}).
\] (3.4b)

**Proof.** The proofs of (3.4a) and (3.4b) are similar to the proofs of Lemmas 3.4 and 3.2, respectively, of Čermák (1983a).

We now present abstract \(H^1\) and \(L^2\) error estimates. This is a generalization of Theorem 1 in Ciarlet & Raviart (1972), which dealt with the case of homogeneous Dirichlet data. A similar exercise is carried out in Theorem 3.1 of Čermák (1983a).

**Lemma 3.2** Assume that (A1) \(\rightarrow\) (A3) hold. The solutions \(u\) and \(u^h\) of (2.1) and (2.18) satisfy
\[
\|\bar{u} - u^h\|_{1,\Omega^h} \leq C \left[ \inf_{\tilde{\xi} \in \tilde{\mathcal{S}}^h} \left( \|\tilde{u} - \tilde{\xi}\|_{1,\Omega^h} + \sup_{\chi \in \mathcal{S}^h} \frac{|\tilde{a}^h(\tilde{\xi}, \chi) - a^h(\tilde{\xi}, \chi)|}{\|\chi\|_{1,\Omega^h}} \right) + \sup_{\chi \in \mathcal{S}^h} \frac{|\tilde{l}^h(\chi) - l^h(\chi)| + |\tilde{a}^h(\bar{u}, \chi) - \tilde{l}^h(\bar{u})|}{\|\chi\|_{1,\Omega^h}} \right],
\] (3.5)

\[
\|\bar{u} - u^h\|_{0,\Omega^h} \leq C \left( \|\bar{u} - u^h\|_{0,\Omega^h} + h \|\bar{u} - u^h\|_{1,\Omega^h} + \sup_{\tilde{\xi} \in \tilde{\mathcal{S}}^h} \frac{|\tilde{a}^h(\bar{u}, \tilde{\xi}) - a^h(\bar{u}, \tilde{\xi})|}{\|\tilde{\xi}\|_{1,\Omega^h}} + |\tilde{l}^h(\bar{\pi}_h \tilde{\xi}) - l^h(\bar{\pi}_h \tilde{\xi})| + |\tilde{a}^h(\bar{u}, \tilde{\xi}) - \tilde{l}^h(\bar{\pi}_h \tilde{\xi})| \right),
\] (3.6)

where \(\bar{u} = Eu\) and \(\tilde{\xi} = Ez\).
Proof. Evidently the inequality
\[ \| \bar{u} - u^h \|_{1, \Omega^*} \leq \| \bar{u} - \xi \|_{1, \Omega^*} + \| \xi - u^h \|_{1, \Omega^*} \]
holds for all \( \xi \in S^h \). Setting \( \chi = u^h - \xi \in S^h \), we obtain from (2.19) and (2.18) that
\[ \| \chi \|_{1, \Omega^*}^2 \leq C^{-1} a^h(\chi, \chi) = C^{-1} \{ \hat{a}^h(\bar{u} - \xi, \chi) + [\hat{a}^h(\xi, \chi) - a^h(\xi, \chi)] + \| t^h(\chi) - \tilde{t}^h(\chi) \| + [\tilde{t}^h(\chi) - a^h(\bar{u}, \chi)] \}. \]

Bounding the first term using (2.16) and dividing throughout by \( \| \chi \|_{1, \Omega^*} \) yields the desired result (3.5).

Observe that
\[ \bar{u}^h = \sup_{\eta \in L^2(\Omega^*)} \frac{|(\bar{u} - u^h, \eta)_{\Omega^*}|}{\| \eta \|_{0, \Omega^*}}. \]  

Let \( \bar{u} \) be defined by \( \bar{u} = \bar{u}^h \) over \( \Omega^h \) and \( \bar{u} = 0 \) over \( \tilde{\Omega} \setminus \Omega^h \), and let \( z \) be defined by
\[ Az = \bar{u} \quad \text{in } \Omega, \quad \sigma \frac{\partial z}{\partial v} + \alpha z = 0 \quad \text{on } \partial \Omega. \]

It follows from elliptic regularity theory that
\[ \| z \|_{2, \Omega} \leq C \| \bar{u} \|_{0, \Omega} \leq C \| \eta \|_{0, \Omega^*}. \]

It follows from (2.14) that
\[ (u^h, \eta)_{\Omega^*} = (\bar{u} - u^h, A\bar{z})_{\Omega^*} + (\bar{u} - u^h, \eta - \bar{A}\bar{z})_{\Omega^*}, \]

where \( \bar{z} = Ez. \) For any \( \chi \in S^h \), the equality
\[ \hat{a}^h(\bar{u} - u^h, \bar{z}) = \hat{a}^h(\bar{u} - u^h, \bar{z} - \chi) + [a^h(u^h, \chi) - \hat{a}^h(u^h, \chi)] + \| t^h(\chi) - \tilde{t}^h(\chi) \|_{\Omega^*} + [\tilde{t}^h(\chi) - a^h(\bar{u}, \chi)] \]  

holds. From (3.4a) with \( w = \bar{z}, \beta = \alpha/\sigma, \) and \( q = 0, (2.9), (3.8), \) and the trace inequalities (1.3), we have
\[ \left| \left( \bar{\sigma} \frac{\partial \bar{z}}{\partial v^h} + \bar{\alpha} \bar{z}, \bar{u} - u^h \right)_{\partial \Omega^*} \right| \leq C \| \eta \|_{0, \Omega^*} \| \bar{u} - u^h \|_{1, \Omega^*}. \]

The term \( \| A\bar{z} \|_{0, \Omega^*} \) can be bounded by \( C \| \bar{z} \|_{2, \Omega^*} \) by noting that
\[ \| \bar{A}\bar{z} \|_{0, \Omega^*} \leq C \| \bar{z} \|_{2, \Omega^*}. \]

The desired result (3.6) follows from (3.7), (3.9), (3.10) with \( \chi = \pi^h \bar{z}, (2.16), (3.12), (2.9), \) and (3.8). \( \square \)

Combining the above lemmas, we obtain the \( H^1 \) error bound. Once again, a similar exercise is carried out in Theorem 3.1 of Čermák (1983a).

