Godeaux and Campedelli surfaces

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0.1 Introduction

This chapter is concerned with minimal surfaces of general type X: I make the blanket assumptions that X is a nonsingular projective surface (for example, defined over $\mathbb{C}$) for which K_X is nef (that is, $K_X C \geq$ for every curve C) and $K_X^2 > 0$. As usual $p_g = h^0(K_X)$ and $q = h^1(\mathcal{O}_X)$.

Definition 0.1.1 *A* Godeaux surface *is a minimal surface of general type X with $p_g = q = 0$ and $K^2 = 1$. A* Campedelli surface *is a minimal surface of general type X with $p_g = q = 0$ and $K^2 = 2$.*

There is up to now no special name for surfaces with $K^2 = 3$; immortality beckons for anyone who does particularly significant work with these. This chapter discusses the known results on Godeaux and Campedelli surfaces, with several interesting classes of examples.

0.1.1 Motivation

Surfaces with $p_g = q = 0$ are interesting for several reasons.

(1) The characterisation of ruled and rational surfaces in terms of plurigenera $P_m = h^0(mK_X)$ was obtained around 1900 by Castelnuovo and Enriques. A surface is rational (birational to $\mathbb{P}^2$) if and only if $p_g = P_2 = q = 0$; a surface is ruled (birational to $C \times \mathbb{P}^1$ for some curve C) if and only if $p_g = P_2 = P_3 = P_4 = P_6 = 0$; these conditions together are of course implied by (in fact equivalent to) $P_{12} = 0$ or $\kappa = -\infty$. This raises the simple-minded question of whether $p_g = q = 0$ might not already be sufficient for rationality. This is false. The first example, around 1910, was Enriques' famous example of a sextic hypersurface in $\mathbb{P}^3$ passing doubly through the 6 edges of a tetrahedron; the modern treatment, unfortunately less picturesque, is as a quotient of a K3 surface by a free action of $\mathbb{Z}/2$. The first examples of surfaces of general type with $p_g = 0$ were discovered by Godeaux and Campedelli in the 1930s. An example of a simply connected Godeaux surface was discovered by Rebecca Barlow [1], [2] around 1980.

More recently, Craighero and Gattazzo [5] have discovered another surface with $p_g = 0$, $K^2 = 1$. Their surface is torsion free (Dolgachev and Werner [6]), and I have little doubt that it is in the same deformation family as the Barlow surface.

(2) The problem of describing all surfaces with fixed invariants seems to be completely intractable, except for very favourable choices of the invariants. The best results known in this direction are Horikawa's results on surfaces with small K^2, which contains a fairly explicit treatment of all surfaces of general type with $p_g \geq 2$ and $K^2 = 2p_g - 4$. Nevertheless, the Godeaux and Campedelli cases $p_g = q = 0$ and $K^2 = 1, 2$ are in some sense the first cases of the geography of surfaces of general type, and it is somewhat embarassing that we are still quite far from having a complete treatment of them. The study of Godeaux and Campedelli surfaces is thus a test case for the study of all surfaces of general type.

(3) Donaldson theory, one of the most substantial advance in mathematics of the 1980s, constructs C^∞ invariants of smooth 4-manifolds, capable of distinguishing between the smooth structure of 4-manifolds with the same homotopy type; in other words, the Donaldson invariants can be used (in favourable cases) to prove that two homotopy equivalent 4-manifolds are not diffeomorphic. Now the invariants of Donaldson theory work very differently for algebraic surfaces with $B_2^+ \geq 3$ (that is, $p_g \neq 0$) and those with $B_2^+ = 1$ (that is, $p_g = 0$). The fact that many papers on Donaldson theory refer to the Barlow surface as the "only known" simply connected surface of general type with $p_g = 0$ is a challenge to construct more; there must be lots lying around, if only anyone was clever enough to find them.

This chapter gives several classes of examples of Godeaux and Campedelli surfaces and discusses some of the known results.

Summary

0.2 The algebraic fundamental group and the main result

Godeaux and Campedelli surfaces are further subdivided according to their torsion subgroup $\operatorname{Tors} X \subset \operatorname{Pic} X$ and their algebraic fundamental group $\pi_1^{\mathrm{alg}}(X)$. The definition and properties of π_1^{alg} of an algebraic variety are discussed below. For Godeaux and Campedelli surfaces, the following result is known.

