EXPONENTIAL AND ALGEBRAIC RELAXATION IN KINETIC MODELS FOR WEALTH DISTRIBUTION

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Two classes of kinetic models for wealth distribution in simple market economies are compared in view of their speed of relaxation towards stationarity in a Wasserstein metric. We prove fast (exponential) convergence for a model with *risky investments* introduced by Cordier, Pareschi and Toscani,[7] and slow (algebraic) convergence for the model with *quenched saving propensities* of Chakrabarti, Chatterjee and Manna.[3] Numerical experiments confirm the analytic results.

Keywords: Econophysics, Maxwell molecules, relaxation, Wasserstein metric.

1. Kinetic Models in Econophysics

One of the founding ideas in the rapidly growing field of econophysics is that the laws of statistical mechanics for particle systems also govern the trade interactions between agents in a simple market. Just as a classical kinetic model is defined by prescribing the collision kernel for the microscopic particle interactions, the econophysical model is defined by prescribing the exchange rules for wealth in trades. In dependence on these "microscopic" rules, the system develops "macroscopic" features in the long-time limit. Such macroscopic correlations are visible for instance in the form of a non-trivial stationary wealth distribution curve.

Many different (and somewhat justifiable) approaches to create a good model exist. Nevertheless, up to now little is known about which factors *should* enter into the exchange rules (in order to make the model realistic), and which *should not* (in order to keep it simple). Typically, the value of a model is estimated *a posteriori* by comparing its predictions with real-world data. For instance, it is widely accepted that the stationary wealth distribution $f_\infty(v)$ (denoting the density of agents with wealth $v > 0$) should possess a *Pareto tail*,

$$F_\infty(v) = \int_v^\infty f_\infty(w)\,dw \propto v^{-\alpha}. \tag{1}$$

The exponent α is referred to as *Pareto index*, named after the economist Vilfredo Pareto,[13] who proposed formula (1) more than a hundred years ago. According to recent empirical data, the wealth distribution among the population in a western country follows the Pareto law, with an index α ranging between 1.5 and 2.5. We refer e.g. to Ref. 1 and the references therein.

Below, we compare two types of econophysical models, which are able to produce Pareto tails.

1.1. *The Cordier-Pareschi-Toscani model*

The *Cordier-Pareschi-Toscani model* (CPT model) has been introduced in Ref. 7, and was intensively studied only recently.[10] When two agents with pre-trade wealths v and w interact, then their post-trade wealth v^* and w^*, respectively, is given by

$$v^* = (\lambda + \eta_1)v + (1-\lambda)w, \qquad w^* = (1-\lambda)v + (\lambda + \eta_2)w. \tag{2}$$

Here $\lambda \in (0,1)$ is the *saving propensity*, which models the fact that agents never exchange their entire wealth in a trade, but always retain a certain fraction λ of it. The quantities η_1 and η_2 are random variables with mean zero, satisfying $\eta_i \geq -\lambda$. They model *risky investments* that each agent performs in addition to trading. A crucial feature of the CPT model is that it preserves the total wealth in the statistical mean,

$$\big\langle v^* + w^* \big\rangle = \big(1 + \langle \eta_1 \rangle\big)\, v + \big(1 + \langle \eta_1 \rangle\big)\, w = v + w, \tag{3}$$

where $\langle \cdot \rangle$ denotes the statistical expectation value. The behavior of the homogeneous Boltzmann equation corresponding to (3) is to a large extend determined[10] by the convex function

$$\mathfrak{S}(s) := (1-\lambda)^s - 1 + \frac{1}{2}\big\langle (\lambda + \eta_1)^s + (\lambda + \eta_2)^s \big\rangle. \tag{4}$$

Clearly, $\mathfrak{S}(1) = 0$ by (3). Provided $\mathfrak{S}'(0) < 0$, the model possesses a unique steady state f_∞. If $\mathfrak{S}(s) < 0$ for all $s > 1$, then f_∞ has an exponentially small tail. On the contrary, if there exits a *non-trivial* root $\bar{s} \in (1, \infty)$ of $\mathfrak{S}$, then f_∞ possesses a Pareto tail (1) of index $\alpha = \bar{s}$.