**Theorem 3.1** Let the assumptions (A1)→(A5), concerning the finite-element
approximation (2.18), hold. The solutions $u$ and $u^h$ of (2.1) and (2.18) satisfy

$$\|\tilde{u} - u^h\|_{1,\Omega^*} \leq Ch^{k-1}(\|u\|_{k+2,\Omega} + \|\tilde{g}\|_{k+1,\Omega^*}),$$

(3.13)

where $\tilde{u} = Eu$.

**Proof.** Choosing $\xi = \pi_h \tilde{u}$ in (3.5), together with (2.21a, c) and (3.2), yields

$$\|\tilde{u} - u^h\|_{1,\Omega^*} \leq C\left(\|\tilde{u} - \pi_h \tilde{u}\|_{1,\Omega^*} + h^{k-1} \|u\|_{k+2,\Omega} + h^{k-1} \|\tilde{g}\|_{k+1,\Omega^*} + \sup_{\chi \in \Omega^*} \frac{|\tilde{a}^h(\tilde{u}, \chi) - \tilde{l}^h(\chi)|}{\|\chi\|_{1,\Omega^*}}\right).$$

(3.14)

From (2.14), (2.15), and (3.1), it follows that

$$|\tilde{a}^h(\tilde{u}, \chi) - \tilde{l}^h(\chi)| = \left|\left(\bar{\partial}_y \tilde{u} + \tilde{g}, \chi\right)_{\Omega^*}\right|$$

$$\leq Ch^{k-1}(\|\tilde{u}\|_{2,\Omega} + \|\tilde{g}\|_{1,\Omega}) \|\chi\|_{0,\Omega^*}$$

$$\leq Ch^{k-1}(\|u\|_{4,\Omega} + \|\tilde{g}\|_{3,\Omega}) \|\chi\|_{1,\Omega^*},$$

(3.15)

where we have used the regularity assumptions (2.10), (2.12), Sobolev's embedding theorem, and the result (3.4b). Combining (3.14) with (3.15) and the approximation property (2.3) yields the desired result (3.13). □

4. $L^2$ error bound

To derive an $L^2$ error bound using (3.6), it is necessary to estimate $\|\tilde{u} - u^h\|_{0,\Omega^\prime,\Omega}$.

**Lemma 4.1** Assuming (A2), we have, for all $w \in H^1(\Omega^\prime)$, the bounds

$$\|w\|_{0,\Omega^\prime,\Omega} \leq C(h^k \|w\|_{1,\Omega^\prime,\Omega} + h^{k+1} \|w\|_{0,\Omega^\prime,\Omega^*}),$$

(4.1a)

$$\|w\|_{0,\Omega^\prime,\Omega^*} \leq C(h^k \|w\|_{1,\Omega^\prime,\Omega} + h^{k+1} \|w\|_{0,\Omega^\prime,\Omega}).$$

(4.1b)

**Proof.** The proof of (4.1a) is given in Lemma 3.2 in Barrett & Elliott (1987). The proof of (4.1b) follows in a similar manner. □

In proving an $L^2$ estimate, the crucial term to bound on the right-hand side of (3.6) is

$$|\tilde{a}^h(\tilde{u}, \pi_h \tilde{z}) - \tilde{l}^h(\pi_h \tilde{z})|.$$

As in the proof of Theorem 3.1 (see (3.15)), this term can be bounded above by

$$Ch^{k-1}(\|u\|_{4,\Omega} + \|\tilde{g}\|_{3,\Omega}) \|\pi_h \tilde{z}\|_{1,\Omega^*}.$$

However, this would lead to a sub-optimal $L^2$ error estimate. Below, we obtain an improved bound for this term. The result (4.2) is the main reason why our proof is shorter than that of Cermák (1983a).

**Lemma 4.2** Let (A1) and (A2) hold. Then, for all $w \in H^2(\Omega)$, it follows that

$$|\tilde{a}^h(\tilde{u}, \pi_h \tilde{w}) - \tilde{l}^h(\pi_h \tilde{w})| \leq Ch^k(\|u\|_{4,\Omega} + \|\tilde{g}\|_{3,\Omega}) \|w\|_{2,\Omega},$$

(4.2)

where $\tilde{u} = Eu$ and $\tilde{w} = Ew$. 
Proof. Evidently we have
\[ |\hat{a}^h(\tilde{u}, \pi_h\tilde{w}) - \tilde{l}^h(\pi_h\tilde{w})| \leq |\hat{a}^h(\tilde{u}, \tilde{w} - \pi_h\tilde{w}) - \tilde{l}^h(\tilde{w} - \pi_h\tilde{w})| + |\hat{a}^h(\tilde{u}, \tilde{w}) - \tilde{l}^h(\tilde{w})|. \] (4.3)

From an argument identical to that used in deriving (3.15), the approximation property (2.3), and (2.9), it follows that
\[ |\hat{a}^h(\tilde{u}, \tilde{w} - \pi_h\tilde{w}) - \tilde{l}^h(\tilde{w} - \pi_h\tilde{w})| \leq Ch^k(\|u\|_{4,\Omega} + \|\tilde{g}\|_{3,\Delta})\|w\|_{2,\Omega}. \] (4.4)

It follows from (2.1) that
\[
\hat{a}^h(\tilde{u}, \tilde{w}) - \tilde{l}^h(\tilde{w}) = [\hat{a}^h(\tilde{u}, \tilde{w}) - a(\tilde{u}, \tilde{w})] + [l(\tilde{w}) - \tilde{l}^h(\tilde{w})]
= \int_{\Omega \setminus \Omega^0} (\tilde{\partial} \tilde{u} \cdot \nabla \tilde{w} + \tilde{c} \tilde{u} \tilde{w} - \tilde{f} \tilde{w}) \, dx - \int_{\partial \Omega \setminus \partial \Omega^0} (\sigma \nabla u \cdot \nabla w + cuw - fw) \, dx
+ \left[ \langle \tilde{a} \tilde{u} - \tilde{g}, \tilde{w} \rangle_{\partial \Omega^0} - \langle \alpha u - g, w \rangle_{\partial \Omega^0} \right].
\] (4.5)