Theorem 0.2.1 *A Godeaux surface X has algebraic fundamental group $\pi_1^{\mathrm{alg}}(X)$ of order ≤ 5.*

Theorem 0.2.2 (Beauville, Reid [10]) *A Campedelli surface X has algebraic fundamental group of order ≤ 9.*

Restatement of Theorems 0.2.1–0.2.2 The algebraic fundamental group π_1^{alg} of an algebraic variety is discussed in the following section. However, Theorems 0.2.1–0.2.2 could be stated and proved without mentioning π_1^{alg}, in the following equivalent form.

Theorem 0.2.3 *Suppose that Y is an algebraic surface and G a finite group of automorphisms of Y acting freely, and such that the quotient variety $X = Y/G$ is a Godeaux (or Campedelli) surface. Then G has order ≤ 5 (respectively ≤ 9).*

0.2.1 The state of current knowledge

Comparable to the idea that, other things being equal, surfaces with small K^2 are easier than those with large K^2, the general experience with these surfaces is that the bigger π_1^{alg} is, the easier it is to study them. The Godeaux surfaces with $\pi_1^{\mathrm{alg}} = \mathbb{Z}/5, \mathbb{Z}/4, \mathbb{Z}/3$ are completely described in [9], and it is known that $\pi_1^{\mathrm{alg}} = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ is impossible. Examples with $\pi_1^{\mathrm{alg}} = \mathbb{Z}/2$ or 0 are known, but a complete description remains a problem. I conjecture that in either case, the moduli space is irreducible (and more-or-less rational). There are lots of examples of Campedelli surfaces with π_1^{alg} of order $4, 5, 7, 8, 9$, and it should be possible to give a complete description if π_1^{alg} has order ≥ 6. It is probable that every group of order ≤ 9 except for the dihedral groups D_8 and $D_6 = S_3$ occurs as π_1^{alg} of a Campedelli surface; the most interesting and almost certainly the hardest questions concern the simply connected case.[1]

More generally, surfaces with $p_g = q = 0$ have $K^2 + c_2 = 12\chi(\mathcal{O}_X) = 12$, and $c_2 \geq 3$, so that $K^2 \leq 9$. Each of the possible values $K^2 = 1, \dots, 9$ occurs, but there is little in the way of a systematic study of the resulting surfaces.

0.2.2 Background to π_1^{alg}

Let X be a topological space and $\pi\colon \widetilde{X} \to X$ its universal cover; the topological fundamental group $\pi_1 = \pi_1^{\text{top}}$ acts naturally on X in such a way that $X = \widetilde{X}/\pi_1$. If $N \subset \pi_1$ is a normal subgroup of finite index then $Y = \widetilde{X}/N$ is a finite etale cover $p\colon Y \to X$ with Galois group $G = \pi_1^{\text{top}}/N$.

Now suppose that the topological space X is an algebraic variety over $\mathbb{C}$; then it can be proved that Y has a natural structure of algebraic variety such that $X = Y/G$ as an algebraic variety (that is, $\mathcal{O}_X = (p_*\mathcal{O}_Y)^G$). In other words, algebraic geometry can see the finite etale covers $p\colon Y \to X$ even if it cannot see the whole of $\widetilde{X}$.

This motivates the definition of the *algebraic fundamental group*:

$$\pi_1^{\text{alg}}(X) = \varprojlim \operatorname{Gal}(Y/X),$$

where the inverse limit runs over all finite etale covers $\pi\colon Y \to X$. This definition is due to Grothendieck in SGA1, and makes sense for an arbitrary scheme X.

The explanation just given shows that for a variety over $\mathbb{C}$, the algebraic fundamental group is the profinite completion of the topological fundamental group, $\pi_1^{\text{alg}} = \widehat{\pi}_1$.

The algebraic fundamental group is a subgroup of the Galois group $\operatorname{Gal}(\overline{k(X)})$. To be more precise, let $k(X)$ be the function field of X and $K = \overline{k(X)}$ its (separable) algebraic closure. A finite extension $k(X) \subset L \subset K$ is *etale* (unramified) over X if the normalisation X_L of X in L is an etale cover

[1]What I wrote here is out of date. The question is settled by the work of Naie and the Korean geometers Yongnam Lee and Jongil Park and their students.

$X_L \to X$. Let $k(X)^{\text{et}}$ be the union of all etale extensions of $k(X)$; then $\pi_1^{\text{alg}} = \text{Gal}(k(X)^{\text{et}}/k(X))$. The Galois group of any infinite algebraic field extension $K \subset L$ is automatically a profinite group, because it is determined in terms of automorphisms of finite subextensions.