Theorem 2.1 below states that (in both cases) any solution $f(t)$ converges to f_∞ exponentially fast in suitable Fourier and Wasserstein metrics.

1.2. *The Chakrabarti-Chatterjee-Manna model*

The *Chakrabarti-Chatterjee-Manna model* (CCM model) was introduced in Ref. 3, and heavily investigated in the last decade. While the saving propensity λ is a *global quantity* in the CPT model, in the CCM model it is a characteristics of the individual agents. The "state" of an agent is now described by his wealth *and* his personal saving propensity. The latter does not change with time. In a trade between two agents with wealth v, w and saving propensities λ, μ, respectively, wealth is exchanged according to

$$v^* = \lambda v + \epsilon\Delta, \quad w^* = \mu w + (1-\epsilon)\Delta \quad \text{with} \quad \Delta = (1-\lambda)v + (1-\mu)w. \tag{5}$$

Here ϵ is a random variable in $(0, 1)$. The key ingredient for this model is the (time-independent) density $g(\lambda)$ of saving propensities among the agents. The homogeneous Boltzmann equation associated to the rules (5) has been heavily investigated numerically in terms of Monte Carlo simulations;[2–6] we present further simulation results here. Also, some theoretical investigations exist.[6,11,14] At least in the deterministic case $\epsilon \equiv 1/2$, the wealth distribution of the steady state is explicitly known,[12]

$$f_\infty(w) = \frac{C}{w^2}\, g\Big(1 - \frac{C}{w}\Big). \tag{6}$$

In the non-deterministic case, the choice of the random quantity ϵ has seemingly little influence[2] on the shape of f_∞. Thus, by prescribing g suitably, cf. section 3, steady states with a Pareto tail of arbitrary index α can be generated. Furthermore, we prove below (see Theorem 2.2) that for the majority of initial conditions the Wasserstein distance between the solution and the corresponding steady state with a Pareto tail can at best decay algebraically in time. For the proof, we shall need no properties of the CCM model other than the pointwise conservation of wealth,

$$v^* + w^* = v + w, \tag{7}$$

which is much stricter than conservation in the mean (3). The argument is based on the result from Ref. 10, that in pointwise conservative models initially finite moments of the solution diverge at most at algebraic rate.

1.3. *Other approaches*

Many econophysical models for wealth distribution — including the ones mentioned above — are still very basic as the agents just trade randomly with each other, do not adapt their saving strategy, and so on. Slightly more realistic economic models have been proposed, which admit the agents to have a little bit of intelligence, or at least trading preferences, see e.g. the collected works in Ref. 5 for an overview on recent developments. Finally, we mention that one may consider mean-field equations or hydrodynamic limits[9] instead of the full kinetic model.

2. Analytical Estimation of Convergence Rates

2.1. *Preliminaries*

We consider weak solutions f to the homogeneous Boltzmann equation,

$$\begin{aligned} &\frac{d}{dt}\int_0^\infty \Phi(v)df(v) \\ &\qquad = \frac{1}{2}\left\langle \int_0^\infty \int_0^\infty \big(\Phi(v^*)+\Phi(w^*)-\Phi(v)-\Phi(w)\big)df(v)df(w)\right\rangle, \quad (8) \end{aligned}$$

where Φ is a regular test function, v, w denote the pre-collisional, and v^*, w^* the post-collisional wealths, according to the rules (2) and (5), respectively. Further, we assume that f is a probability density with mean wealth equal to one. (Notice that both models preserve mass and mean wealth.) In order to measure the convergence to equilibrium, $f(t,v) \to f_\infty(v)$ for $t \to \infty$, we introduce the following distances.

Definition 2.1. Let two probability densities f and g on $\mathbb{R}_+$ be given, both with first moment equal to one, and finite moments of some order $\bar{s} \in (1,2]$.