From Lemma 4.1 and the trace inequality (1.3a) we have, for all \( v \in H^1(\tilde{\Omega}) \), the bounds
\[ \|v\|_{0,\partial \Omega^0} \leq Ch_{1,\Omega} \|v\|_{1,\Omega}, \quad \|v\|_{0,\partial \Omega^0} \leq Ch_{1,\Omega} \|v\|_{1,\Omega}, \]
which imply, for all \( v \in H^2(\tilde{\Omega}) \), that
\[ \|v\|_{1,\partial \Omega^0} + \|v\|_{1,\partial \Omega^0} \leq Ch_{1,\Omega} \|v\|_{2,\Omega}. \] (4.6)

Therefore the first two (integral) terms on the right-hand side of (4.5) can be bounded by \( Ch^k(\|u\|_{2,\Omega} + \|A\tilde{u}\|_{2,\Omega})\|w\|_{2,\Omega} \leq Ch^k\|u\|_{4,\Omega}\|w\|_{2,\Omega} \).

Let
\[ J = 1 + |\nabla \psi|^2, \quad J^h = 1 + |\nabla \psi|^2, \]
where, for convenience, the subscript \( \tau \) has been suppressed in the notation of (A2). Adopting the notation \( w = w(X, \psi(X)) \) and \( w(h) = (X, \psi^h(X)) \) etc and setting \( p = (\tilde{a} \tilde{u} - \tilde{g}) \tilde{w} \), we have
\[
\left| \int_{\partial \Omega^0} p \, ds^h - \int_{\partial \Omega} p \, ds \right| = \left| \sum_{\tau \in \mathcal{B}^h} \int_{\Delta_\tau} [p(h)J^h - pJ] \, dX \right|
\leq \sum_{\tau \in \mathcal{B}^h} \left| \int_{\Delta_\tau} \left( \int_{\psi(X)} \frac{\partial}{\partial Y} p(X, Y) \, dY \right) J^h \, dX \right|
+ \sum_{\tau \in \mathcal{B}^h} \left| \int_{\Delta_\tau} (p(J^h - J) \, dX \right|
\leq C \sum_{\tau \in \mathcal{B}^h} \left[ m(\Delta_\tau)\|\psi^h - \psi\|_{0,\infty,\Delta_\tau} \right] \left( \int_{\Delta_\tau} \left[ \int_{\psi(X)} \left( \frac{\partial}{\partial Y} p(X, Y)^2 \right) \, dY \right] \, dX \right)^{\frac{1}{2}}
+ C \sum_{\tau \in \mathcal{B}^h} m(\Delta_\tau)^{\frac{1}{2}} \|J^h - J\|_{0,\infty,\Delta_\tau} \left( \int_{\Delta_\tau} p^2 \, dX \right)^{\frac{1}{2}}. \] (4.7)

From (2.5a,b) it follows that
\[ \|J^h - J\|_{0,\infty,\Delta_\tau} \leq Ch^k. \]
Noting that \( \sum_{t \in \mathcal{T}} m(\Delta_t) = O(1) \), we can see that the right-hand side of (4.7) can be bounded by

\[
Ch^k(\|p\|_1, \Omega, \omega + \|p\|_{1, \Omega, \omega}) + Ch^k \|p\|_{0, \omega} \\
\leq C \|\tilde{u} - \tilde{u}\|_{1, \omega, \omega}[\hat{h}^k(\|\tilde{u}\|_{1, \Omega, \omega} + \|\tilde{w}\|_{1, \Omega, \omega}) + Ch^k \|\tilde{w}\|_{0, \omega}] \\
\leq Ch^k(\|u\|_{3, \omega} + \|\tilde{w}\|_{3, \omega}) \|w\|_{2, \omega},
\]

using (4.6), (1.3a), and (2.9). Hence the desired result (4.2) holds. \( \square \)

Combining the above lemmas we obtain the \( L^2 \) error bound.

**Theorem 4.1** Let the assumptions \( (A1) \rightarrow (A5) \), concerning the finite-element approximation, hold. The solutions \( u \) and \( u^h \) of (2.1) and (2.18) satisfy

\[
\|\tilde{u} - u^h\|_{0, \omega} \leq Ch^k(\|u\|_{k+2, \omega} + \|\tilde{u}\|_{k+1, \omega}),
\]

where \( \tilde{u} = Eu \).

**Proof.** First note that the approximation property (2.3) implies, for any integer \( k' \) such that \( k \geq k' \geq 2 \) and for any \( v \in H^k(\tau) \), that

\[
\|\pi_h v\|_{k', \tau} \leq C \|v\|_{k', \tau} \quad \forall \tau \in T^h.
\]

Hence the numerical integration bounds (2.21b, d), (4.9), (4.2), (2.9), (2.3), and (3.6) imply that

\[
\|\tilde{u} - u^h\|_{0, \omega} \leq C(\|\tilde{u} - u^h\|_{0, \omega} + \|\tilde{u} - u^h\|_{1, \omega} + \|\tilde{u}\|_{k+2, \omega} + \|\tilde{u}\|_{k+1, \omega} + \|\tilde{w}\|_{k+1, \omega}).
\]

The desired result (4.8) then follows from (4.1a), (3.13), (3.2), (1.3a) and (2.9). \( \square \)

5. Numerical integration assumptions

In this section, we justify the numerical integration assumptions \( (A3) \rightarrow (A5) \).

