MA246

Number Theory

Workbook 4 (without solutions)

Continued Fractions
(Carry On Upending)

Summer 2013

(originally written and devised by
Trevor Hawkes and Alyson Stibbard;
revised in 2010 by John Cremona)

Aims of these workbooks:

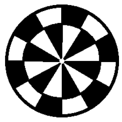
(a) To encourage you to teach yourself mathematics from written material,

(b) To help you develop the art of independent study — working either alone, or co-operatively with other students,

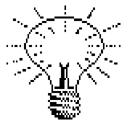
(c) To help you learn a mathematical topic, in this case Number Theory, through calculation and problem-solving.

Copies of this workbook, both with and without solutions, can be found on Mathstuff.

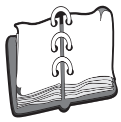
Icons in this Workbook



The 'Section Targets' box contains an idea of what you should aim to get out of the current section. Perhaps you might return to this at the end to evaluate your progress.



Reaching this icon in your journey through the workbook is an indication that an idea should be starting to emerge from the various examples you have seen.



Material here includes reference either to earlier workbooks, or to previous courses such as foundations/Sets and Groups.

A caution. Watch your step over issues involved here.

Are You Ready?

To understand the material and do the problems in each section of this workbook, you will need to be on good terms with:

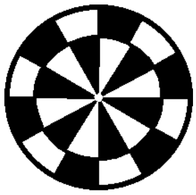
Section 1: • The Euclidean Algorithm
Section 2: • Properties of Sequences

Note: You will need a pocket calculator for some of the questions in the workbooks, and are encouraged to use one for this purpose and to experiment with results and ideas in the course. Calculators are NOT needed and are NOT allowed in tests or in the examination.

These workbooks were orginally written and devised by *Trevor Hawkes and and Alyson Stibbard*. *Ben Carr* designed the LaTeX template and *Rob Reid* converted their drafts into elegant print. Over the years, other lecturers and students have corrected a number of typos, mistakes and other infelicities. In 2010 *John Cremona* made some substantial revisions.

Send corrections, ask questions or make comments at the module forum. You can join the MA246 forum by going to `http://forums.warwick.ac.uk/wf/misc/welcome.jsp` and signing in, clicking the *browse* tab, and then following the path: Departments > Maths > Modules > MA2xx modules > MA246 Number Theory.

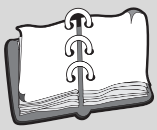
1 Finite Continued Fractions for Rationals



Section Targets We explore the connection between Euclid's Algorithm and continued fractions for rational numbers. In particular, we show that

- every finite continued fraction represents a rational number, and
- every rational number can be represented as a finite continued fraction in an essentially unique way.

We also investigate some interesting algebra associated with numbers q_i, called *partial quotients*.



The Euclidean Algorithm
Recall (from Foundations, and the last section of Workbook 1) how Euclid's Algorithm (EA) gives an efficient method of computing the highest common factor of two integers, that avoids having to find their factorisations.

(1.1) Important Notation. It will be helpful to keep in mind the following standard notation that describes the steps of the Euclidean Algorithm (EA) applied to two integers a and b, where $b > 0$.

$$\begin{aligned}
a &= q_0 b + r_1 && 0 \le r_1 < b \\
b &= q_1 r_1 + r_2 && 0 \le r_2 < r_1 \\
r_1 &= q_2 r_2 + r_3 && 0 \le r_3 < r_2 \\
\vdots &\ \ \vdots \quad \vdots \ \vdots \ \vdots \ \vdots \ \vdots \\
r_{t-2} &= q_{t-1} r_{t-1} + r_t && 0 \le r_t < r_{t-1} \\
r_{t-1} &= q_t r_t
\end{aligned} \tag{1.a}$$

The last non-zero remainder, r_t, is equal to the highest common factor of a and b.
Note: to make this true when b divides a we set $t = 0, a = r_{-1}$ and $b = r_0$.
This would be a good time to revise your EA technique, including the tabular layout which we will be making good use of in this Workbook (see Section 4 of Workbook 1).

Continued Fractions

A *continued fraction* is a fraction of the form,

$$q_0 + \cfrac{1}{q_1 + \cfrac{1}{q_2 + \cfrac{1}{q_3 + \ddots}}}$$

where $q_0 \in \mathbb{Z}$ and $q_i \in \mathbb{N}$ for all $i \geq 1$. The q_is are called *partial quotients*. We will consider both finite and infinite continued fractions (according to whether the set of partial quotients is finite or countably infinite).

(1.2) Questions on a Concrete Example.

(a) Write the following continued fraction as a rational number a/b:

$$1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4}}}$$

(b) Apply the EA to the pair $a = 43$ and $b = 30$.

(c) List the q_is that appear in ((b)) and compare them with the partial quotients in ((a)).

Answers to (1.2)

(a)

(b)

(c)

(1.3) A Worked Example. Use the Euclidean Algorithm to convert the rational number 89/49 into a continued fraction.

Saving paper and ink
In the compact format of WB1§4 this looks like

(r)	(q)
89	1
49	1
40	4
9	2
4	4
1	

The column of q values gives the partial quotients directly, with rather less writing than on the right!

Solution. Applying the EA to the pair $a = 89$ and $b = 49$ yields

$$\begin{aligned} 89 &= 1 \times 49 + 40 && \text{(1.b)} \\ 49 &= 1 \times 40 + 9 && \text{(1.c)} \\ 40 &= 4 \times 9 + 4 && \text{(1.d)} \\ 9 &= 2 \times 4 + 1\,. && \text{(1.e)} \end{aligned}$$

From Equation (1.b) we get

$$\frac{89}{49} = 1 + \frac{40}{49} = 1 + \frac{1}{\left(\frac{49}{40}\right)}\,.$$

From (1.c) we obtain

$$\frac{49}{40} = 1 + \frac{9}{40}$$

and hence

$$\frac{89}{49} = 1 + \cfrac{1}{1 + \cfrac{9}{40}} = 1 + \cfrac{1}{1 + \cfrac{1}{\left(\frac{40}{9}\right)}} \qquad \text{(1.f)}$$

Since (1.d) and (1.e) yield

$$\frac{40}{9} = 4 + \frac{4}{9} = 4 + \frac{1}{\left(\frac{9}{4}\right)} = 4 + \cfrac{1}{2 + \cfrac{1}{4}},$$

substitution in (1.f) yields

$$\frac{89}{49} = 1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{2 + \cfrac{1}{4}}}}\,,$$

Check directly that this is true.

Illustration
In this notation, the continued fraction in the preceding Worked Example is written

$$1 + \frac{1}{1+}\frac{1}{4+}\frac{1}{2+}\frac{1}{4}$$

Compact Notation for Continued Fractions. To save space on the page, from now on we will use the notation

$$q_0 + \frac{1}{q_1+}\frac{1}{q_2+}\cdots\frac{1}{q_n}$$

instead of the sprawling

$$q_0 + \cfrac{1}{q_1 + \cfrac{1}{q_2 + \ddots + \cfrac{1}{q_{n-1} + \cfrac{1}{q_n}}}}$$

It is important to get some practice at converting rationals into continued fractions (CFs).

Fibonacci Numbers
Fibonacci numbers are defined recursively by the equations $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n = 1, 2, 3, \ldots$. Write down the first ten Fibonacci numbers and think about the development of the rational number

$$\frac{F_{n+1}}{F_n}$$

as a continued fraction.

(1.4) Question on CF development. As in Worked Example (1.3) (and the sidebar next to it), use the EA to develop the following rationals as continued fractions,

$$\frac{21}{13}, \quad -\frac{11}{31}, \quad \frac{20}{31}, \quad \frac{42}{26}\,.$$

Answers to (1.4)

continued. . .

Your work with these continued fraction developments of rational numbers should persuade you of the following:

Lowest Terms
Your work with 21/13 and 42/26 in (1.4) above should convince you that "lowest terms" representations can always be used to develop a rational number a/b as a CF. The EA applied to $a' = ad$ and $b' = bd$ is obtained from Equations (1.a) by multiplying a, b and all the r_i's by d and leaving the partial quotients $q_0,\ q_1, \ldots,\ q_t$ unchanged.