0.2.3 Known relations between π_1^{alg} and π_1^{top}

A theorem of Xiao Gang states that the natural homomorphism

$$\pi_1(X) \to \widehat{\pi}_1(X) = \pi_1^{\text{alg}}(X)$$

is injective for all surfaces of general type with $K_X^2 < 3\chi(\mathcal{O}_X) - 10$. A group G with the property $G \hookrightarrow \widehat{G}$ is *residually finite*. Injectivity just means that a nonzero element of G is nonzero in some finite quotient group. Although Xiao's theorem can probably be extended to surfaces with $K_X^2 < 4\chi(\mathcal{O}_X)$ (see [12]), it is not true that π_1 of an algebraic surface is always residually finite: Domingo Toledo [14] has constructed an example of an algebraic surface for which π_1 contains a normal subgroup that is free of infinite rank, but maps to the identity under any homomorphism to a finite group.

Whereas the fundamental group is the most basic invariant in topology and homotopy theory, in algebraic geometry, covers with specified ramification may be just as important as etale covers. The definition of π_1^{alg} in terms of Galois theory has the advantage that it can cover these generalisations with no extra effort. "The fundamental group is not really fundamental" would be a reasonable slogan in some contexts of algebraic geometry.

0.2.4 Proof of Theorem 0.2.1

Let X be a Godeaux surface, $p\colon Y \to X$ a finite Galois etale cover of degree n. Then since $T_Y = p^*T_X$ it follows that $K_Y = p^*(K_X)$ and $c_2(Y) = p^*(c_2(X))$, so that

$$K_Y^2 = n, \ \chi(\mathcal{O}_Y) = n\chi(\mathcal{O}_X) = n, \quad \text{that is,} \quad p_g(Y) = n - 1 + q(Y).$$

Thus what I have to prove is equivalent to the following:

Theorem 0.2.4 *Let Y be a minimal surface of general type, with $K_Y^2 = n$, $p_g(Y) = n - 1 + q(Y)$, and suppose that a group G of order n acts freely on Y. Then $n \leq 5$.*

Proof The inequality $K^2 \geq 2p_g - 4$ of Max Noether and Horikawa gives

$$n \geq 2(n - 1 + q(Y)) - 4,$$

so that $n \leq 6$. Suppose that $n = 6$; then $K_Y^2 = 6$ and $p_g = 5$, so that $K^2 = 2p_g - 4$. Then by Horikawa's results, $|K_Y|$ defines a double cover $\varphi_{K_Y}: Y \to \mathbb{F}_1 \subset \mathbb{P}^4$ from Y to $\mathbb{F}_1$ embedded in $\mathbb{P}^4$ as a rational normal scroll of degree 3. Now the composite $f = p \circ \varphi$ of $\varphi_{K_Y}: Y \to \mathbb{F}_1$ and the projection $p\colon \mathbb{F}_1 \to \mathbb{P}^1$ is a fibre space $Y \to \mathbb{P}^1$ with fibres of genus 2.

Since p is constructed in a unique way from Y, it follows that it is invariant under G, or more precisely, G also has an action on $\mathbb{P}^1$ so that $f\colon Y \to \mathbb{P}^1$ is equivariant. The following result then contradicts the assumption that G acts freely on Y.

Lemma 0.2.5 *If a fibre space $f\colon Y \to \mathbb{P}^1$ of curves of genus g has compatible automorphisms $\alpha \in \operatorname{Aut} Y$ and $\beta \in \operatorname{Aut} \mathbb{P}^1$, such that α generates a cyclic group $G = \langle\alpha\rangle = \mathbb{Z}/r \subset \operatorname{Aut} Y$ acting freely on Y, then r divides $g - 1$. (If $g = 1$ the statement is vacuous.)*

Corollary 0.2.6 *A surface with a canonically defined pencil of curves of genus 2 over $\mathbb{P}^1$ has no fixed point free automorphism. This holds in particular for all the Horikawa surfaces with $K^2 = 2p_g - 4$, and in fact for all regular surfaces with $K^2 < \frac{8}{3}p_g - 14$ (compare [11] or [15]).*

Proof β certainly has a fixed point $P \in \mathbb{P}^1$, so that α preserves some fibre $F = p^{-1}P$. If $g = 0$ then F is a tree of rational curves, so that there can be no fixed point free automorphism.