5.1 Fitted Mesh

We discuss first the case of a fitted mesh using \( (k - 1) \)-regular simplicial Lagrangian isoparametric elements. Thus each element \( \tau \) is the image of the unit \( n \)-simplex \( \bar{t} \) by the unique mapping \( F_\tau : \bar{t} \rightarrow \mathbb{R}^n \), where \( F_\tau \in \hat{P}(k - 1)^n \) and \( \hat{P}(r) \) is the space of polynomials of degree \( \leq r \) in \( n \) variables on \( \bar{t} \). The finite-dimensional subspace \( S^h \), satisfying (A1) and (A2), is then defined by

\[
S^h = \{w^h \in C^0(\hat{\Omega}^h) : w^h|_\tau \in P_{k-1} \quad \forall \tau \in T^h\},
\]

where

\[
P_{k-1} = \{p_\tau \in \mathbb{R}^r : p_\tau = \hat{p} \circ F_\tau^{-1} \text{ for some } \hat{p} \in \hat{P}(k - 1)\}.
\]

Let \( v \) be any function defined on the element \( \tau \). Then \( \tilde{v}(\bar{\tau}) = v(F_\tau(\bar{\tau})) \) defines a function \( \tilde{v} \) on \( \bar{\tau} \). Following Ciarlet & Raviart (1972) and Nedoma (1979), we
employ isoparametric numerical integration and have at our disposal a quadrature formula of order $d_1$ over the reference set $\hat{\tau}$; that is,
\[
\int_{\hat{\tau}} \theta(\hat{x}) \, d\hat{\tau} \text{ is approximated by } \sum_r \hat{w}_{\hat{t},r} \theta(\hat{b}_{\hat{t},r}) \tag{5.3}
\]
for some specified points $\hat{b}_{\hat{t},r} \in \hat{\tau}$ and positive weights $\hat{w}_{\hat{t},r}$.

An integral over the element $\tau$, that is,
\[
\int_{\tau} v(x) \, dx = \int_{\hat{\tau}} \theta(\hat{x}) J_t(\hat{x}) \, d\hat{\tau}, \tag{5.4}
\]
where $J_t$ is the Jacobian of the mapping $F_t$, is then approximated by
\[
\sum_r \hat{w}_{\hat{t},r} J_t(\hat{b}_{\hat{t},r}) \theta(\hat{b}_{\hat{t},r}) = \sum_r w_{t,r} v(b_{t,r}), \tag{5.5}
\]
where $w_{t,r} = \hat{w}_{\hat{t},r} J_t(\hat{b}_{\hat{t},r})$ and $b_{t,r} = F_t(\hat{b}_{\hat{t},r})$. We note that the sampling points $\hat{b}_{\hat{t},r}$ may be chosen so that, for $h$ sufficiently small, $b_{t,r} \in \hat{\Omega}$ and so assumption (A5)(i) holds. If we denote by $(\ast, \ast)^h$ the approximation of $(\ast, \ast)_{\hat{\Omega}}$ by isoparametric numerical integration, then
\[
a_t^h(\ast, \ast) = (\hat{\sigma} \nabla \ast, \nabla \ast)^h + (\hat{\epsilon} \ast, \ast)^h, \quad l_t^h(\ast) = (\hat{f}, \ast)^h. \tag{5.6a,b}
\]

**Lemma 5.1** Let $T^h$ be a $(k-1)$-regular triangulation of $\Omega^h$. Let the quadrature formula over $\hat{\tau}$ be of degree $d_1 \geq \max\{1, 2k-4\}$. Then, if $\sigma \in C^{k+1}(\hat{\Omega})$ and $\epsilon \in C^k(\hat{\Omega})$, it follows, for all $w \in H^k(\hat{\Omega})$ and for all $w^h$ and $\chi \in S^h$, that
\[
|\hat{a}_t^h(w^h, \chi) - a_t^h(w^h, \chi)| \leq Ch^{-1} \|w\|_{k, \omega^h} + \|w - w^h\|_{1, \omega^h} \|\chi\|_{1, \omega}\), \tag{5.7a}
\]
\[
|\hat{a}_t^h(w^h, \chi) - a_t^h(w^h, \chi)| \leq Ch^{-1} \|w\|_{k, \omega^h} + \|w - w^h\|_{1, \omega^h} \left(\sum_{t \in T^h} \|\chi\|^2_{2, r}\right)^{1/2}, \tag{5.7b}
\]
\[
|\hat{l}_t^h(\chi) - l_t^h(\chi)| \leq Ch^{-1} \|\hat{f}\|_{k, \omega^h} \|\chi\|_{1, \omega^h}, \tag{5.8a}
\]
\[
|\hat{l}_t^h(\chi) - l_t^h(\chi)| \leq Ch \|\hat{f}\|_{k, \omega^h} \left(\sum_{t \in T^h} \|\chi\|^2_{2, r}\right)^{1/2}. \tag{5.8b}
\]

**Proof.** In the case $\epsilon = 0$, the results (5.7a,b) follow from Theorem 2.2 of Nedoma (1979). It is a straightforward extension of that proof to show that the results also hold for any $\epsilon \in C^k(\hat{\Omega})$. The results (5.8a,b) follow from Theorem 2.1 of Nedoma (1979).

As well as having to perform integrals over $\Omega^h$, we have to perform integrals over $\partial \Omega^h$. For each $t \in B^h$, we assume (without loss of generality) that $\partial \Omega^h_t$ is the image of the face $\hat{\xi}_1 = 0$ of $\hat{\tau}$ under the mapping $F_t$. Once again, we employ isoparametric numerical integration and have at our disposal a quadrature formula of order $d_2$ over the face $\hat{\xi}_1 = 0$ of $\hat{\tau}$, which we shall denote by $\partial_t \hat{\tau}$; that is,
\[
\int_{\partial_t \hat{\tau}} \theta(\hat{x}) \, d\hat{\tau} \text{ is approximated by } \sum_r \hat{w}_{\partial_t \hat{\tau},r} \theta(\hat{b}_{\partial_t \hat{\tau},r}) \tag{5.9}
\]
for some specified points $\hat{b}_{\partial_t \hat{\tau},r} \in \partial_t \hat{\tau}$ and positive weights $\hat{w}_{\partial_t \hat{\tau},r}$. 