(1.5) Theorem *If Equations (1.a) denote the Euclidean algorithm for natural numbers a and b, then the continued fraction for the rational number a/b is*

$$q_0 + \frac{1}{q_1+}\frac{1}{q_2+}\cdots\frac{1}{q_t}.$$

Alternatively, the sequence of partial quotients for a/b are precisely the numbers in the quotient sequence for a, b as defined in Workbook 1 Section 4.

Proof. If you still need convincing after your work on the preceding examples, it is easy to cook up a formal proof by induction on t, which is one less than the number of equations in (1.a). Start the induction at $t = 0$, which corresponds to the case $r_1 = 0$ and $a/b = q_0$, yielding a continued fraction with one partial quotient and no fractional part. If $t \geq 1$, we get

$$\frac{a}{b} = q_0 + \frac{1}{\left(\dfrac{b}{r_1}\right)} \tag{1.g}$$

and the EA applied to the pair b and r_1 yields the last t equations of (1.a) with partial quotients $q_1, \ldots, q_t$. By induction, the continued fraction expansion of b/r_1 is

$$q_1 + \frac{1}{q_2+}\frac{1}{q_3+}\cdots\frac{1}{q_t}$$

and plugging this into Equation (1.g) gives the stated conclusion of the theorem. □

Uniqueness

According to Theorem 1.5 the continued fraction associated with the rational number a/b is determined by the partial quotients $q_0,\ q_1, \ldots,\ q_t$. The final equation in (1.a) is

$$r_{t-1} = q_t r_t$$

Since $0 < r_t < r_{t-1}$, it follows that $q_t > 1$. The final term $1/q_t$ of the CF can be replaced with

$$\frac{1}{(q_t-1)+}\,\frac{1}{1}$$

without changing its value since

$$\frac{1}{q_t} = \frac{1}{(q_t-1)+\frac{1}{1}}\,.$$

This alternative ending can be excluded by insisting that the final partial quotient of a continued fraction is at least 2; with this stipulation, continued fractions for rationals are unique.

(1.6) Question on the Uniqueness of Finite CFs. Let $q_0,\ q_1, \ldots,\ q_t$ and $r_0,\ r_1, \ldots,\ r_s$ be natural numbers with $q_t \geq 2$ and $r_s \geq 2$. Show that if

$$q_0 + \frac{1}{q_1+}\,\frac{1}{q_2+}\cdots\frac{1}{q_t} = r_0 + \frac{1}{r_1+}\,\frac{1}{r_2+}\cdots\frac{1}{r_s} \qquad \text{(1.h)}$$

then $s = t$ and $q_i = r_i$ for all $i = 0,\ 1, \ldots,\ t$.

Hint: If

$$x = q_0 + \frac{1}{q_1+}\cdots\frac{1}{q_t},$$

show that $q_0 = [x]$, the integral part of x.

Answer to (1.6)

The Partial Quotients
The numbers $q_0, q_1, \ldots, q_t$ appearing in the CF labelled (1.j) are called its *partial quotients*. Although we require that q_0 is an integer and q_i ($i \geq 1$) is a natural number when continued fractions are under scrutiny, the expression (1.j) makes perfectly good algebraic sense when the q_i's are arbitrary real numbers or even variables. Therefore we keep our options open.

Multivariable Polynomials
You are no doubt familiar with polynomials involving a single variable, e.g. $5x^5 + 4x^3 - 2$. At heart, polynomials are functions that involve only the operations of multiplication and addition. Hence we can define multivariable polynomials. For instance,

$$3x_1^2 x_2 x_4^3 - 2x_2 x_3 + x_5^4$$

is a polynomial in the variables $x_1, x_2, \ldots, x_5$. We say it has degree 6, the degree of a polynomial being the maximum number of variables present in a single term.

Hint
Use the fact that

$$x = q_0 + \frac{1}{y},$$

where

$$y = q_1 + \frac{1}{q_2+}\frac{1}{q_3}$$

We now focus on the numerator and denominator of the general finite continued fraction:

$$q_0 + \frac{1}{q_1+}\frac{1}{q_2+}\cdots\frac{1}{q_t} \qquad (1.j)$$

(1.7) Worked Example. Transform the expression labelled (1.j) into a quotient of two polynomials in the q_i's for the cases $t = 0, 1, 2$.

Solutions. For $t = 0$, we get $q_0/1$.
For $t = 1$, it becomes

$$q_0 + \frac{1}{q_1} = \frac{q_0 q_1 + 1}{q_1}.$$

For $t = 2$, we obtain

$$q_0 + \cfrac{1}{q_1 + \cfrac{1}{q_2}} = q_0 + \frac{q_2}{q_1 q_2 + 1} = \frac{q_0 q_1 q_2 + q_0 + q_2}{q_1 q_2 + 1}$$

(1.8) Question on this Example. Use the worked example to transform

$$x = q_0 + \frac{1}{q_1+}\frac{1}{q_2+}\frac{1}{q_3}$$

into a similar form.

Answer to (1.8)

Harry Davenport
Our treatment of continued fractions, as well as the notation, closely follows that used by H. Davenport in his elegant introduction to Number Theory called *Higher Arithmetic* published by Cambridge University Press (copies are on sale in the University Bookshop).

Notation It is clear from (1.7) and (1.8) that the CF

$$q_0 + \frac{1}{q_1+} \cdots \frac{1}{q_t}$$

can be written as the quotient of two polynomial expressions in the variables $q_0, q_1, \ldots, q_t$. Denote the *numerator* of this quotient by

$$[q_0, q_1, \cdots, q_t]$$

Thus, from (1.7), we see that $[q_0] = q_0$, $[q_0, q_1] = q_0q_1+1$, and $[q_0, q_1, q_2] = q_0q_1q_2+q_0+q_2$. Your calculation in (1.8) will provide the formula for $[q_0, q_1, q_2, q_3]$. Observe that for $t = 1, 2$ in the Worked Example, the *denominator* of the quotient for

$$q_0 + \frac{1}{q_1+} \cdots \frac{1}{q_t}$$

has the form $[q_1, q_2, \cdots, q_t]$. Check this out in your calculation for $t = 3$. We can prove this in general and give a useful recursion formula for $[q_0, \cdots, q_t]$ in the following result.

Convention
We define the "empty bracket" expression $[\,]$ to be 1 to make sense of the formula in Part (b) when $t = 0$.
The first terms of the sequence given by applying the recurrence with $t = 1$ and $t = 2$ are

$$[q_0, q_1] = q_0[q_1] + [\,] = q_0q_1 + 1$$

and $[q_0, q_1, q_2] =$

$$\begin{aligned} &q_0[q_1, q_2] + [q_2] \\ &= q_0(q_1q_2 + 1) + q_2 \\ &= q_0q_1q_2 + q_0 + q_2. \end{aligned}$$

(1.9) Proposition *Let* $[q_0, q_1, \cdots, q_t]$ *denote the numerator when the CF*

$$q_0 + \frac{1}{q_1+} \cdots \frac{1}{q_t}$$

is simplified as a quotient of two polynomials in variables $q_0, q_1, \cdots, q_t$. *Then*

(a) $[q_0, q_1, \cdots, q_t]$ *is defined recursively by the initial conditions* $[\,] = 1$, $[q_0] = q_0$, *and the recursion formula*

$$[q_0, q_1, \cdots, q_t] = q_0[q_1, \cdots, q_t] + [q_2, \cdots, q_t],$$

for $t \geq 1$.

(b) $q_0 + \dfrac{1}{q_1+} \cdots \dfrac{1}{q_t} = \dfrac{[q_0, \cdots, q_t]}{[q_1, \cdots, q_t]}$.

for $t = 0, 1, 2, \ldots$.

Proof. We prove statements (a) and (b) simultaneously by complete induction on the number of terms in the square bracket; both statements certainly hold for one term (see the convention in the left-hand margin). Assume true for brackets with up to k terms for i some $k \geq 1$. Let

$$x = q_0 + \frac{1}{q_1+} \cdots \frac{1}{q_k}$$

and

$$y = q_1 + \frac{1}{q_2+} \cdots \frac{1}{q_k}$$

and observe that $x = q_0 + 1/y$. By induction,

$$y = \frac{[q_1, \cdots, q_k]}{[q_2, \cdots, q_k]}$$

and therefore

$$\begin{aligned} x &= q_0 + \frac{[q_2, \cdots, q_k]}{[q_1, \cdots, q_k]} \\ &= \frac{q_0[q_1, \cdots, q_k] + [q_2, \cdots, q_k]}{[q_1, \cdots, q_k]}. \end{aligned}$$

Thus the numerator of x is given by the recursion formula in Part (a) and the denominator has the form asserted in Part (b). This completes the induction step. Hence, by induction, the Proposition is true for all $t \geq 0$. □

(1.10) Question on $[q_0, q_1, \cdots, q_t]$. Using your answer to (1.8) evaluate $[1, 2, 3, 4]$ and $[2, 3, 4]$. Compare the fraction

$$\frac{[1, 2, 3, 4]}{[2, 3, 4]}$$

with your answer to (1.2)(a).