- For $s \in [1,\bar{s}]$, the *Fourier distance* d_s is defined by
$$d_s(f,g) := \sup_{\xi\in\mathbb{R}\setminus\{0\}} \big(|\xi|^{-s}\,\big|\widehat{f}(\xi)-\widehat{g}(\xi)\big|\big), \quad (9)$$
where $\widehat{f}$ and $\widehat{g}$ denote the Fourier transforms of f and g.
- The *Wasserstein-one-distance* is defined by
$$W(f,g) := \int_{\mathbb{R}_+} \big|F(v)-G(v)\big|\,dv, \quad (10)$$
where F and G denote the distribution functions of f and g,
$$F(v) = \int_v^\infty f(w)\,dw, \qquad G(v) = \int_v^\infty g(w)\,dw.$$

Equivalently, the Wasserstein distance between f and g can be defined as the infimum of the costs for transportation,

$$W(f,g) := \inf_{\pi\in\Pi} \int_{\mathbb{R}_+\times\mathbb{R}_+} |v-w|\,d\pi(v,w). \tag{11}$$

Here Π is the collection of all probability measures on $\mathbb{R}_+ \times \mathbb{R}_+$ with marginals $f(x)\,dx$ and $g(x)\,dx$, respectively. The infimum in (11) is in fact a minimum, and is realized by some *optimal transport plan* π_{opt}. For details, see Ref. 8 and references therein.

2.2. *Exponential convergence for the CPT model*

Theorem 2.1. *Assume η_1 and η_2 in the CPT model (2) are such that $\mathfrak{S}'(1) < 0$ with $\mathfrak{S}$ defined in (4). Then there exists a unique steady state f_∞ for (8), which is of mean wealth one. Further, there is some $\bar{s} \in (1,2]$ for which $\lambda := \mathfrak{S}(\bar{s}) < 0$, and any solution $f(t)$ to (8) — with initially bounded $\bar{s}$th moment — is exponentially attracted by f_∞:*

$$d_{\bar{s}}\big(f(t), f_\infty\big) \le d_{\bar{s}}\big(f(0), f_\infty\big)\exp(\lambda t), \tag{12}$$

$$W\big(f(t), f_\infty\big) \le C \exp\left(\frac{(\bar{s}-1)\lambda}{\bar{s}(2\bar{s}-1)}t\right), \tag{13}$$

with some finite, time-independent constant $C > 0$.

Estimate (12) is a consequence of Theorem 3.3 in Ref. 10. Estimate (13) is new, and follows from (12) and estimate (14) below. We remark that (12) is relatively easy to obtain, working on the Fourier representation of the Boltzmann equation (8), and using the homogeneity properties of d_s. On the contrary, a direct proof of (13) seems difficult. Notice that we cannot resort to the more convenient Wasserstein-*two*-metric here since the second moment of f_∞ might be infinite.

Lemma 2.1. *Assume that two probability densities f and g have first moment equal to one, and some moment of order $s \in (1,2]$ bounded. Then there exists a constant $C > 0$, depending only on s and the values of the sth moments of f and g, such that*

$$W(f,g) \le C d_s(f,g)^{\frac{s-1}{s(2s-1)}}. \tag{14}$$

Conversely, one has

$$d_1(f,g) \le W(f,g), \tag{15}$$

even if no moments of f and g above the first are bounded.

Proof. To prove (14), we extend the proof of Theorem 2.21 in Ref. 8, corresponding to $s = 2$ in the theorem above. Define

$$M = \max\left\{ \int_{\mathbb{R}} |v|^s f(v)\,dv,\ \int_{\mathbb{R}} |v|^s g(v)\,dv \right\}.$$

Starting from the definition of the Wasserstein distance in (10), we estimate

$$\begin{aligned}
&W(f,g) = \int_{\mathbb{R}_+} \big|F(v) - G(v)\big|\,dv \\
&\le \int_0^R \big|F(v) - G(v)\big|\,dv + R^{1-s}\int_R^\infty v^{s-1}\big|F(v) - G(v)\big|\,dv \qquad (16)\\
&\le R^{1/2}\bigg(\int_{\mathbb{R}_+} \big|F(v) - G(v)\big|^2\,dv\bigg)^{1/2} + R^{1-s}\int_R^\infty v^{s-1}\big|F(v) - G(v)\big|\,dv,
\end{aligned}$$

where the parameter $R = R(t) > 0$ is specified later. By Parseval's identity,

$$\begin{aligned}
&\int_{\mathbb{R}_+} \big|(F-G)(v)\big|^2\,dv = \int_{\mathbb{R}} \big|\widehat{(F-G)}(\xi)\big|^2\,d\xi \\
&= \int_{\mathbb{R}} \big|(i\xi)^{-1}\big(\widehat{f}(\xi) - \widehat{g}(\xi)\big)\big|^2\,dx \le d_s(f,g)^2 \int_{|\xi|<r} |\xi|^{2(s-1)}\,d\xi + 4\int_{|\xi|\ge r} \xi^{-2}\,d\xi \\
&= (2s-1)^{-1} r^{2s-1} d_s(f,g)^2 + 8r^{-1} \le C_1 d_s(f,g)^{1/s}.
\end{aligned}$$