An integral over $\partial \Omega^h$, that is,
\[ \int_{\partial \Omega^h} v(s) \, ds = \int_{\partial \Omega^h} \delta(s) \left( \sum_{i=1}^{n} [J_i^{(i,j)}(s)]^2 \right)^{\frac{1}{2}} \, ds, \tag{5.10} \]
where $J_i^{(i,j)}$ is the $(i, j)$ cofactor of the Jacobian $J_i$, is then approximated by
\[ \sum_{r} w_{\partial \Omega^h,r} \left( \sum_{i=1}^{n} [J_i^{(i,j)}(b_{\partial \Omega^h,r})]^2 \right)^{\frac{1}{2}} \delta(b_{\partial \Omega^h,r}) = \sum_{r} w_{\partial \Omega^h,r} v(b_{\partial \Omega^h,r}), \tag{5.11} \]
where
\[ w_{\partial \Omega^h,r} = w_{\partial \Omega^h,r} \left( \sum_{i=1}^{n} [J_i^{(i,j)}(b_{\partial \Omega^h,r})]^2 \right)^{\frac{1}{2}}, \quad b_{\partial \Omega^h,r} = F_r(b_{\partial \Omega^h,r}). \]

We denote by $(\cdot, \cdot)^h$ the approximation of $(\cdot, \cdot)_{\partial \Omega^h}$ by isoparametric numerical integration. Following Čermák (1983a), we set
\[ a_k^h(\cdot, \cdot, \cdot) = \langle \pi_k \alpha, \cdot, \cdot \rangle^h, \quad l_k^h(\cdot) = \langle \pi_k \tilde{g}, \cdot \rangle^h, \tag{5.12a,b} \]
where $\pi_k$ is the induced interpolation operator from $C^0(\partial \Omega^h)$ into $\mathcal{S}^k$, where $\mathcal{S}^k = \mathcal{S}^k(h) \oplus \mathcal{S}_o^k$ and $\mathcal{S}_o^k = \{ \chi \in S^k : \chi = 0 \text{ on } \partial \Omega^h \}$, satisfying: for any integer $k'$ with $k \geq k' \geq 2$ and for $p \in [2, \infty]$,
\[ \|w - \pi_k w\|_{0,p,\partial \Omega^h} \leq C k' \|w\|_{k',p,\partial \Omega^h}, \quad \forall w \in W^{k',p}(\partial \Omega^h), \tag{5.13} \]
where $C$ is independent of $h$ and $w$. We note that the definitions (5.12) satisfy the assumption (A5) (ii).

**Lemma 5.2** Let $T^h$ be a $(k - 1)$-regular triangulation of $\Omega^h$ and $\tilde{\alpha} \in C^{k+1}(\tilde{\Omega})$. Then, if the quadrature formula over $\partial \Omega^h$ is of degree $d_2$, it follows, for all $w \in H^k(\tilde{\Omega})$ and for all $w^h$ and $\chi \in \mathcal{S}^h$, that (i)
\[ |\tilde{a}_2^h(w^h, \chi) - a_2^h(w^h, \chi)| \leq C (h^{k-1} \|w\|_{k,\mathcal{O}} + \|w - w^h\|_{1,\mathcal{O}}) \|\chi\|_{1,\mathcal{O}}, \tag{5.14a} \]
\[ |l_2^h(\chi) - l_2^h(\chi)| \leq C k' \|\tilde{\chi}\|_{k+1,\mathcal{O}} \|\chi\|_{1,\mathcal{O}}, \tag{5.14b} \]
for $d_2 \geq 2k - 3$, and that (ii)
\[ |\tilde{a}_2^h(w^h, \chi) - a_2^h(w^h, \chi)| \leq C h^{k-1} \|w\|_{k,\mathcal{O}} + \|w - w^h\|_{1,\mathcal{O}} \left( \sum_{\tau \in T^h} \|\chi\|_{2,\tau}^2 \right)^{\frac{1}{2}}, \tag{5.15a} \]
\[ |l_2^h(\chi) - l_2^h(\chi)| \leq C h^{k} \|\tilde{\chi}\|_{k+1,\mathcal{O}} \left( \sum_{\tau \in T^h} \|\chi\|_{2,\tau}^2 \right)^{\frac{1}{2}}, \tag{5.15b} \]
for $d_2 \geq \max \{2, 2k - 3\}$.

**Proof.** We have that
\[ |\tilde{a}_2^h(w^h, \chi) - a_2^h(w^h, \chi)| \leq \left| \langle (\tilde{\alpha} - \pi_k \tilde{\alpha}) w^h, \chi \rangle_{\partial \Omega^h} \right| + \left| \langle (\pi_k \tilde{\alpha}) w^h, \chi \rangle_{\partial \Omega^h} - \langle (\pi_k \tilde{\alpha}) w^h, \chi \rangle_{\partial \Omega^h} \right|, \tag{5.16a} \]
\[ |l_2^h(\chi) - l_2^h(\chi)| \leq \left| \langle \tilde{g} - \pi_k \tilde{g}, \chi \rangle_{\partial \Omega^h} \right| + \left| \langle \pi_k \tilde{g}, \chi \rangle_{\partial \Omega^h} - \langle \pi_k \tilde{g}, \chi \rangle_{\partial \Omega^h} \right|. \tag{5.16b} \]
The first terms on the right-hand sides of (5.16a,b) can be bounded by
\[ Ch^k \|w^h\|_{1,\omega} \|\chi\|_{1,\omega} \|\tilde{a}\|_{k,\omega,\partial\Omega^h}, \]
using (5.13) with \( p = \infty \) and (1.3), and by \( Ch^k \|\tilde{g}\|_{k+1,\omega} \|\chi\|_{1,\omega} \), using (5.13) with \( p = 2 \) and (1.3), respectively. The second terms on the right-hand sides of (5.16a,b) can be bounded by extending the proofs of Nedoma (1979) for isoparametric numerical integration over \( \Omega^h \) in the natural manner to \( \partial\Omega^h \). We omit the technical details.

**Remark 5.1.** We note the Remark 3.2 in Čermák (1983a), where it is stated that \( d_2 \geq 2k - 3 \) is sufficient to retain optimal convergence for the error in \( L^2 \). This is achieved by bounding the quadrature errors (5.15a,b) in terms of \( \|\chi\|_{1,\partial\Omega^h} \) instead of \( (\sum_{\tau \in T^h} \|\chi\|_{2,\tau}^2)^{\frac{1}{2}} \). To obtain the \( L^2 \) error estimate, one has then to bound \( \|\pi_h\tilde{g}\|_{1,\partial\Omega^h} \) above by \( C \|\tilde{g}\|_{2,\partial\Omega^h} \). This is achieved using an argument involving mollifiers, see Čermák (1981, 1983a) for details. Since we have not been able to extend this approach to the unfitted mesh case, we have settled for the simplified bounds (5.15a,b) throughout this paper.