Answer to (1.10)

You may have noticed that the recursion giving the successive numerators and denominators is similar to the recursive construction of the sequences denoted (u) and (v) in Workbook 1. This is no coincidence!

(1.11) Worked Example for Euler's rule.

(a) Given variables q_0, q_1, q_2, q_3, q_4 form the following sum:

(i) The product of all the variables written in order +

(ii) all terms obtained by deleting a pair of consecutive variables $q_i q_{i+1}$ from this product +

(iii) all terms that can be obtained by deleting 2 distinct pairs of consecutive variables from the product.

(You should get one term from (i), four terms from (ii), and three terms from (iii).)

(b) Use (1.7) and (1.8) together with the recursion formula in Proposition 1.9(a) to compute $[q_0, q_1, q_2, q_3, q_4]$.

(c) Compare your answers in ((a)) and ((b)).

Solution.

(a)(i) $q_0 q_1 q_2 q_3 q_4$.

(ii) By deleting from this product first $q_0 q_1$, then $q_1 q_2$, $q_2 q_3$, and $q_3 q_4$ in turn, we get the contribution

$$q_2 q_3 q_4 + q_0 q_3 q_4 + q_0 q_1 q_4 + q_0 q_1 q_2$$

from (ii).

(iii) There are only three ways of removing two disjoint pairs from the product in (i); for example, deleting $q_0 q_1$ and $q_3 q_4$ leaves q_2. Hence the contribution from (ii) is $q_0 + q_2 + q_4$.

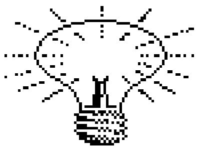
Now adding together the contributions from (i), (ii) and (iii) gives the desired polynomial with 8 terms.

Continued. . .

(b) Using our calculations in (1.7) and (1.8) with the recursion formula of (1.9)(a) gives

$$\begin{aligned}&[q_0, q_1, q_2, q_3, q_4]\\ &= q_4[q_0, q_1, q_2, q_3] + [q_0, q_1, q_2]\\ &= q_4(q_0q_1q_2q_3 + q_0q_1 + q_0q_3 + q_2q_3 + 1)\\ &\quad + (q_0q_1q_2 + q_0 + q_2)\\ &= q_0q_1q_2q_3q_4 + q_0q_1q_4 + q_0q_3q_4 + q_2q_3q_4\\ &\quad + q_0q_1q_2 + q_0 + q_2 + q_4\end{aligned}$$

which is exactly the same as the polynomial we found in ((a)).



Why does Euler's rule work?
The rule follows easily by induction using the recursion formula

$$[q_0, q_1, \cdots, q_t] = q_0[q_1, \cdots, q_t] + [q_2, \cdots, q_t]\,.$$

The terms coming from $[q_2, \cdots, q_t]$ correspond to those from which the pair q_0q_1 is omitted, while contributed by $q_0[q_1, \cdots, q_t]$ are precisely those from which the pair q_0q_1 is not omitted.

(1.12) Euler's Rule. The polynomial $[q_0, q_1, \cdots, q_t]$ is the sum of the following terms: the product $q_0q_1 \cdots q_t$ together with all terms obtained by deleting a consecutive pair q_iq_{i+1} from this product, next all terms obtained by omitting two consecutive pairs, then likewise three consecutive pairs, and so on, always provided that when $t + 1$ is even the final term obtained by deleting $\frac{t+1}{2}$ consecutive pairs (i.e. all the variables) is by convention set equal to 1.

(1.13) Question using Euler's rule. Use Euler's rule to calculate $[q_0, q_1, \ldots, q_5]$ and $[q_5, q_4, \ldots, q_0]$. Compare your answers. (Bear in mind that $q_iq_j = q_jq_i$.)

Answer to (1.13)

New Recursion Formula
The alternative version of the recursion formula given in (1.14)(b) is more useful than the original version of (1.9)(a). This is because we usually tinker with the end rather than the beginning when manipulating CFs.

(1.14) Corollary to Euler's Rule. *For all* $t \geq 0$*,*

(a) $[q_0, q_1, \ldots, q_t] = [q_t, q_{t-1}, \ldots, q_0]$

(b)
$$[q_0, q_1, \ldots, q_t] = q_t[q_0, q_1, \ldots, q_{t-1}] + [q_0, q_1, \ldots, q_{t-2}]$$

Proof.

(a) Since $q_i q_{i+1} = q_{i+1} q_i$ for all i, it is clear that Euler's rule applied to the product $q_t q_{t-1} \cdots q_1 q_0$ yields the same set of terms $q_{i_1} q_{i_2} \cdots q_{i_r}$ as when they are applied to its mirror image $q_0 q_1 \cdots q_{t-1} q_t$ since these rules are left-right symmetric.

(b) First relabel the variables in the recursion formula of (1.14)(b) according to the permutation $q_0 \to q_t,\ q_1 \to q_{t-1}, \ldots, q_t \to q_0$, and then apply Part (a).

□

Summary of Section 1

The most important result in this section is the one-to-one correspondence (or bijection) between the set of rational numbers a/b and the set of finite continued fractions of the form

$$q_0 + \frac{1}{q_1+}\frac{1}{q_2+}\cdots\frac{1}{q_t}$$

where $q_0 \in \mathbb{Z}$, $q_1, q_2, \ldots, q_t \in \mathbb{N}$, and $q_t \geq 2$.

- Given the rational number a/b (with $a \in \mathbb{Z}$ and $b \in \mathbb{N}$), the partial quotients in the corresponding continued fraction are precisely the numbers in the quotient sequence obtained when applying the Euclidean algorithm to a and b; thus they are the numbers q_i appearing in Equations (1.a).

- Now given the continued fraction above, the key to rewriting it as a rational number lies in the polynomials $[q_0, q_1, \ldots, q_t]$, defined recursively by the initial conditions

  $$[\,] = 1, \quad [q_0] = q_0, \quad [q_0, q_1] = q_0 q_1 + 1$$

  and the recursion formula

  $$[q_0, q_1, \ldots, q_t] = q_0[q_1, q_2, \ldots, q_t] + [q_2, q_3, \ldots, q_t]$$

  for $t \geq 2$. You get the rational number represented by the continued fraction

  $$q_0 + \frac{1}{q_1+}\cdots\frac{1}{q_t}$$

  by substituting the values of q_i in the formula

  $$\frac{[q_0, q_1, \ldots, q_t]}{[q_1, q_2, \ldots, q_t]}.$$

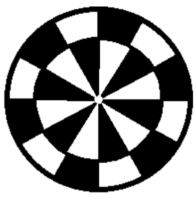
  Finally, Euler's rule gives a direct way of writing down the polynomial $[q_0, q_1, \ldots, q_t]$ without recourse to the recursion formula.

2 Infinite Continued Fractions for Irrational Numbers

Section Targets In Section 1 we described a one-to-one correspondence between rational numbers and finite continued fractions. Our main objective in this section is to do the same for *irrational* numbers and *infinite* continued fractions. In addition to the machinery of Section 1, we will need some basic facts about convergent sequences: an irrational number will be represented as the limit of an infinite sequence of rational numbers

$$\frac{[q_0, q_1, \ldots, q_n]}{[q_1, q_2, \ldots, q_n]} = q_0 + \frac{1}{q_1+}\frac{1}{q_2+}\cdots\frac{1}{q_n}$$

constructed in turn out of an infinite sequence $q_0, q_1, q_2, q_3, \ldots$ of partial quotients q_i with $q_0 \in \mathbb{Z}$ and $q_i \in \mathbb{N}$ when $i \geq 1$. (This is analogous to the definition of an infinite decimal $b \cdot a_1 a_2 a_3 \ldots$ as the limit of the sequences of rational numbers $b \cdot a_1 a_2 \ldots a_n$).