Write $F = p^*P$ for the scheme theoretic fibre. Then if $g > 1$ it follows that $K_Y F = 2g - 2 > 0$, and clearly, α must have finite order, say r. (Because α preserves an ample divisor class on Y of the form $K_Y + f^*(\text{ample}.)$ If α has no fixed points near F then the quotient by α is a finite etale cover $\pi\colon Y \to X$ of degree r, and both $K_Y = \pi^*(K_X)$ and $F = \pi^* G$ are pullbacks of divisors on X. Therefore $2g - 2 = K_Y F = rK_X G$; also $G^2 = 0$, so that $K_X G$ is even. Thus r divides $g - 1$. Q.E.D.

The proof of Theorem 0.2.2 is basically similar, although a bit more complicated. See [10].

0.3 Standard example of Godeaux surfaces

0.3.1 The $\mathbb{Z}/5$ Godeaux surface

In the notation of the preceding proof, if $n = 5$ then $K_Y^2 = 5$, $p_g = 4$ and $q(Y) = 0$, that is, Y has the numerical invariants of a quintic hypersurface. By Horikawa's results there are two possibilities:

(I) The canonical linear system $|K_Y|$ is free, and the associated morphism $\varphi_{K_Y}\colon Y \to Y_5 \subset \mathbb{P}^3$ is birational to a quintic $Y_5 \subset \mathbb{P}^3$.

(II) K_Y has a unique base point P.

In case (II), G must fix the point P, which is a contradiction. Hence only (I) can hold. In suitable homogeneous coordinates $x_1, \dots, x_4$ on $\mathbb{P}^3$, the action of $\mathbb{Z}/5$ on $\mathbb{P}^3$ is given by $x_i \mapsto \varepsilon^i x_i$ for $i = 1, \dots, 4$ (where ε is a primitive 5th root of 1), and Y is defined by a quintic invariant under G, for example $x_1^5 + \cdots + x_4^5 = 0$. The quotient $X = Y/(\mathbb{Z}/5)$ is a nonsingular projective surface with K ample, and it is easy to calculate its invariants: $K_X^2 = K_Y^2/5$ and $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_Y)/5 = 1$. Also $H^1(\mathcal{O}_X) \subset H^1(\mathcal{O}_Y) = 0$, so that X is a Godeaux surface.

For $\mathbb{Z}/4$ Godeaux surfaces see [9] or Ex. 3.

0.3.2 Example of a Campedelli surface with $\pi_1 = 3\mathbb{Z}/2$

Consider the action of $G = 3\mathbb{Z}/2 = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$ on a 7-dimensional vector space V which is the sum of the 7 different nontrivial character spaces. That is, write $x_{001}, x_{010}, \dots, x_{111}$ for coordinates on V, with the group action given by

$$g_{ijk}(x_{i'j'k'}) = (-1)^{ii'+jj'+kk'} x_{i'j'k'}$$

for $g_{ijk} \in G$. In other words, every element $e \neq g \in G$ acts by $\operatorname{diag}(\pm, \dots, \pm)$ with 3 pluses and 4 minuses. Then G acts on $\mathbb{P}^6$, and one checks at once that every nonzero element $g \in G$ has fixed locus $\operatorname{Fix} g = \mathbb{P}^2 \cup \mathbb{P}^3$.

Any diagonal quadratic form $q = \sum a_i x_i^2$ is of course invariant under G. It is easy to see that a general choice of 4 diagonal quadrics Q_1, Q_2, Q_3, Q_4 intersect transversally in a nonsingular surface Y disjoint from $\operatorname{Fix} g$ for all $g \in G$. Then $\pi\colon Y \to X = Y/G$ is an etale quotient, and $p_g(Y) = 7$, $q(Y) = 0$ and $K_Y^2 = 16$ implies that $p_g(X) = 0$, $K_X^2 = 2$, that is, X is a Campedelli surface with $\pi_1 = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

There are lots of variations on this example, obtained by making different groups of order 8 act on $Y = \bigcap_{i=1}^4 Q_i \subset \mathbb{P}^6$ (see [10] and compare [4]). Each of $\mathbb{Z}/8$, $\mathbb{Z}/2 \oplus \mathbb{Z}/4$, $\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and the quaternion group $H_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ can act freely. The dihedral group D_8 of order 8 does not have a free action on $Y = \bigcap_{i=1}^4 Q_i \subset \mathbb{P}^6$, for easy reasons of representation theory. For more details see Exercises 1–8.