Therefore, the assumption (A4) holds for a fitted mesh using \((k - 1)\)-regular simplicial Lagrangian isoparametric elements by combining Lemmas 5.1 and 5.2. Thus it remains to justify the assumption (A3) for a fitted mesh.

**Lemma 5.3** Let \( T^h \) be a \((k - 1)\)-regular triangulation of \( \Omega^h \). Let \( d_1 = \max \{1, 2k - 4\} \) and \( d_2 \geq 2k - 3 \) be the orders of the quadrature formulae over \( \hat{\xi} \) and \( \partial_1\hat{\xi} \). Let \( \tilde{a}, \tilde{c}, \) and \( \tilde{a} \) belong to \( C^1(\hat{\Omega}) \). Then, if either \( \tilde{c}_0 > 0 \) or \( \tilde{a}_0 > 0 \), it follows that (A3) holds for \( h \) sufficiently small.

**Proof.** We sketch a proof due to Ženišek (1987). One can show, for all \( w^h, \chi \in S^h \), that
\[
|\tilde{a}^h_1(w^h, \chi) - a^h_1(w^h, \chi)| \leq Ch^k \|w^h\|_{1,\omega} \|\chi\|_{1,\omega} \]
\[
|\tilde{a}^h_2(w^h, \chi) - a^h_2(w^h, \chi)| \leq Ch^k \|w^h\|_{1,\omega} \|\chi\|_{1,\omega}.
\]
Hence there exists a constant \( C_1 > 0 \) such that
\[
a^h(\chi, \chi) - \tilde{a}^h(\chi, \chi) \geq -C_1 h^k \|\chi\|_{1,\omega}^2 \quad \forall \chi \in S^h. \tag{5.17}
\]
From the assumption on \( \tilde{c}_0 \) and \( \tilde{a}_0 \) and Lemma 2.1, it follows that there exists a constant \( C_2 > 0 \) such that
\[
\tilde{a}^h(\chi, \chi) \geq C_2 \|\chi\|_{1,\omega}^2 \quad \forall \chi \in S^h. \tag{5.18}
\]
Therefore (5.17) and (5.18) give, for \( h < \frac{1}{2}(C_2/C_1)^2 \), that
\[
a^h(\chi, \chi) \geq \frac{1}{2}C_2 \|\chi\|_{1,\omega}^2 \quad \forall \chi \in S^h. \quad \square
\]

### 5.2 Unfitted Mesh (k = 2)

In this case, all of the elements \( \tau \) have straight sides or faces. However, since the mesh is unfitted, we have to approximate integrals over the subregions \( \Omega^h \cap \tau \) and \( \partial\Omega^h \cap \bar{\tau} \) for those elements \( \tau \in B^h \). For \( k \geq 3 \), the boundary \( \partial\Omega^h \) will (in
general) be curved, and it is difficult to obtain simple quadrature formulae satisfying (A4). When \( k = 2 \), the subregion \( \Omega^h \cap \tau \) is either a simplex or a union of 2 (resp. 3) simplices for \( n = 2 \) (resp. 3)—see Fig. 5.1—so that

\[
\Omega^h \cap \tau = \bigcup_{i=1}^{n} t_i, \quad \partial \Omega^h \cap \tau = \gamma = \partial \Omega^h, \tag{5.19a,b}
\]

where the \( t_i (i = 1, \ldots, n) \) are simplices in \( \mathbb{R}^n \) (with \( n = 2 \) or 3), up to \( n - 1 \) of which may be empty, and \( \gamma \) is a simplex in \( \mathbb{R}^{n-1} \) being the boundary face of a \( t_i \in \Omega^h \cap \tau, \, 1 \leq i \leq n \). We have quadrature formulae over the unit simplex \( \hat{t} \) of \( \mathbb{R}^n \) and the unit simplex \( \hat{\gamma} \) of \( \mathbb{R}^{n-1} \) which induce the quadrature rules over the simplices \( t \) and \( \gamma \):

\[
\int_{t} v(x) \, dx = L_t(v) = J_t \sum_{t} \hat{w}_{t_i} \hat{\theta}(\hat{b}_{t_i}), \tag{5.20a}
\]

\[
\int_{\gamma} v(x) \, dx = L_{\gamma}(v) = J_{\gamma} \sum_{\gamma} \hat{w}_{\gamma_i} \hat{\theta}(\hat{b}_{\gamma_i}), \tag{5.20b}
\]

where \( J_t \) is the Jacobian of the unique affine mapping \( F_t: \hat{t} \to t \) and \( J_{\gamma} \) is the Jacobian of the unique affine mapping \( F_{\gamma}: \hat{\gamma} \to \gamma \). With

\[
\Omega^h = \bigcup_{\tau \in T^h} \Omega^h \cap \tau = \bigcup_{t} t, \quad \partial \Omega^h = \bigcup_{\tau \in B^h} \partial \Omega^h \cap \tau = \bigcup_{\gamma} \gamma,
\]

we let \((v, w)^h \) denote approximation of \((v, w)^h \) by \( \Sigma_h l_t(vw) \), and let \( \langle v, w \rangle^h \) denote the approximation of \( \langle v, w \rangle^h \) by \( \Sigma_h l_t(vw) \).