The last estimate follows by optimizing in the previous line with respect to $r > 0$. The constant C_1 depends only on $s > 1$. This gives a bound on the first term in (16) above. We estimate the second term, integrating by parts:

$$\begin{aligned}
&\int_R^\infty v^{s-1}\big|F(v) - G(v)\big|\,dv \le \int_R^\infty v^{s-1}\big(F(v) + G(v)\big)\,dv \\
&= \frac{1}{s}\int_R^\infty v^s\big(f(v) + g(v)\big)\,dv + \Big(v^s\big(F(v) + G(v)\big)\Big)\Big|_R^\infty \\
&\le \frac{2M}{s} + \lim_{r\to+\infty}\Big(r^s\big(F(r) + G(r)\big)\Big).
\end{aligned}$$

The last expression is easily estimated by Chebyshev's inequality, i.e.,

$$\lim_{r\to\infty}\big(r^s F(r)\big) \le \lim_{r\to\infty}\big(r^s\,\mathrm{P}_f\,[v > r]\big) \le \lim_{r\to\infty}\int_r^\infty v^s f(v)\,dv = 0,$$

since the sth moment of f is finite. In summary, (16) yields

$$W(f,g) \le C_1^{1/2} R^{1/2} d_s(f,g)^{1/(2s)} + 2s^{-1} M R^{1-s}.$$

Optimizing this over R yields the desired inequality (14).

The other inequality (15) is derived from the alternative definition (11) of $W(f,g)$. With π_{opt} being the optimal transport in (11),

$$\begin{aligned} d_1(f,g) &= \sup_{\xi\neq 0}\Big(|\xi|^{-1}\Big|\int_{\mathbb{R}_+} e^{-iv\xi} f(v)\,dv - \int_{\mathbb{R}_+} e^{-iw\xi} g(w)\,dw\Big|\Big) \\ &\leq \sup_{\xi\neq 0}\Big(|\xi|^{-1}\int_{\mathbb{R}_+}\big|e^{-iv\xi} - e^{-iw\xi}\big|\,d\pi_{opt}(v,w)\Big) \\ &\leq \int_{\mathbb{R}_+}\sup_{\xi\neq 0}\Big(\frac{|1-e^{i(v-w)\xi}|}{|v-w||\xi|}\Big)|v-w|\,d\pi_{opt}(v,w) = \sup_{x\in\mathbb{R}}\Big(\frac{|1-e^{ix}|}{|x|}\Big)W(f,g). \end{aligned}$$

In view of the elementary inequality $|1-\exp(ix)| \leq |x|$ for $x\in\mathbb{R}$, this yields the claim (15). □

2.3. *Algebraic convergence for the CCM model*

Theorem 2.2. *Assume the steady state f_∞ for the CCM model possesses a Pareto tail of index $\alpha > 1$. Let $f(t)$ be a solution of the associated Boltzmann equation* (8)*, whose initial condition $f(0)$ has a finite moment of some order $n > \alpha$. Then*

$$W\big(f(t), f_\infty\big) \geq c t^{-\frac{n(\alpha-1)}{n-\alpha}} \tag{17}$$

with some time-independent constant $c > 0$.