It is also interesting to consider a group of order 16 acting on a complete intersection of 4 quadrics $Y = \bigcap_{i=1}^4 Q_i \subset \mathbb{P}^6$ such that certain group elements have isolated fixed points on Y. In this way, using Lemma 0.4.3 below, Barlow [3] constructed Godeaux surfaces with $\pi_1 = \mathbb{Z}/2$ and $\mathbb{Z}/4$.

0.3.3 Example of a Campedelli surface with $\pi_1 = \mathbb{Z}/7$

For this example, I first construct a class of surfaces $Y \subset \mathbb{P}^5$ with the following properties:

(i) Y is canonically embedded, that is, $\mathcal{O}_Y(K_Y) = \mathcal{O}_Y(1)$, so that $p_g(Y) = 6$, and of degree 14, that is, $K_Y^2 = 14$.

(ii) It is projectively normal (that is, $H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(k)) \to H^0(Y, \mathcal{O}_Y(k))$ is surjective for every k), and projectively Cohen–Macaulay, that is $H^1(\mathcal{O}_Y(k)) = 0$ for every k, and in particular $q(Y) = h^1(\mathcal{O}_Y) = 0$.

Under these assumptions, the canonical ring

$$R(Y, K_Y) = \bigoplus_{n \geq 0} H^0(Y, nK_Y),$$

which equals the homogeneous coordinate ring $k[x_1, \dots, x_6]/I_Y$ of $Y \subset \mathbb{P}^5$ is a Gorenstein ring of codimension 3. The famous structure theorem of Buchsbaum and Eisenbud asserts that the ideal of relations of such a ring can be written in *Pfaffian format*. In my case, this means the following: there is a skew 7×7 matrix $M = (l_{ij})$ with entries l_{ij} linear forms in the homogeneous coordinates of $\mathbb{P}^5$, and the ideal I_Y is generated by the 7 cubic forms Pf_i obtained as the 6×6 Pfaffians. That is, delete the ith row and jth column of M to obtain a skew 6×6 matrix M_{ij}; then the determinant of M_{ij} is a product of two Pfaffians:

$$\det M_{ij} = \pm \mathrm{Pf}_i \mathrm{Pf}_j$$

as a polynomial identity in the entries of M, and in particular, the diagonal minors are perfect squares: $\det M_{ii} = \mathrm{Pf}_i^2$. One shows that if the entries l_{ij} of M are sufficiently general then $Y : (\mathrm{Pf}_i = 0)$ has the stated properties.

Now to construct Y with an action of $\mathbb{Z}/7$, I have to choose the matrix M carefully. In homogeneous coordinates $x_1, \dots, x_6$ on $\mathbb{P}^5$, I take the action of $\mathbb{Z}/7$ to be generated by $\alpha\colon x_i \mapsto \varepsilon^i x_i$, where ε_i is a primitive 7th root of 1. Let M be the skew 7×7 matrix given by

$$M = \begin{pmatrix} 0 & x_1 & x_3 & x_2 & x_6 & x_4 & x_5 \\ & 0 & x_4 & \lambda_3 x_3 & 0 & -\lambda_5 x_5 & -x_6 \\ & & 0 & x_5 & \lambda_2 x_2 & 0 & -\lambda_1 x_1 \\ & & & 0 & x_1 & \lambda_6 x_6 & 0 \\ & & & & 0 & x_3 & \lambda_4 x_4 \\ & -\mathrm{sym} & & & & 0 & x_2 \\ & & & & & & 0 \end{pmatrix}$$

where $\lambda_1 \dots, \lambda_6$ are parameters. It is not hard to see that every 2×2 minor of M is an eigenform of the group action, and hence so is every minor and Pfaffian. Then one calculates easily that

$$\begin{aligned} \mathrm{Pf}_0 &= x_1x_2x_4 + \dots \\ \mathrm{Pf}_1 &= -\lambda_5 x_5^3 - \lambda_1\lambda_6 x_1^2 x_6 + (1 - \lambda_1\lambda_5) x_1x_2x_5 - \lambda_1\lambda_3 x_1x_3x_4 + x_2x_4^2 \\ &\quad - \lambda_3 x_2 x_3^2 + (1+\lambda_6) x_4x_5x_6 + \lambda_6 x_3 x_6^2 \\ &\quad \dots \end{aligned}$$

The group action maps $\mathrm{Pf}_i \mapsto \varepsilon^i \,\mathrm{Pf}_i$. It is extremely difficult to see that for sufficiently general values of $\lambda_1, \dots, \lambda_6$, the equations $\mathrm{Pf}_0 = \mathrm{Pf}_1 = \dots = \mathrm{Pf}_6$ define a nonsingular surface $Y \subset \mathbb{P}^5$. I only know how to do this by a computer algebra calculation (see [13]).