(2.1) Some Notation and Terminology. Given an integer q_0 and an infinite sequence of natural numbers $q_1, q_2, \ldots$ we will fix the notation

$$A_n = [q_0, q_1, \ldots, q_n] \quad \text{and} \quad B_n = [q_1, q_2, \ldots, q_n]$$

for $n = 0, 1, 2, \ldots$. The ratio A_n/B_n is called the nth *convergent* of the sequence. Since $A_n \in \mathbb{Z}$ and $B_n \in \mathbb{N}$, the convergents $A_0/B_0,\ A_1/B_1,\ A_2/B_2, \ldots$ are all rational numbers. Later we will see that A_n and B_n are always coprime, so the rationals A_n/B_n are automatically in lowest terms.

Dual View

We will retain the options of regarding A_n and B_n as polynomials in variables $q_0, q_1, \ldots, q_n$ and of allowing the q_i's to take values in $\mathbb{R}$; however, when we substitute actual values for $q_0 \in \mathbb{Z}$ and $q_1, q_2, \ldots \in \mathbb{N}$, then evidently $A_n \in \mathbb{Z}$ and $B_n \in \mathbb{N}$.

(2.2) Questions about Convergence.

(a) Recall that $A_0/B_0 = q_0/1$. Write out A_1/B_1 and A_2/B_2 as ratios of polynomials in the q_i's.

(b) Which result in Section 1 gives rise to the two recursion formulas,

$$A_n = q_n A_{n-1} + A_{n-2} \tag{2.a}$$
$$B_n = q_n B_{n-1} + B_{n-2} \tag{2.b}$$

for $n \geq 2$?

(c) If A_{n-2}/B_{n-2} and A_{n-1}/B_{n-1} are two consecutive convergents, the next convergent A_n/B_n is therefore

$$\frac{q_n A_{n-1} + A_{n-2}}{q_n B_{n-1} + B_{n-2}}. \tag{2.c}$$

The first two convergents corresponding to the sequence $3, 7, 15, 1, 292, \ldots$ are

$$\frac{A_0}{B_0} = \frac{3}{1}, \quad \frac{A_1}{B_1} = \frac{q_0 q_1 + 1}{q_1} = \frac{22}{7}$$

Use (2.c) repeatedly to work out the next three convergents, and express the rationals A_3/B_3 and A_4/B_4 as decimals. Guess which real number has an infinite continued fraction beginning

$$3 + \frac{1}{7+}\frac{1}{15+}\frac{1}{1+}\frac{1}{292+}\cdots$$

Answers to (2.2)

(a)

(b)

(c)

continued...

(2.3) Extend the sequences (A_n) and (B_n) backwards to $n = -1$ and $n = -2$, by setting $A_{-2} = 0$, $A_{-1} = 1$, $B_{-2} = 1$, $B_{-1} = 0$. Now use (2.a) and (2.b) with $n = 0$ and $n = 1$ to (re)compute values for A_0, A_1 and B_0, B_1. Are they correct?

Notice that the numerator and denominator sequences (A_n) and (B_n) satisfy the *same* recurrence since (2.a) and (2.b) are the same, but they start with *different* initial values: $A_0 = q_0$ and $A_1 = q_0 q_1 + 1$ while $B_0 = 1$ and $B_1 = q_1$.

Answer to (2.3)

EEA revisited

Compare with the Extended Euclidean Algorithm (EEA) layout from Workbook 1 Section 4: the sequence we are denoting here by (A_n) is exactly the sequence (v) we had there, and similarly $(B_n) = (u)$.

We will stick to the new names (A_n), (B_n) for the sequences here, but you will probably find that the columnar layout you used in WB1§4 is helpful for evaluating them.

Hint
Use the tabular layout, with columns labelled (q), $(B) = (u)$ and $(A) = (v)$.

(2.4) Spot the Numbers. Work out the early convergents A_n/B_n for the sequences

$$1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, \ldots$$

and

$$2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \ldots,$$

expressing each as a rational number and then as a decimal. Any ideas? Let the inspiration strike!

Answers to (2.4)

(2.5) Questions about $A_nB_{n-1} - B_nA_{n-1}$.

(a) Regarding A_0, A_1, B_0, B_1 as polynomials in q_0 and q_1, work out and simplify the polynomial

$$A_1B_0 - B_1A_0\,.$$

(b) Let $q_0 = 2,\ q_1 = 1,\ q_2 = 2,\ q_3 = 1$. In general, set $q_n = 1$ if n is odd and $q_n = 2$ if n is even. Calculate the value of the expression

$$A_nB_{n-1} - B_nA_{n-1}$$

for $n = 2, 3, 4$ and 5.

(c) Make a conjecture about the value of $A_nB_{n-1} - B_nA_{n-1}$.

Answers to (2.5)

(a)

(b)

(c)

We will now prove what we hope is your conjecture. Let

$$D_n = A_n B_{n-1} - B_n A_{n-1}$$

and observe that $D_1 = 1$ by Part (a) of the preceding question. Assume inductively that $D_k = (-1)^{k+1}$. Then, using Equations (2.a) and (2.b), we obtain

$$\begin{aligned}
D_{k+1} &= A_{k+1}B_k - B_{k+1}A_k \\
&= (q_{k+1}A_k + A_{k-1})B_k - (q_{k+1}B_k + B_{k-1})A_k \\
&= A_{k-1}B_k - B_{k-1}A_k = -D_k \\
&= -(-1)^{k+1} = (-1)^{(k+1)+1}
\end{aligned}$$

Thus the induction hypothesis holds for $k+1$ if it holds for k, and since it holds for $k = 1$, we have proved the following result by induction.

(2.6) Proposition *If A_n/B_n denotes the nth convergent associated with a sequence $q_0, q_1, q_2, \ldots$, then*

$$A_n B_{n-1} - B_n A_{n-1} = (-1)^{n+1} \qquad (2.d)$$

for all $n \in \mathbb{N}$.

(2.7) Question on the sequence $B_1, B_2, \ldots$
Let $q_i \in \mathbb{N}$ for $i \geq 1$. Prove that $\{B_n\}_{n=1}^{\infty}$ is a strictly increasing sequence of natural numbers, and conclude that $B_n \to \infty$ as $n \to \infty$.

Answer to (2.7)

(2.8) Corollary

(a) *For all* $n \in \mathbb{N}$,

$$\frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}} = \frac{(-1)^{n+1}}{B_{n-1}B_n}.$$

(b) *The even convergents,*

$$\frac{A_0}{B_0}, \frac{A_2}{B_2}, \frac{A_4}{B_4}, \ldots$$

form a strictly increasing sequence of rational numbers.

(c) *The sequence of odd convergents*

$$\frac{A_1}{B_1}, \frac{A_3}{B_3}, \frac{A_5}{B_5}, \ldots$$

is strictly decreasing.

Proof.

(a) Divide Equation (2.d) by $B_{n-1}B_n$.

(b) By Part (a)

$$\begin{aligned} \frac{A_{2n}}{B_{2n}} - \frac{A_{2n-2}}{B_{2n-2}} &= \frac{(-1)^{2n}}{B_{2n-2}B_{2n-1}} + \frac{(-1)^{2n+1}}{B_{2n-1}B_{2n}} \\ &= \frac{1}{B_{2n-1}}\left(\frac{1}{B_{2n-2}} - \frac{1}{B_{2n}}\right) > 0 \end{aligned}$$

by your work on Question 2.7.

(c) A similar argument proves the statement about the decreasing odd convergents. □

If we now substitute $n = 2m + 1$ in (2.8)(a) and apply (2.8)(c), we obtain

$$\frac{A_{2m}}{B_{2m}} = \frac{A_{2m+1}}{B_{2m+1}} - \frac{1}{B_{2m}B_{2m+1}} < \frac{A_{2m+1}}{B_{2m+1}} < \frac{A_1}{B_1}$$

Thus the sequence of even convergents is increasing and bounded above by A_1/B_1, and therefore converges to a limit, l say. Similarly the sequence of odd convergents is decreasing and bounded below, and also tends to a limit, L say. Since

$$\left|\frac{A_{2n+1}}{B_{2n+1}} - \frac{A_{2n}}{B_{2n}}\right| = \frac{1}{B_{2n+1}B_{2n}} \longrightarrow 0 \text{ as } n \to \infty$$

it follows that $L = l$. We have therefore proved the following theorem.