Proof. By definition of the Pareto tail (1), one has $F_\infty(v) \geq 2\varepsilon v^{-\alpha}$ for $v \gg 0$ with some $\varepsilon > 0$. On the other hand, Theorem 3.2 in Ref. 10 (we refer also to the discussion of Example 8 in section 4.2 in Ref. 10) yields that the n-th moment $M_n(t)$ of $f(t)$ satisfies $M_n(t) \leq Ct^n$ for $t \gg 0$ with a finite constant $C > 0$. Consequently,

$$F(t;v) = \int_v^\infty f(t;w)\,dw \leq v^{-n}\int_0^\infty w^n f(t;w)\,dw \leq Cv^{-n}t^n.$$

Hence $F_\infty(v) - F(t;v) \geq \varepsilon v^{-\alpha}$ for all $v \geq V(t) := (Ct^n/\varepsilon)^{1/(n-\alpha)}$. By definition of the Wasserstein distance,

$$W\big(f(t), f_\infty\big) \geq \varepsilon\int_{V(t)}^\infty v^{-\alpha}dv \geq c\cdot t^{-\frac{n(\alpha-1)}{n-\alpha}},$$

where $c > 0$ depends on ε, n, α and C, but *not* on t. □

3. Numerical Experiments

In order to verify the analytically estimated bounds on the relaxation behavior, we have performed a series of kinetic Monte Carlo simulations for both the CPT and the CCM models. We compare numerical results for systems consisting of $N = 200$, $N = 1000$ and $N = 5000$ agents, respectively. In these rather basic simulations, pairs of agents are randomly selected for binary collisions, and exchange wealth according to the trade rules (2) and (5), respectively. One *time step* corresponds to N such interactions.

As parameters for the CPT model, we have chosen a saving propensity of $\lambda \equiv 0.7$ and independent random variables η_1, η_2 attaining the values ± 0.5 with probability $1/2$ each. The non-trivial root of $\mathfrak{S}(s)$ in (4) is $\bar{s} \approx 3.7$. In the CCM model, we assign the saving propensities by means of $\lambda = (1-\omega)^{2.5}$, with $\omega \in (0,1)$ being a uniformly distributed random variable. We restricted simulations to the deterministic situation $\epsilon \equiv 1/2$. In all our experiments, every agent possesses unit wealth initially.

In order to compute a good approximation of the steady state, the simulation is carried out for about 10^5 time steps, and then the wealth distribution is averaged over another 10^4 time steps. The thus obtained reference state is used in place of the (unknown) steady wealth distribution when calculating the decay of the Wasserstein distance in Fig. 1 and Fig. 2, respectively. The evolution of the wealth distributions over time for $N = 1000$ agents is illustrated in Fig. 3 and Fig. 4. The agents are sorted by wealth, and the wealth distribution at different time steps is compared to the approximate steady state.

Some words are in order to explain the results. The first remark concerns the seemingly poor approximation of the steady state in the CPT model, with a residual Wasserstein distance of the order $10^{-1} \dots 10^{-2}$. The reason for this behavior lies in the essentially statistical nature of this model, which *never* reaches equilibrium in finite-size systems, due to persistent thermal fluctuations. Strictly speaking, a comparision with the CMM model is misleading here, since simulations for the latter are performed in the purely deterministic setting $\epsilon \equiv 1/2$. Second, the almost perfect exponential (instead of algebraic) decay displayed in Fig. 2 is due to finite-size effects of the system. The decrease of the exponential rates when the system size N increases strongly indicates that in the theoretical limit $N \to \infty$ relaxation is indeed sub-exponential as expected. We stress that — in contrast — the decay rate in Fig. 1 for the CPT model is independent of the system size.

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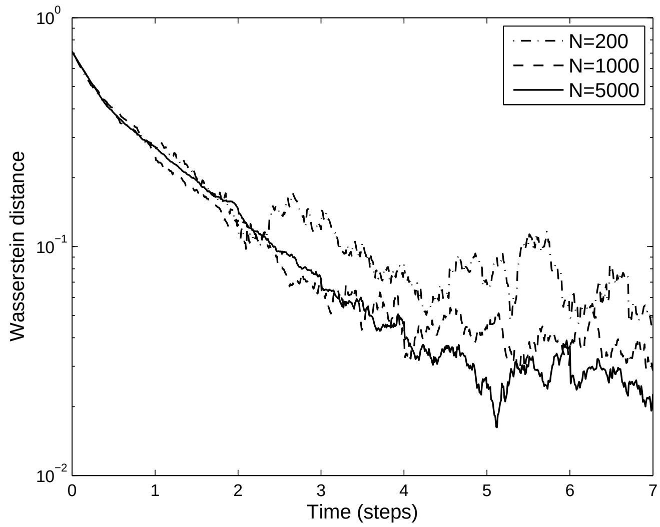

Fig. 1. Decay of the Wasserstein distance to the steady state in the CPT model.