The columns of M should be thought of as labelled by 0 and $1, 3, 2, 6, 4, 5$, to be invariant under the symmetry group $(\mathbb{Z}/7)^\times$ of $\mathbb{Z}/7$. In fact, if $\lambda_1 = \dots = \lambda_6$, then it is easy to check that the permutation $\beta\colon x_1 \mapsto x_3 \mapsto x_2 \mapsto \cdots$ of the coordinates x_i takes $\mathrm{Pf}_1 \mapsto \mathrm{Pf}_3 \mapsto \dots$, so that construction has the symmetry group of order 42 generated by α and β.

0.4 The Burniat–Inoue examples

Inoue's form of the Burniat construction starts from three elliptic curves E_1, E_2, E_3, each marked with an action of $(\mathbb{Z}/2)^{\oplus 2}$ generated by the following two commuting involutions σ_i, τ_i: a "minus" involution $\sigma_i \colon E_i \to E_i$, corresponding to a double cover $\pi_i \colon E_i \to \mathbb{P}^1$, and a translation involution $\tau_i \colon E_i \to E_i$. The composite $\tau_i\sigma_i$ is of course another "minus" involution. I consider the group $H \simeq 3\mathbb{Z}/2$ of involutions of the Abelian 3-fold $A = E_1 \times E_2 \times E_3$ generated by the 3 elements:

$$\begin{gathered}(\mathrm{id}_1, \tau_2, \sigma_3\tau_3)\\ (\sigma_1\tau_1, \mathrm{id}_2, \tau_3)\\ (\tau_1, \sigma_2\tau_2, \mathrm{id}_3)\end{gathered}$$

(note the cyclic symmetry (123)). Each generator acts as a translation τ_i in one of the three components, and therefore has no fixed points on $E_1 \times E_2 \times E_3$; the same holds for each composite of 2 generators, for example $(\sigma_1\tau_1, \tau_2, \sigma_3)$. The composite of all 3 generators is $(\sigma_1, \sigma_2, \sigma_3)$, the "minus" of the Abelian 3-fold, and so has 64 fixed points.

Now look for an H-invariant divisor $Z \subset E_1 \times E_2 \times E_3$. Choose the divisor class of Z to be $\bigotimes_{i=1}^{3} \pi_i^* \mathcal{O}_{\mathbb{P}^1}(1)$, where $\pi_i \colon E_i \to \mathbb{P}^1$. In this section, I leave all the nonsingularity calculation until the end. I check in 0.4.2 below that there is an invariant 1-dimensional linear system whose general element Z is nonsingular and misses the 64 fixed points of $(\sigma_1, \sigma_2, \sigma_3)$, so that the quotient $Y = Z/H$ is a nonsingular surface.

Claim 0.4.1 $p_g(Y) = q(Y) = 0$, $K_Y^2 = 6$. *(Of course, $\pi_1(Y)$ is infinite.)*

Proof An easy calculation shows that a nonsingular divisor in an Abelian 3-fold $Z \subset A$ satisfies $c_1^2 = c_2$, that is, $K_Z^2 = 6\chi(\mathcal{O}_Z)$. Indeed, A has trivial tangent bundle $T_A = 3\mathcal{O}_A$, so that the usual restriction sequence gives

$$0 \to T_Z \to \mathcal{O}_A^{\oplus 3} \to \mathcal{O}_A(Z) \to 0.$$

Hence the total Chern class of Z is $c(T_Z) = (1+Z)^{-1} = 1 - Z + Z^2$, and thus

$$K_Z = -c_1(T_Z) = Z_{|Z}, \quad \text{and} \quad c_2(Z) = {Z^2}_{|Z} = c_1^2(Z).$$

Now Z is twice a principal polarisation of A, so that $K_Z^2 = Z^3 = 2^3 \cdot 3! = 48$, and therefore the quotient $Y = Z/H$ has $K_Y^2 = 6$, $\chi(\mathcal{O}_Y) = 1$.