Notation
We will denote the limit α of the sequence A_n/B_n by the infinite continued fraction,

$$q_0 + \frac{1}{q_1+}\frac{1}{q_2+}\cdots$$

(2.9) Theorem *Let $q_0, q_1, q_2, \ldots$ be a sequence with $q_0 \in \mathbb{Z}$ and $q_i \in \mathbb{N}$ for $i = 1, 2, \ldots$. Then the sequence of rational numbers*

$$\frac{A_n}{B_n} = q_0 + \frac{1}{q_1+}\frac{1}{q_2+}\cdots\frac{1}{q_n}$$

converges to a limit α.

Our Goal
Keep our section target in mind. We are looking for a bijection between irrational numbers and infinite sequences $q_0, q_1, q_2, \ldots$ defining infinite continued fractions.

So far we have shown how to obtain a real number

$$\lim_{n\to\infty} \frac{A_n}{B_n}$$

from an infinite sequence $q_0, q_1, q_2, \ldots$ of partial quotients q_i. We will now focus on the reverse question:

Starting with a real number α, how can we find a continued fraction which represents it?

Of course, when α is rational, we have already found the answer in Section 1. If $\alpha = a/b$, we obtain a *finite* continued fraction representation for α of the form

$$q_0 + \frac{1}{q_1+}\cdots\frac{1}{q_t}$$

where the q_is are the partial quotients that appear when the Euclidean algorithm is applied to the pair a and b.
To find a clue for dealing with the case where α is irrational, look at two consecutive steps in the EA:

$$r_{s-1} = q_s r_s + r_{s+1} \quad (2.e)$$
$$r_s = q_{s+1} r_{s+1} + r_{s+2} \quad (2.f)$$

with $r_{s+1}, r_{s+2} > 0$. Dividing (2.e) by r_s and (2.f) by r_{s+1}, we obtain

$$\frac{r_{s-1}}{r_s} = q_s + \frac{r_{s+1}}{r_s}, \quad \text{and}$$
$$\frac{r_s}{r_{s+1}} = q_{s+1} + \frac{r_{s+2}}{r_{s+1}}.$$

Let's write $\alpha_s = r_{s-1}/r_s$ and $\alpha_{s+1} = r_s/r_{s+1}$. Since $r_{s-1} > r_s > r_{s+1} > 0$. it follows that q_s is the integral part, and r_{s+1}/r_s the fractional part of α_s; furthermore α_{s+1} is the inverse of this fractional part.

Fractional Part
Since $q_0 = [\alpha]$ and α is irrational, the fractional part $\alpha - q_0$ lies in the open interval $(0, 1)$, and so

$$\alpha_1 = \frac{1}{\alpha - q_0}$$

is greater than 1.

We can proceed in exactly the same way when α is irrational. Let $q_0 = [\alpha] \in \mathbb{Z}$, and write

$$\alpha = q_0 + \frac{1}{\alpha_1} \quad \text{with } \alpha_1 > 1\,.$$

If α_1 were rational, then $q_0 + 1/\alpha_1 (= \alpha)$ would also be rational, which we have assumed not to be the case. If we define α_2 to be the inverse of the fractional part of α_1, we can write

$$\begin{aligned} \alpha_1 &= q_1 + \frac{1}{\alpha_2} \quad \text{with } \alpha_2 > 1, \\ \alpha_2 &= q_2 + \frac{1}{\alpha_3} \quad \text{with } \alpha_3 > 1 \end{aligned}$$

and so on.

After $n + 1$ steps we therefore obtain

$$\alpha = q_0 + \frac{1}{q_1 +} \cdots \frac{1}{q_n +} \frac{1}{\alpha_{n+1}} \tag{2.g}$$

Foresight!
At this point, it is clear why we insisted earlier that the partial quotients appearing in continued fractions should be allowed to take any real values. Here the final term $\frac{1}{\alpha_{n+1}}$ is irrational.

All the computations we carried out in Section 1 are still valid for Equations (2.g), and in particular

$$\begin{aligned} \alpha &= \frac{[q_0, q_1, \ldots, q_n, \alpha_{n+1}]}{[q_1, q_2, \ldots, q_n, \alpha_{n+1}]} \\ &= \frac{\alpha_{n+1} A_n + A_{n-1}}{\alpha_{n+1} B_n + B_{n-1}} \end{aligned} \tag{2.h}$$

where A_n/B_n is the nth convergent defined earlier in terms of $q_0, q_1, \ldots, q_n$.

Hint
Appeal to Proposition 2.6.

(2.10) Question on Equation (2.h). Deduce from (2.h) that

$$\left| \alpha - \frac{A_n}{B_n} \right| < \frac{1}{B_n B_{n+1}}\,. \tag{2.i}$$

Answer to (2.10)

Since $B_n \to \infty$, it follows from the inequality (2.i) that the following result is true.

(2.11) Theorem *Let $q_0 \in \mathbb{Z}$ and $q_i \in \mathbb{N}$ for $i \geq 1$. The infinite sequence $q_0, q_1, q_2, \ldots$ defined recursively by:*

$$\begin{aligned} q_0 &= [\alpha] \\ q_1 &= [\alpha_1] \text{ where } \alpha_1 = \frac{1}{\alpha - q_0} \end{aligned}$$

and

$$q_n = [\alpha_n] \text{ where } \alpha_n = \frac{1}{\alpha_{n-1} - q_{n-1}}$$

for $n \geq 2$, yields an infinite continued fraction

$$q_0 + \frac{1}{q_1+}\frac{1}{q_2+}\cdots$$

which converges to α.

In Theorem 2.9 we saw how to give a precise meaning to

$$q_0 + \frac{1}{q_1+}\frac{1}{q_2+}\cdots$$

for *any* choice of $q_0 \in \mathbb{Z}$ and $q_i \in \mathbb{N}$ when $i \geq 1$. It is defined to be the real number α that is the limit of the convergents A_n/B_n. If we now take this α and develop it as a continued fraction in the way just described in Theorem 2.11, we get back exactly the same sequence $q_0, q_1, q_2, \ldots$ of partial quotients, which is the answer to our best hopes. To see why, first observe that the infinite continued fraction

$$\beta = \frac{1}{q_1+}\frac{1}{q_2+}\cdots$$

lies strictly between 0 and 1; this is because it has the form

$$\frac{1}{q_1 + \dfrac{1}{\gamma}},$$

where

$$\gamma = \frac{1}{q_2+}\frac{1}{q_3+}\cdots$$

which is the limit of convergents

$$\frac{1}{q_2+}\frac{1}{q_3+}\cdots\frac{1}{q_n}.$$

Since these convergents lie between 0 and 1, we have $\gamma \in [0, 1]$, and therefore $\beta \in (0, 1)$. Since $\alpha = q_0 + \beta$, the upshot is that $q_0 = [\alpha]$. Next, when we write $\alpha = q_0 + 1/\alpha_1$, we obtain

$$\alpha_1 = q_1 + \frac{1}{q_2+}\frac{1}{q_3+}\cdots,$$

and by the same argument $q_1 = [\alpha_1]$. Similar reasoning gives $q_2 = [\alpha_2]$, $q_3 = [\alpha_3]$, etc., and proves our contention that the two maps

$$\text{irrational } \alpha \quad \stackrel{(2.11)}{\longrightarrow} \quad q_0 + \frac{1}{q_1+}\frac{1}{q_2+}\cdots$$

and

$$\text{infinite CF } q_0 + \frac{1}{q_1+}\frac{1}{q_2+}\cdots \quad \stackrel{(2.9)}{\longrightarrow} \quad \alpha \in \mathbb{R}$$

are mutually inverse. We have therefore found the one-to-one correspondence sought at the outset.

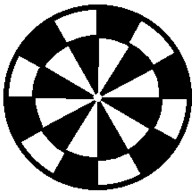
Summary of Section 2

In Sections 1 and 2 we have constructed two bijections:

$$\begin{aligned} \mathbb{Q} &\longleftrightarrow \text{finite CFs} \\ \mathbb{R}\backslash\mathbb{Q} &\longleftrightarrow \text{infinite CFs}\,. \end{aligned}$$

Taken together, these guarantee that *every* real number has a unique representation as either a finite or an infinite continued fraction.

3 Periodic Continued Fractions



Section Targets. We learn in *Analysis* in the first term at Warwick that the rational numbers are characterised by the property of having recurring decimal representations (if you regard a terminating decimal as having recurring zeros). In Section 1 we saw that the rationals are also characterised by the property of having *finite* continued fraction representations. In this section we ask the question:

Which real numbers have recurring continued fractions?