The simplices \( t \) and \( \gamma \) are not regular as \( h \to 0 \), so the results of Lemmas 5.1–5.3 need to be checked. We note the seminorm inequalities with \( m \geq 0 \) and \( p \in [1, \infty] \) (see Ciarlet, 1978: Theorem 3.1.2):

\[
|\theta|_{m,p,t} \leq Ch^{m} J_t^{-1/2} |v|_{m,p,t} \quad \forall v \in W^{m,p}(t), \tag{5.21a}
\]

\[
|\hat{\theta}|_{m,p,\hat{t}} \leq Ch^{m} J_{\hat{t}}^{-1/2} |\hat{v}|_{m,p,\hat{t}} \quad \forall \hat{v} \in W^{m,p}(\hat{t}); \tag{5.21b}
\]

and, for all polynomials \( \hat{q} \) on \( \hat{t} \),

\[
|\hat{q}|_{m,\omega,\hat{t}} \leq C |\hat{q}|_{m,\hat{t}}, \quad |\hat{q}|_{m,\omega,\hat{\gamma}} \leq C |\hat{q}|_{m,\hat{\gamma}}. \tag{5.21c,d}
\]

**Fig. 5.1.** The case \( n = 2 \). Here, \( \Omega^h \cap \tau \) is the unshaded region; in (i) it comprises just one subtriangle, and in (ii) it comprises two subtriangles.
With \( h_t \) and \( h_\gamma \) the diameters of \( t \) and \( \gamma \), respectively, then we know only that 
\[
|J_t| \leq h_t^n \quad \text{and} \quad |J_\gamma| \leq h_\gamma^{n-1},
\]
whereas, in the case of regular elements, we have additionally that 
\[
Ch_t^n \leq |J_t| \quad \text{and} \quad Ch_\gamma^{n-1} \leq |J_\gamma|.
\]

We use the notation
\[
E_t(v) = \int v(x) \, dx - I_t(v), \quad E_\gamma(v) = \int_\gamma v(x) \, dx - I_\gamma(v).
\]

**Lemma 5.4** Let the quadrature formula \( I_\gamma(*) \) be exact for linear polynomials. It follows, for all \( \chi, \xi \in S^h \), that
\[
|E_t(\xi \chi)| \leq C h^2 \| f \|_{L^2} \| \chi \|_{H^1}, \quad (5.22a)
\]
\[
|E_t(\xi^2 \chi)| \leq C h^2 \| \xi \|_{L^2} \| \chi \|_{H^1}, \quad (5.22b)
\]
\[
|E_t(\xi \nabla \xi \cdot \nabla \chi)| \leq C h^2 \| \nabla \xi \|_{L^2} \| \chi \|_{H^1}. \quad (5.22c)
\]

**Proof.** (a) Since the formulae are exact for linear polynomials it follows from the Bramble–Hilbert lemma (see Ciarlet (1978: p. 192 & 4.1.42)) that
\[
|E_t(\xi \chi)| = |I_t(\xi \chi)| \leq C h^2 \| \xi \|_{L^2} \| \chi \|_{H^1}, \quad (5.23)
\]
by noting that \( D^2 \xi = 0 \). It follows from (5.23) and (5.21c) that
\[
|E_t(\xi \chi)| \leq C h^2 \| \xi \|_{L^2} \| \chi \|_{H^1}, \quad (5.24)
\]
and applying (5.21a) with \( p = 2 \) to each of the seminorms in (5.24) yields (5.22a).

(b) Applying the Bramble–Hilbert lemma in a similar way, we obtain
\[
|E_t(\xi^2 \chi)| \leq C h^2 \| \xi \|_{L^2} \| \chi \|_{H^1}, \quad (5.25)
\]

Using (5.21c) for each of the \( |\xi|_{m,\omega,t} \) terms in (5.25) and directly applying (5.21a) to each of the resulting seminorms yields (5.22b).

(c) Noting that \( \nabla \xi \cdot \nabla \chi \) is a constant on \( \tau \), and that the quadrature rule is exact for linear polynomials, we obtain
\[
|E_t(\xi \nabla \xi \cdot \nabla \chi)| = \left| \int \nabla \xi \cdot \nabla \chi (1 - \Pi_t^1) \tilde{\sigma} \, dx \right| \leq \|(1 - \Pi_t^1) \tilde{\sigma}\|_{0,\omega,t} \| \xi \|_{1,t} \| \chi \|_{1,t},
\]
where \( \Pi_t^1 \) denotes the linear interpolation operator on \( t \). From interpolation theory, we have
\[
\|(1 - \Pi_t^1) \tilde{\sigma}\|_{0,\omega,t} \leq C h^2 \| \tilde{\sigma}\|_{2,\omega,t},
\]
and so we obtain (5.22c). □
Now set
\[ a_h^2(\cdot, \cdot) = \langle (\Pi_h^1 \tilde{\alpha}) \cdot, \cdot \rangle^h, \quad l_h^n(\cdot) = \langle (\Pi_h^1 g) \cdot \rangle^h, \] (5.26a,b)
where \( \Pi_h^1 w \) is the linear interpolate of \( w \) at the vertices of the simplex \( \gamma \). From interpolation theory and (1.3), we have
\[ |\langle (\tilde{\alpha} - \Pi_h^1 \tilde{\alpha}) w^h, \chi \rangle_{\partial \Omega}| \leq Ch^2 \|w^h\|_{1, \Omega} \|\chi\|_{1, \Omega}, \] (5.27a)
\[ |\langle (\tilde{g} - \Pi_h^1 \tilde{g}), \chi \rangle_{\partial \Omega}| \leq Ch^2 \|\tilde{g}\|_{1, \Omega} \|\chi\|_{1, \Omega}. \] (5.27b)

**Lemma 5.5** Let the quadrature rule \( I_\gamma(\cdot) \) be exact for quadratic polynomials. It follows that, for all \( \chi, \xi \in S^h \):
\[ E_\gamma((\Pi_h^1 g) \chi) = 0, \quad |E_\gamma((\Pi_h^1 \tilde{\alpha}) \xi, \chi)| \leq Ch^2 |\tilde{\alpha}|_{1, \omega, y} |\xi|_{1, \omega, y} \] (5.28a,b)