Thus it is enough to show that $q(Y) = 0$. But Y is an ample divisor on the quotient $B = A/H$, and $H^1(\mathcal{O}_B) = 0$. Indeed, $H^1(\mathcal{O}_B)$ equals the invariants of H acting on $H^1(\mathcal{O}_A) = \bigoplus_{i=1}^3 H^1(\mathcal{O}_{E_i}) = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$; but H is generated by 3 elements acting on this by $+, +, -$ and cyclic permutations, so leaves nothing invariant.

0.4.1 First variation

If[2] we choose $E_1 = E_2 = E_3$ the Inoue construction obviously has the cyclic symmetry (123). Thus the surface Y just constructed has an action of $\mathbb{Z}/3$. In 0.4.2 below I check that (123) acts on Y with 3 fixed points $\{P_j\}$, and that the action on the tangent space T_{P_j} at each fixed point diagonalises to $(\varepsilon, \varepsilon^{-1})$, where ε is a primitive 3rd root of 1. Thus the quotient $\overline{X} = Y/(123)$ has 3 Du Val singularities A_2. It is clear that its resolution $X \to \overline{X}$ is a Campedelli surface: indeed, K_Y is ample, and $\pi\colon Y \to X$ is etale in codimension 1, so that $K_Y = \pi^*(K_{\overline{X}})$, and $K_{\overline{X}}$ is ample with $K_{\overline{X}}^2 = 2$.

Claim 0.4.2 $\pi_1^{\mathrm{top}}(X) = \mathbb{Z}/2$.

It is easy to see that $\pi_1^{\mathrm{alg}}(X)$ is at least $\mathbb{Z}/2$. Indeed, by construction, X is an ample divisor on the 3-fold $C = B/(\mathbb{Z}/3)$ with $2K_C = 0$ but $K_C \neq 0$, and therefore $K_{C|X} \in \operatorname{Tors} X$ is a nontrivial 2-torsion class.

Lemma 0.4.3 *Let T be a simply connected topological space and G a transformation group of T. Say that $g \in G$ is* elliptic *if* $\operatorname{Fix} g \neq \emptyset$. *Then*

$$\pi_1(T/G) = G/E,$$

where $E \subset G$ is the normal subgroup generated by elliptic elements.

This is well known, and can be proved either by the homotopy lifting property of the universal cover (the universal cover of T/G is dominated by $T \to G$, so is of the form T/H, where $H \subset G$ is a subgroup, that you show must be normal and contain the elliptic elements), or by similar ideas in Galois theory. (For details see Rebecca Barlow's thesis [1].)

[2] From here on the paper is not reliable. This material is also out of date. The questions here have been settled by work of Bauer and Catanese and their school.

Proof of the claim First, $Z \subset A$ is an ample divisor, so by the Lefschetz theorems, $\pi_1(Z) \xrightarrow{\cong} \pi_1(A)$; it follows that the inverse image of Z in the universal cover $\mathbb{C}^3 = \widetilde{A}$ is the universal $\widetilde{Z}$ cover of Z, and in particular, is connected and simply connected. Now $\overline{X}$ is the quotient of $\widetilde{Z}$ by a complex crystallographic group G: the group contains the cocompact lattice $\mathbb{Z}^6 = \pi_1(A)$ as a normal subgroup of index 3×2^3, and the quotient group $H = G/\mathbb{Z}^6$ (the "point group" in crystallographic terms) consists of the possible linear parts of elements of G. By construction, H is generated by matrixes

$$\operatorname{diag}(-1,1,1),\ \operatorname{diag}(1,-1,1),\ \operatorname{diag}(1,1,-1), \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Thus the linear parts of elements of G consists of cyclic permutations of 3 coordinates z_1, z_2, z_3 of $\mathbb{C}^3$ with $\pm$ signs attached. Obviously, such a matrix M with a nontrivial cyclic permutations has the 3 eigenvalues

$$\begin{cases} 1, \varepsilon, \varepsilon^2 & \text{if } \det M = 1; \\ -1, -\varepsilon, -\varepsilon^2 & \text{if } \det M = -1; \end{cases}$$

Claim 0.4.4 *Suppose that the linear part $\overline{g}$ of $g \in G$ satisfies*

1. *$\overline{g}$ is a nontrivial cyclic permutation;*
2. $\det \overline{g} = 1$*;*
3. $g^3 = \mathrm{id}_{\mathbb{C}^3}$.