We will eventually give a simple answer to this question, but will approach the question indirectly by first describing exactly which numbers have purely *periodic* CF representations, that is to say, periodic *with no delay.* The description is not an obvious one, but is intriguing nonetheless.

CF Lingo
The correct term for a *recurring* continued fraction is *periodic.*

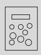
(3.1) Worked Example. Develop the irrational number $\sqrt{3}$ as an infinite continued fraction.

Solution: Set $\alpha = \sqrt{3} \approx 1 \cdot 7$ and so $[\alpha] = 1$. Thus $q_0 = 1$ and

$$\alpha = 1 + \frac{1}{\alpha_1}\,.$$

It follows that

$$\begin{aligned} \alpha_1 &= \frac{1}{\sqrt{3}-1} \\ &= \frac{\sqrt{3}+1}{(\sqrt{3}-1)(\sqrt{3}+1)} \\ &= \frac{\sqrt{3}+1}{2} \approx 1 \cdot 4\,. \end{aligned}$$



Use your calculator
When calculating the continued fractions for an irrational number α, use your pocket calculator to work out the integral part q_i of α_i. By the way, the irrational numbers, $\alpha_0, \alpha_1, \alpha_2, \ldots$, defined by

$$\begin{aligned} \alpha_0 &= \alpha\,, \quad \text{and} \\ \alpha_{i+1} &= \frac{1}{\alpha_i - [\alpha_i]} \end{aligned}$$

for $i = 0, 1, 2, \ldots$ are usually called the *complete quotients.*

We'll show an easier way to do this below: finding the CF for an irrational of the form $\sqrt{D}$ can easily be done without using a calculator at all!

Continued. . . Therefore $q_1 = [\alpha_1] = 1$ and

$$\begin{aligned}\alpha_2 &= \frac{1}{\alpha_1 - q_1} = \frac{2}{\sqrt{3}-1} \\ &= \frac{2(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)} \\ &= \sqrt{3}+1 \approx 2\cdot 7\,.\end{aligned}$$

Hence $q_2 = [\alpha_2] = 2$ and

$$\alpha_3 = \frac{1}{\alpha_2 - 2} = \frac{1}{\sqrt{3}-1} = \alpha_1\,.$$

Since α_3 is the same as α_1, the calculations now repeat themselves, and so

$$\begin{aligned}\sqrt{3} &= q_0 + \frac{1}{q_1+}\frac{1}{q_2+}\frac{1}{q_1+}\frac{1}{q_2+}\cdots \\ &= 1 + \frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{2+}\cdots\end{aligned}$$

Warning

There is no standard notation for periodic CFs. A variety of different symbols are found in standard textbooks. For example

$$[1, \dot{1}, \dot{2}]$$

is used by Hardy and Wright,

$$< 1, 1, 2, 1, 2, \cdots >$$

is used by Niven, Zuckerman, and Montgomery to denote the CF for $\sqrt{3}$.

New Notation. We introduce a new notation to handle periodic continued fractions. In this notation, the continued fraction for $\sqrt{3}$ described in (1.3) becomes $1, \overline{1, 2}$. A general periodic continued fraction has partial quotients of the form

$$d_0, d_1, \ldots, d_m, q_0, q_1, \ldots, q_n, q_0, \ldots$$

where $d_m \neq q_n$ and $q_0, \ldots, q_n$ is the shortest repeating cycle. It is denoted by

$$d_0, d_1, \ldots, d_m, \overline{q_0, q_1, \ldots, q_n} \tag{3.a}$$

where the bar stretches over the periodic cycle.

You may think that a calculator is needed for fact (2), but it is not! Since

$$q_0^2 < D < (q_0+1)^2,$$

all you need to find q_0 is to see which perfect squares D lies between.

Simpler layout. For irrational numbers of the form $\alpha = \sqrt{D}$ we can work out the CF expansion simply and *without* any use of calculators, using only the facts (1) that $\alpha^2 = D$ and (2) that $q_0 < \alpha < q_0 + 1$ where $q_0 = [\sqrt{D}]$.
Take $\alpha = \sqrt{3}$ again. Since $1 < 3 < 4$ we have $q_0 = [\sqrt{3}] = 1$. Now

$$\begin{aligned} \sqrt{3} &= \mathbf{1} + (\sqrt{3}-1) \\ \frac{1}{\sqrt{3}-1} = \frac{\sqrt{3}+1}{2} &= \mathbf{1} + \frac{\sqrt{3}-1}{2} \\ \frac{2}{\sqrt{3}-1} = \frac{\sqrt{3}+1}{1} &= \mathbf{2} + (\sqrt{3}-1) \end{aligned}$$

The bold numbers are the quotients q_n. To see how these are obtained, take the second line for example: since $\sqrt{3}$ is between 1 and 2, it follows that $\sqrt{3}+1$ is between 2 and 3, so $\frac{\sqrt{3}+1}{2}$ is between 1 and 2. We stop when the fractional part repeats that of the first line (in this case, $\sqrt{3}-1$ repeats).
So we see that $\sqrt{3} = 1, \overline{1, 2}$.

Best approximations

Since the sequence of convergents A_n/B_n converge to α, we can use CFs to find rational approximations to irrational numbers. This is called "Diophantine Approximation".

It is a remarkable fact that CF convergents are guaranteed to give the "best possible" rational approximations. (You may like to try to formulate a precise definition of what that should mean!) In the previous section we saw this in action with the CF expansions of e (which has a nice pattern, but is not periodic) and of π (whose CF expansion appears completely random). The well-known approximation 22/7 to π is one of its convergents; and the remarkably good approximation 355/113 is as good as it is since it comes from truncating the expansion just before a very large partial quotient, 292.

Finding convergents. Once we have the CF expansion, and in particular the first few partial quotients q_n, we can easily compute the convergents A_n/B_n using the formulas of the previous section. As always, the simplest method is to use the tabular method.
With $\alpha = \sqrt{3} = 1, \overline{1, 2}$ the calculation goes like this:

(q)	(B)	(A)	A/B
	1	0	
1	0	1	
1	1	1	1
2	1	2	2
1	3	5	5/3
2	4	7	7/4
⋮	11	19	19/11

So $A_4/B_4 = 19/11 = 1.7272\ldots$, compared with $\sqrt{3} = 1.7320\ldots$.

(3.2) Questions on CFs for irrationals. Find the continued fractions for the following (using the simpler layout):
(a) $\sqrt{2}$; (b) $3+\sqrt{10}$; (c) $\sqrt{37}$; (d) $\sqrt{13}$.
In each case, find the convergents A_n/B_n for n up to 4.

Answers to (3.2)

(a)

(b)

continued. . .

(c)

(d)

(3.3) Worked Example. Which irrational number α is represented by the periodic continued fraction $\overline{1, 2, 3}$? Show that it satisfies the equation

$$7x^2 - 8x - 3 = 0\,. \qquad (3.b)$$

Calculating Convergents
Recall that $A_0 = q_0, A_1 = q_0 q_1 + 1$, and

$$A_n = q_n A_{n-1} + A_{n-2}$$

for $n \geq 2$. Also, $B_0 = 1, B_1 = q_1$, and

$$B_n = q_n B_{n-1} + B_{n-2}$$

for $n \geq 2$.