**Proof.** Since \( E_\gamma(\cdot) \) vanishes on quadratics, (5.28a) holds, and we deduce from the Bramble–Hilbert lemma that, with \( \eta = \Pi_h^1 \tilde{\alpha} \),
\[ |E_\gamma(\eta \xi \chi)| \leq C J_\gamma |\eta|_{1, \omega, \gamma} |\xi|_{1, \omega, \gamma} |\chi|_{1, \omega, \gamma} \]
\[ \leq C J_\gamma |\eta|_{1, \omega, \gamma} |\xi|_{1, \omega, \gamma} |\chi|_{1, \omega, \gamma} \]
\[ \leq C J_\gamma h^3 |\eta|_{1, \omega, y} |\xi|_{1, \omega, y} |\chi|_{1, \omega, y}, \]
where we have used the equivalence of norms on polynomials defined over \( \gamma \) and (5.21b). Noting that
\[ |\Pi_h^1 \tilde{\alpha}|_{1, \omega, y} \leq |\tilde{\alpha}|_{1, \omega, y}, \quad J_\gamma \leq h^{n-1}, \quad |\xi|_{1, \omega, y} \leq C |\xi|_{1, \omega, y}, \]
and since \( \tau \) is regular as \( h \to 0 \), the bound
\[ |\xi|_{1, \omega, \gamma} \leq Ch^{-\nu} |\xi|_{1, \omega, \gamma} \]
yields the desired result (5.28b). □

We now choose integration rules over \( \tau \) and \( \gamma \). With \( \{\hat{a}_i\}_{i=1}^{n+1} \) being the vertices of \( \tau \) and \( \{\hat{\alpha}_i\}_{i=1}^{n+1} \) the vertices of \( \gamma \), we define
\[ I_\gamma(v) = \frac{1}{n+1} \sum_{i=1}^{n+1} \theta(\hat{a}_i), \] (5.29a)
\[ I_\tau(v) = \frac{1}{8} J_\tau \left[ \theta(\hat{a}_1) + 4 \theta(\frac{1}{2}(\hat{a}_1 + \hat{a}_2)) + \theta(\hat{a}_2) \right] \quad (n = 2) \]
\[ I_\tau(v) = \frac{1}{4} J_\tau m(\gamma) \sum_{i \neq j} \theta(\frac{1}{2}(\hat{a}_i + \hat{a}_j)) \quad (n = 3) \] (5.29b)

It is clear that the assumptions of Lemmas 5.4 and 5.5 are satisfied. The assumption (A4) is an immediate consequence of these lemmas and (5.27). Further, since the vertices of \( \tau \) lie in \( \bar{\Gamma} \) and the vertices of \( \gamma \) lie on \( \partial \Omega \), assumption (A5) is verified. It remains to check the coercivity assumption (A3). This assumption is an immediate consequence of the following lemma and (1.4).
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Lemma 5.6 Let $I_1(\cdot)$ and $I_r(\cdot)$ be defined by (5.29). It follows that, for all $\chi \in S^h$:
\[ I_1(\partial \nabla \chi \cdot \nabla \chi) \geq \sigma_0 |\chi|_{1,v}, \quad I_r(\ell \chi^2) \geq c_0 |\chi|_{0,v}^2, \quad (5.30a,b) \]
\[ I_r((\Pi_1^0)^* \chi^2) \geq \bar{a}_0 |\chi|_{0,v}^2, \quad (5.30c) \]

Proof. (a) Since $\nabla \chi$ is constant on $t$,
\[ I_1(\partial \nabla \chi \cdot \nabla \chi) = \int_t (\Pi_1^0 \partial) |\nabla \chi|^2 \, dx \]
\[ \geq \sigma_0 |\nabla \chi|_{0,v}^2, \]
and (5.30a) is proved.
(b) It follows from (5.29a) that
\[ I_1(\ell \chi^2) \geq c_0 I_1(\chi^2) = c_0 \int_t \Pi_1^0 (\chi^2) \, dx \]
\[ \geq c_0 \int_t \chi^2 \, dx, \]
where the last inequality follows from the convexity of $\chi^2$.
(c) It follows from (5.29b) that
\[ I_r((\Pi_1^0)^* \chi^2) \geq \bar{a}_0 I_r(\chi^2) \]
and, since the formula is exact for quadratic polynomials and $\chi$ is linear, we obtain (5.30c). □

Therefore the assumptions (A3) $\rightarrow$ (A5) hold for an unfitted mesh when using linear elements and the quadrature formulae (5.29).

6. Numerical example

We now report on a numerical example using an unfitted mesh. The problem chosen was
\[ \nabla^2 u = 4 \text{ in } \Omega \equiv \{(x, y): x^2 + y^2 \leq 1\}, \quad \frac{\partial u}{\partial y} + u = 3 \text{ on } \partial \Omega. \]

This has the solution $u = x^2 + y^2$. Due to symmetry, the problem was solved in a single quadrant. For our trial space, we took piecewise linears on uniform right-angled triangles; these resulted from a uniform partition of the complete square $[0, 1] \times [0, 1]$ into squares with sides of length $h = 1/J$, and then into triangles by bisection from the SW to the NE vertices. The computational domain $\Omega^h$ was obtained by replacing $\partial \Omega$ by its chord in each triangle it intersects, as described previously. The results obtained from the approximation (2.18) using the quadrature rules (5.29) are presented in Table 1. Clearly the analysis in the previous sections is confirmed.

For piecewise linears on a fitted mesh, we recall the result of Čermák (1981, 1983a)—see Remark 5.1 of this paper—that the condition $d_2 \geq 1$ is sufficient to
retain an optimal rate of convergence in the $L^2$ norm. Whereas, for an unfitted mesh, we have only been able to prove that $d_2 \geq 2$ is sufficient—i.e. Simpson's rule for $n = 2$: see (5.29b). To see if the condition $d_2 \geq 2$ is also necessary, we repeated the above calculations using the trapezoidal rule ($d_2 = 1$),

$$L_2(v) = \frac{1}{2}L_2(\theta(\bar{a}_1) + \theta(\bar{a}_2)),$$

instead of (5.29b). The results obtained were the same as those in Table 1, except for some changes in the sixth decimal place for $J = 4$ and 8. Thus it appears that the results of Section 5.2 may not be optimal, in that $d_2 \geq 1$ is sufficient to retain an optimal rate of convergence in $L^2$.

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**References**