Then $\operatorname{Fix}(g : \mathbb{C}^3)$ *is* 1*-dimensional, and* $\operatorname{Fix}(g : \widetilde{Z}) \neq \emptyset$.

This is obvious: under the stated conditions, the linear part of g is a matrix M whose linear part with eigenvalues of $1, \varepsilon, \varepsilon^2$. Thus the linear part has a 1-dimensional fixed set. (3) implies that there is no translation part.

0.4.2 Nonsingularity proofs

I first set up some notation: the elliptic curve $E : y^2 = (x^2 - a^2)(x^2 - b^2)$ has two involutions $\sigma\colon (x,y) \mapsto (x,-y)$ and $\tau\colon (x,y) \mapsto (-x,-y)$. Obviously σ has 4 fixed points $y = 0$, $x = \pm a, \pm b$, and τ has no fixed points, but take $(a,0) \mapsto (-a,0)$. The third element $\sigma\tau$ of the four-group has fixed points

$x = 0, y = \pm ab$ and $x = \infty, x^2 y = 1/ab$. My coordinate x is of course inhomogeneous: I write simply $1, x$ for the basis of sections of $\pi^*(\mathcal{O}_{\mathbb{P}^1}(1))$ on E corresponding to $\pi\colon E \to \mathbb{P}^1$, and leave the reader to figure out what goes on at $x = \infty$.

For coordinates on the product $A = E \times E \times E$ (or $E_1 \times E_1 \times E_2$), I take $(x, y); (x', y'); (x'', y'')$. The only nonidentity element of H that has fixed points on A is σ, σ, σ, which fixes the 64 points where $y = y' = y'' = 0$ and $x, x', x'' \in \{\pm a, \pm b\}$ are independent choices.

Now the action of $H = (\mathbb{Z}/2)^{\oplus 3}$ on the complete linear system $|Z|$ on A is given by the linear action of H on the trilinear forms $(1, x) \otimes (1, x') \otimes (1, x'')$. Since the 3 generators of H act on x, x', x'' by $+, -, -$ and cyclic permutations, the invariant subspace is based by the two elements $\{1, xx'x''\}$. Thus the invariant surface Z is defined by $xx'x'' = c$ for some constant $c \in \mathbb{P}^1$, and to ensure the irreducibility of Z, it will be prudent to take $c \neq 0, \infty$. To encourage confidence in the nonsingularity calculation, note that $xx'x'' = c$ simply defines a "general" hyperplane section of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$.

Claim 0.4.5 *For $c \neq ??$, $Z : (xx'x'' = c) \subset A$ is nonsingular and misses the 64 fixed points of σ, σ, σ.*

Homework to Chapter 0.4

1. Prove that any algebraic action of $\mathbb{Z}/2$ on $\mathbb{P}^3$ is given in suitable linear coordinates by $(x_0, x_1, x_2, x_3) \mapsto (\pm x_0, \pm x_1, \pm x_2, \pm x_3)$. [Hint: Argue that $g^*\mathcal{O}(1) \simeq \mathcal{O}(1)$, then $g^* x_i \in H^0(\mathcal{O}(1))$, so that g takes x_i into a matrix A times x_i. After scaling, you get $A^2 = 1$, and can reduce it to diagonal form.]

2. Let $G = \mathbb{Z}/2$ act on $\mathbb{P}^3$ by $(x_0, x_1, x_2, x_3) \mapsto (x_0, x_1, -x_2, -x_3)$. Prove that a surface $X = X_d : (f_d = 0) \subset \mathbb{P}^3$ is invariant under G if and only if f_d is a $\pm$ eigenform. For general choice of f_d, determine the fixed locus of $\mathbb{Z}/2$ on X and the singularities of the quotient X/G.

3. Godeaux with $\mathbb{Z}/4$.

4. Campedelli with $\mathbb{Z}/8$ and $\mathbb{Z}/4 \oplus \mathbb{Z}/2$.

5. Campedelli with $\mathbb{Z}/3 \oplus \mathbb{Z}/3$.
6. Existence of Campedelli with $\mathbb{H}_8$ (compare Beauville [4]).
7. Existence of Campedelli with $\mathbb{Z}/9$ and $2\mathbb{Z}/3$ (compare Xiao [] and Mendes Lopes–Pardini []).
8. Nonexistence of Campedelli with D_8.
9. Nonexistence of Campedelli with $\pi_1 = D_6 = S_3$ (compare Naie [])).

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