Solution. Set

$$\begin{aligned} \alpha &= \overline{1, 2, 3} = 1 + \frac{1}{2+}\frac{1}{3+}\frac{1}{1+}\frac{1}{2+}\frac{1}{3+}\cdots \\ &= 1 + \frac{1}{2+}\frac{1}{3+}\frac{1}{\alpha} = \frac{[1, 2, 3, \alpha]}{[2, 3, \alpha]} \end{aligned}$$

by Equations (2.h). Furthermore, we have

$$\frac{[1, 2, 3, \alpha]}{[2, 3, \alpha]} = \frac{\alpha A_2 + A_1}{\alpha B_2 + B_1},$$

where A_n / B_n denotes the nth convergent of

$$1 + \frac{1}{2+}\frac{1}{3}$$

for $n = 0, 1, 2$. Since

$$A_0 = 1,\ A_1 = 3,\ A_2 = 10,$$

$$B_0 = 1,\ B_1 = 2,\ B_2 = 7,$$

it follows that

$$\alpha = \frac{10\alpha + 3}{7\alpha + 2}$$

whence $\alpha(7\alpha + 2) = 10\alpha + 3$, and therefore

$$7\alpha^2 - 8\alpha - 3 = 0.$$

This proves that α is a root of the quadratic equation labelled (3.b). Since $\alpha = 1 + 1/\alpha_1$, where

$$\alpha_1 = 2 + \frac{1}{3+}\frac{1}{1+}\cdots > 0,$$

we conclude that $\alpha > 1$. Since the roots of (3.b) are

$$\frac{8 \pm \sqrt{64 + 84}}{14} \approx 1 \cdot 4 \quad \text{and} \quad -0 \cdot 3$$

it follows that α is equal to the positive root

$$\frac{(8 + \sqrt{148})}{14}\,.$$

Conjugate Roots

The two roots of the quadratic equation

$$ax^2 + bx + c = 0$$

are

$$\frac{-b \pm \sqrt{D}}{2a},$$

where $D = b^2 - 4ac$ is called the *discriminant* of the equation. The two roots are called *conjugates* of each other. Any algebraic expression involving one of the roots can be transformed into the corresponding expression for its conjugate by replacing $\sqrt{D}$ by $-\sqrt{D}$ wherever it appears.

Symmetric Relationship

Notice that it is also true that $\frac{-1}{\alpha}$ is a root of (3.c).

(3.4) Question related to (3.3). Let β be the number with periodic CF $\overline{3,2,1}$, the reverse of the periodic sequence for α in the Worked Example.

(a) Show that β satisfies the quadratic equation

$$3x^2 - 8x - 7 = 0\,. \qquad (3.c)$$

(b) Find β.

(c) Show that $-1/\beta$ is the second (negative) root of Equation (3.b).

Answer to (3.4)

New Terms

(i) A *quadratic irrational* is a real irrational root of a quadratic equation with integer coefficients. Such numbers have the form

$$x + y\sqrt{D}$$

where $x, y \in \mathbb{Q}$ and D is a positive integer which is not a perfect square.

(ii) A *purely periodic* CF is one with no delay and so can be written

$$\overline{q_0, q_1, \ldots, q_n}$$

Two Observations.

(a) The examples of quadratic irrationals in (3.1) and (3.2) have periodic continued fractions (sometimes after a delay).

(b) The purely periodic CFs of (3.3) and (3.4)

- are quadratic irrationals
- are greater than 1
- possess conjugates lying in the interval $(-1, 0)$.

These turn out to be defining properties for purely periodic CFs. We begin by showing that a purely periodic continued fraction always represents a reduced quadratic irrational in the following sense.

(3.5) Definition. Let α be a quadratic irrational number. We will call α *reduced* if

(a) $\alpha > 1$, and

(b) its conjugate α' satisfies $-1 < \alpha' < 0$.

Periodic CFs

Since periodic continued fractions are infinite, we know they are irrational by our work in Sections 1 and 2.

Let $\overline{q_0, q_1, \ldots, q_n}$ be a purely periodic continued fraction representing α, say. The periodicity means that

$$\begin{aligned} \alpha &= q_0 + \frac{1}{q_1+} \cdots \frac{1}{q_n+} \frac{1}{\alpha} \\ &= \frac{[q_0, q_1, \ldots, q_n, \alpha]}{[q_1, q_2, \ldots, q_n, \alpha]} \\ &= \frac{\alpha A_n + A_{n-1}}{\alpha B_n + B_{n-1}} \end{aligned} \tag{3.d}$$

where

$$A_n = [q_0, q_1, \ldots, q_n], \; A_{n-1} = [q_0, q_1, \ldots, q_{n-1}],$$
$$B_n = [q_1, q_2, \ldots, q_n], B_{n-1} = [q_1, q_2, \ldots, q_{n-1}].$$

We want to compare α with another irrational number β represented by the purely periodic CF

$$\overline{q_n, q_{n-1}, \ldots, q_1, q_0}$$

obtained by reversing the order of the periodic sequence for α.

Hint
Use the fact that

$$[q_0, q_1, \ldots, q_n] = [q_n, q_{n-1}, \ldots, q_1, q_0]$$

proved in Corollary 1.14.

(3.6) Questions about β. By the argument used above,

$$\beta = \frac{[q_n, q_{n-1}, \ldots, q_1, q_0, \beta]}{[q_{n-1}, q_{n-2}, \ldots, q_1, q_0, \beta]}.$$

Show that

$$\beta = \frac{\beta A_n + B_n}{\beta A_{n-1} + B_{n-1}}. \tag{3.e}$$

Answer to (3.6)

By (3.d) we have

$$\alpha(\alpha B_n + B_{n-1}) = \alpha A_n + A_{n-1}$$

and so

$$B_n \alpha^2 + (B_{n-1} - A_n)\alpha - A_{n-1} = 0.$$

Therefore α is a root of the quadratic equation

$$B_n x^2 + (B_{n-1} - A_n)x - A_{n-1} = 0 \tag{3.f}$$

From (3.e) we similarly obtain

$$A_{n-1}\beta^2 + (B_{n-1} - A_n)\beta - B_n = 0 \tag{3.g}$$

and substituting $x = -1/\beta$ in (3.f) yields

$$\begin{aligned} & B_n\beta^{-2} - (B_{n-1} - A_n)\beta^{-1} - A_{n-1} \\ &= -\beta^{-2}(A_{n-1}\beta^2 + (B_{n-1} - A_n)\beta - B_n) \\ &= 0 \quad \text{by (3.g)}. \end{aligned}$$

Therefore $-1/\beta$ is also a root of (3.f). Since q_0 and q_n are natural numbers, we have $\alpha > 1$ and $\beta > 1$, and it follows that $-1 < -1/\beta < 0$.

Symmetry Again
Note also that $-1/\alpha$ is the conjugate of β.

Consequently α and $-1/\beta$ are the distinct (irrational) roots of (3.f), and so we can conclude that α is a reduced quadratic irrational. Thus we have proved the following.

(3.7) Proposition *A purely periodic continued fraction represents a reduced quadratic irrational number.*

RQI for short
Let's agree to abbreviate the term 'reduced quadratic irrational number' to RQI.

Our next target is the remarkable fact that every reduced quadratic irrational has a purely periodic continued fraction. The proof is a bit longer than the typical proofs we have met so far in this course, and it might be helpful to summarise the strategy of the proof before we get down to the nitty-gritty, which involves a certain amount of algebra that may obscure the wood from the trees. Until further notice therefore, α will denote a reduced quadratic irrational (RQI), that is to say a root of a quadratic equation with integer coefficients a, b and c:

$$ax^2 + bx + c = 0$$

satisfying:

$$\begin{array}{ll} (\mathbf{R1}) & \alpha > 1, \text{ and} \\ (\mathbf{R2}) & -1 < \alpha' < 0 \end{array}$$

where α' is the conjugate of α (i.e. the second root of the equation).

Simpler notation
The algebra will be more transparent if we work with the quantities B, C and D instead of a, b and c.

Thus, if we set,

- $D = b^2 - 4ac$
- $B = -b$ (3.h)
- $C = 2a$

we know that

$$\alpha = \frac{B + \sqrt{D}}{C} \text{ and } \alpha' = \frac{B - \sqrt{D}}{C} \qquad (3.\text{i})$$

where the integer D is positive and not a perfect square since α is assumed to be irrational.

Square Root Sign
We use the convention that $\sqrt{D}$ denotes the *positive* real number whose square is D.

The proof that α has a purely periodic CF goes like this:
Step 1: Express conditions R1 and R2 in terms of algebraic properties of B, C and D; in particular, show that B and C are natural numbers bounded above by $2\sqrt{D}$.
Step 2: Next begin the continued fraction development in the familiar way by writing,

$$\begin{aligned} \alpha &= q_0 + \frac{1}{\alpha} \quad (q_0 = [\alpha]) \\ \alpha_1 &= q_1 + \frac{1}{\alpha_2} \quad (q_1 = [\alpha_1]) \\ \vdots &\ \vdots \qquad \vdots \end{aligned}$$

and so on, to obtain the partial quotients $q_0, q_1, \ldots$ and the *complete quotients* $\alpha_0 (= \alpha), \alpha_1, \alpha_2, \ldots$. At any given stage we have the exact equation

$$\begin{aligned} \alpha &= q_0 + \frac{1}{q_1 +} \cdots \frac{1}{q_{n-1} +} \frac{1}{\alpha_n} \\ &= \frac{[q_0, \ldots, q_{n-1}, \alpha_n]}{[q_1, \ldots, q_{n-1}, \alpha_n]} . \end{aligned}$$

Terminology
Remember we call the (irrational) numbers $\alpha_0, \alpha_1, \ldots$ as the *complete quotients* of the CF development.