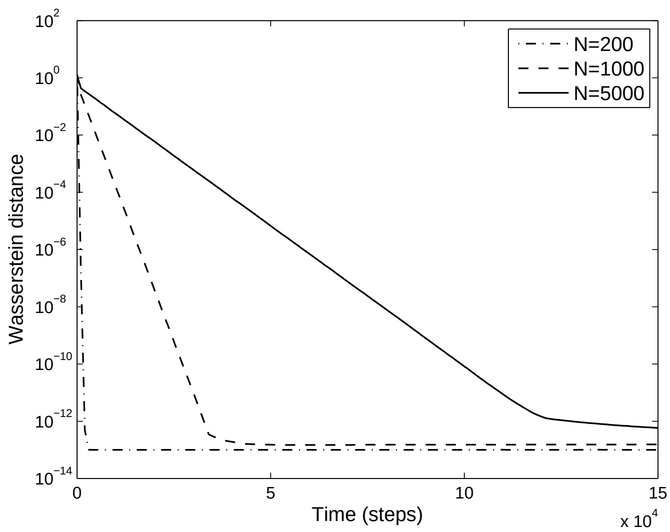

Fig. 2. Decay of the Wasserstein distance to the steady state in the CCM model.

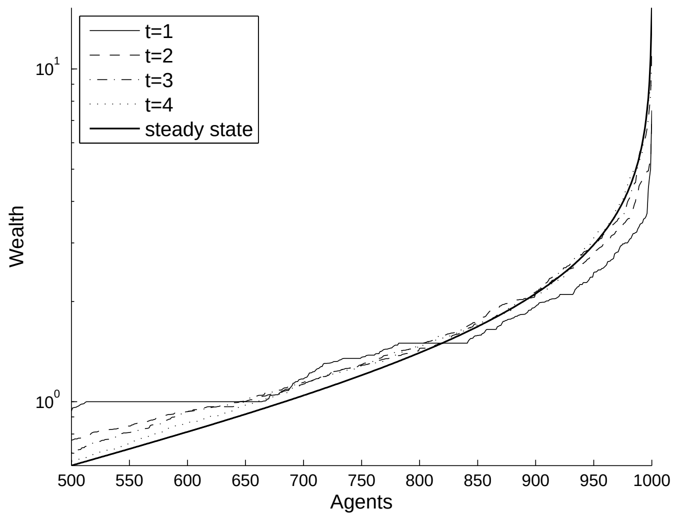

Fig. 3. Evolution of the wealth distribution towards the steady state for the upper half of the population in the CPT model ($N = 1000$).

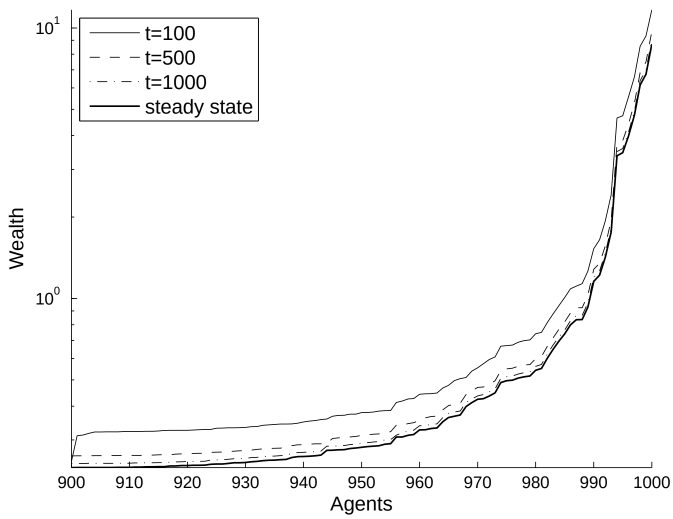

Fig. 4. Evolution of the wealth distribution towards the steady state for the upper tenth part of the population in the CCM model ($N = 1000$).