The key to this step is to show that each complete quotient α_i is again a reduced quadratic irrational with the same discriminant D as α.
Step 3: The upper bounds for the natural numbers B and C in terms of D proved in Step 1 imply that there are finitely many possibilities for the complete quotients $\alpha_0, \alpha_1, \alpha_2, \ldots$ associated with α, and so there must eventually be a repetition. The final step in the proof is to show that if $\alpha_m = \alpha_{m+r}$ for some $r \geq 1$ and $m > 0$, then $\alpha_{m-1} = \alpha_{m+r-1}$. Applying this as often as necessary leads to the conclusion that $\alpha_0 = \alpha_r$ and hence that α has a purely periodic continued fraction.

We now look at each step in detail. The first is straightforward algebraic manipulation involving Conditions R1 and R2 and equations (3.h) and (3.i).

(3.8) Questions for Step 1.

(a) Deduce from (3.h) that

$$\frac{B^2 - D}{C} = 2c \ (\in \mathbb{Z})$$

and conclude that

$$\textbf{(P1)} \qquad C \text{ divides } B^2 - D\,.$$

(b) From Conditions R1 and R2 deduce that $\alpha - \alpha' > 0$, and conclude that $C > 0$, in other words that

$$\textbf{(P2)} \qquad C \text{ is a natural number.}$$

(c) From Conditions R1 and R2, deduce that $\alpha + \alpha' > 0$, and now conclude that

$$\textbf{(P3)} \qquad B \text{ is a natural number.}$$

(d) Since $\alpha' < 0$, we have $B < \sqrt{D}$. Now use the fact that $\alpha > 1$ to deduce that $C < B + \sqrt{D}$ and conclude that

$$\textbf{(P4)} \qquad B < \sqrt{D} \text{ and } C < 2\sqrt{D}\,.$$

Answers to (3.8)

(a)

(b)

(c)

(d)

Quadratic Irrational
Equation (3.j) tells us that

$$(C_1\alpha_1 - B_1)^2 = D$$

and so α_1 is a root of the quadratic equation,

$$C_1^2x^2 - 2B_1C_1x + B_1^2 - D = 0\,.$$

Step 2: In this step, we aim to show that the complete quotient α_1, like α, is a RQI with discriminant D. By definition,

$$\begin{aligned}\frac{1}{\alpha_1} &= \alpha - q_0 = \frac{B+\sqrt{D}}{C} - q_0 \\ &= \frac{B - q_0C + \sqrt{D}}{C}\,.\end{aligned}$$

Set $B_1 = -B + q_0C$, and observe that

$$\alpha_1 = \frac{C}{-B_1+\sqrt{D}} = \frac{C(B_1+\sqrt{D})}{-B_1^2+D}\,.$$

Since $B_1 \equiv B \pmod{C}$ and C divides $D - B^2$, it also divides $-B_1^2 + D$. Therefore we can write

$$-B_1^2 + D = CC_1$$

for some $C_1 \in \mathbb{Z}$. Hence

$$\alpha_1 = \frac{B_1+\sqrt{D}}{C_1} \tag{3.j}$$

and so its conjugate α' is given by

$$\alpha_1' = \frac{B_1-\sqrt{D}}{C_1}\,.$$

The upshot of this is that α_1 is a quadratic irrational with discriminant D, the same as α.

(3.9) Question about Conjugates. Explain why

$$\alpha_1' = -\frac{1}{q_0 - \alpha'} \tag{3.k}$$

and deduce that

$$\alpha_1 > 1 \quad \text{and} \quad -1 < \alpha_1' < 0\,,$$

in other words that α_1 is an RQI.

Hints

(a) The real numbers form a vector space (of uncountable dimension!) over the rational numbers.

(b) $\sqrt{D}$ is irrational.

Answer to (3.9)

We can now repeat this argument as often as we like to conclude that $\alpha_1, \alpha_2, \ldots$ are all RQIs with discriminant D; in particular, α_n has the form

$$\alpha_n = \frac{B_n + \sqrt{D}}{C_n}$$

where B_n and C_n are natural numbers satisfying

$$B_n < \sqrt{D} \quad \text{and} \quad C_n < 2\sqrt{D}\,. \qquad (3.l)$$

by Properties P2, P3 and P4 in Question 3.8. Since there are less than $2D$ pairs (B_n, C_n) satisfying the inequalities (3.l), there are less than $2D$ possibilities for α_n. Hence there must be a repetition in the list

$$\alpha = \alpha_0, \alpha_1, \ldots, \alpha_{2D}\,.$$

Step 3: We have seen from Step 2 that

$$\alpha_m = \alpha_n \tag{3.m}$$

for different m and n satisfying $0 \leq m < n \leq 2D$. We will now show that if $m > 0$, then $\alpha_{m-1} = \alpha_{n-1}$. It follows that the smallest m for which (3.m) holds is $m = 0$, and therefore that periodicity gets under way without delay, in other words, that α is purely periodic. For $n = 0, 1, 2, \ldots$ set

$$\beta_n = -\frac{1}{\alpha'_n} > 1$$

By the earlier argument,

$$\alpha'_n = q_n + \frac{1}{\alpha'_{n+1}}$$

or, in terms of the newly-defined βs,

$$\beta_{n+1} = q_n + \frac{1}{\beta_n}\,.$$

Thus, q_n is the integral part both of α_n and β_{n+1}. If $\alpha_m = \alpha_n$, then $\alpha'_m = \alpha'_n$, and so $\beta_m = \beta_n$. But then q_{m-1}, as the integral part of β_m, must be equal to the integral part q_{n-1} of β_n. Consequently,

$$\alpha_{m-1} = q_{m-1} + \frac{1}{\alpha_m} = q_{n-1} + \frac{1}{\alpha_n} = \alpha_{n-1}$$

as asserted above. This completes the proof of our long-term goal that an RQI has a purely periodic CF. Putting this fact together with Proposition 3.7 gives the following main result.

(3.10) Theorem *A real number α has a purely periodic continued fraction if and only if*

(a) *it is an irrational root of a quadratic equation with integer coefficients,*

(b) *it is greater than 1, and*

(c) *it has a conjugate α' satisfying*

$$-1 < \alpha' < 0\,.$$

We will continue our story of continued fractions in Workbook 5 and will leave you, as a cliff-hanger, with the question we began with:

Which real numbers have periodic continued fractions?

Meanwhile, here are some exercises to reinforce your understanding of the material in this section.

(3.11) Concluding Exercises.

(a) Let $\alpha = 3 + \sqrt{13}$. Is α a RQI? Work out its CF.

(b) Which irrational number is represented by the purely periodic CF $\overline{8, 1, 2, 4, 2, 1}$?

(c) Prove that the highest common factor of $A_n = [q_0, q_1, \ldots, q_n]$ and $B_n = [q_1, q_2, \ldots, q_n]$ is 1 for $n = 0, 1, 2, \ldots$ (thus the convergents A_n/B_n are rational numbers *in their lowest terms.*)

(d) Work out the first five convergents of

$$\frac{1}{2+}\,\frac{1}{6+}\,\frac{1}{10+}\,\frac{1}{14+}\cdots,$$

where the partial quotients are in arithmetic progression. Compare your answer for A_4/B_4 with

$$\frac{e-1}{e+1}.$$

Summary of Section 3

This was a very goal-directed section. Our sole aim was to characterize the real numbers α that are represented by *purely periodic* continued fractions. Along the way, we noted that such an α

- is irrational (because periodic continued fractions are infinite)
- is a root of a quadratic equation with integer coefficients
- is greater than 1 and so has the form

$$\alpha = \frac{B + \sqrt{D}}{C}$$

  where B, C and D are natural numbers and D is not a perfect square.
- has a conjugate

$$\alpha' = \frac{B - \sqrt{D}}{C}$$

  in the open interval $(-1, 0)$.

The second half of the section was devoted to showing the remarkable fact that these properties (of being a *reduced quadratic irrational*) completely describe the real numbers that have purely periodic continued fractions.